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FASCINATING FRACTIONS

ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Н. М. Бескин

ЗАМЕЧАТЕЛЬНЫЕ ДРОБИ

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Contents

Preface

This booklet is intended for high-school students interested in mathematics. It is concerned with approximating real numbers by rational ones, which is one of the most captivating topics in arithmetic.

In the last decade, some young mathematicians, and not only young mathematicians, have shown a negligent attitude towards "classical" and "pure" mathematics in contrast with "modern" and "applied" mathematics. This stance is fully unjustified.

First, mathematics rests on a foundation of numerous classical theories, facts and findings which must be known to every mathematician. For instance, the theory of continued fractions, a part of classical pure mathematics, is widely used nowadays to calculate numerical values of functions by means of computers.

Second, while science develops, many of its theories become obsolete and "dry up", like some branches of a tree. Quite a few do, yes, but not all of them. There are theories which survived centuries (or even millenia) and still retained their significance.

Continued fractions represent one of the most perfect creations of 17-18th century mathematicians: Huygens, Euler, Lagrange, and Legendre. The properties of these fractions are really striking.

The following should be borne in mind when reading this booklet.

Topics easily understandable are presented in normal print, while those more difficult are given in small print. Proofs of some theorems given in small print may be omitted safely. These theorems will necessarily be taken for granted.

However, mathematics is not just reading for entertainment. A future mathematician as well as a physicist or an engineer has to acquire skill in dealing with mathematical constructions and proofs. So take a pencil and a sheet of paper and study carefully the topics given in small print. You may succeed in simplifying some proofs or finding better ones.

The theory of continued fractions is vast. This booklet covers only its fundamentals, but it contains everything that may be useful for a layman interested in mathematics. Professional mathematicians need to know much more.

Nikolai Beskin

Chapter 1

Two Historical Puzzles

1.1. Archimedes' Puzzle

1.1.1. Archimedes' Number. Many people believe that only a distant journey, preferably to outer space or the ocean bottom, could enable them to meet anything extraordinary, for the everyday life is so familiar that can show up no unusual facets.

What a delusion it is! Our surroundings are full of puzzles which go unnoticed because they seem to be habitual.

This chapter tells us a story of two enigmatic, yet familiar, episodes from the history of mathematics.

High-school students the world over know from the course in geometry a symbol π which denotes the ratio of the circumference of a circle to its diameter.

The letter π is the first letter of the Greek word περιφερεια which means "circle". An English mathematician Jones was the first to introduce the symbol π in 1706. In 1736 Euler adopted this notation instead of the symbol p he previously used. Since then the symbol π has come into general use.

From the most ancient times mathematicians sought a value for the number π. Archimedes determined its approximate value as 22/7*. This fact is so well known that hardly anybody suspects that it conceals a mystery. Who ever asks

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* Actually Archimedes gave a different formulation to this result in his book *On the Measurements of the Circle.* He determined for π its bounds: $3\frac{10}{71} < \pi < 3\frac{1}{7}$. To quote Archimedes, "The circumference of any circle equals three times the diameter plus an excess which is less than one seventh of the diameter but greater than $\frac{10}{71}$ of it."

Although the value of π is closest to $3\frac{10}{71}$ as compared to $3\frac{1}{7}$, the simpler value $3\frac{1}{7}$ is the one in general use.

why Archimedes chose a fraction with 7 for denominator? What would happen if π were approximated by a fraction with denominator 8?

This question proves to be of extreme interest.

1.1.2. Approximation. Mathematicians often encounter a problem of replacing an object (a number, a function, a figure, etc.) by some other object of the same nature, which is in some sense sufficiently near to, but simpler than, the original one. This replacing is called the *approximation*. In the general

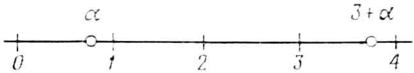

Fig. 1

case it requires that a set of objects be singled out and the sense of the phrase "sufficiently close to" be defined. We shall not discuss this general problem and restrict ourselves to the approximation of real numbers.

Let us consider the set of all real numbers. The conventional notation for this set is **R**. Real numbers may be of complicated nature, e.g. irrational numbers, or be cumbersome, e.g. fractions with large denominators.

It is worth explaining why cumbersomeness of a fraction is evaluated by its denominator. (We remind that a fraction is a number $\frac{p}{q}$ where p and q are *integers*, and $q \neq 0$; therefore, $\frac{\sqrt{3}}{3}$ and $\frac{\pi}{2}$ are not fractions.) If we are mainly interested not in the magnitude of a real number α but in its arithmetic nature, we need to know the position of α between two consecutive integers n and $n+1$. The addition of an integer to the number α will not change the arithmetic nature of α (this statement does not hold for that branch of arithmetic which deals with integers). Figure 1 shows two numbers α and $3+\alpha$ identically located within segments [0, 1] and [3, 4] (the term "segment" is defined on p. 41). For instance, the numbers $\frac{391}{4} = 97\frac{3}{4}$ and $\frac{3}{4}$ are identically located within the corresponding segments [97, 98] and [0, 1], and thus there are no reasons to regard the former as being more complicated than the latter. This implies that an analysis of the nature of the numbers within the segment [0, 1] would be quite sufficient since the same pattern is reproduced within each segment $[n, n+1]$. This is why we are concerned only with the denominator when evaluating the cumbersomeness of a fraction.

Let us single out a subset of fractions with a given denominator q from the set $\mathbf{R}$ of all real numbers. The distance between a number α and a fraction $\frac{p}{q}$ is $\left|\alpha - \frac{p}{q}\right|$. Now we can give an interpretation of the problem of the approximation of real numbers as follows: *to approximate a real number α by a fraction with q denominator which is the closest to α among all fractions with q denominator.*

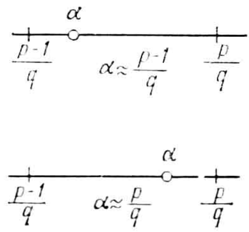

Fig. 2

If we mark all fractions with q denominator on the number line, the number α will fall within an interval between two fractions or coincide with one of them. The latter case is trivial, and we can write that

$$\frac{p-1}{q} < \alpha < \frac{p}{q}.$$

Of these two fractions the one nearest to α is chosen as its approximation (Fig. 2).

It could happen that α is the middle point of the segment $\left[\frac{p-1}{q}, \frac{p}{q}\right]$. This and only this case implies that there exist two solutions of the problem. *For the sake of definiteness*, we choose to adopt the left-end point of the segment as the approximation of α.

It is clear, therefore, that a fraction with any denominator can approximate the number α, that is, the choice of q denominator is a matter of preference.

Approximation is employed when it is necessary to use a rational number instead of an irrational number. It is also applicable to replacement of rational numbers by less cumbersome ones, i.e. by numbers with smaller denominators. For instance, the approximation of the number $\frac{2936}{7043}$ by the fraction with denominator 12 is

$$\frac{2936}{7043} \simeq \frac{5}{12},$$

since

$$\frac{5}{12} < \frac{2936}{7043} < \frac{6}{12},$$

where $\frac{2936}{7043}$ is nearer to $\frac{5}{12}$ than to $\frac{6}{12}$.

The approximation of real numbers by decimal fractions has long been in general use. However, decimals were yet unknown in Archimedes' time*, and he could choose fractions with arbitrary denominators to approximate the number π. Why did he prefer fractions with denominator 7? Could it be purely accidental?

1.1.3. Error of Approximation. A real number α is approximated by a fraction $\frac{p}{q}$ with an *error*

$$\Delta = \alpha - \frac{\hat{p}}{q},$$

where $\frac{\hat{p}}{q}$ stands for the end point of the segment $\left[\frac{p-1}{q}, \frac{p}{q}\right]$ which is the closest to α.

The error is thus the exact value of α minus its approximation.

Therefore, the error is positive if $\frac{\hat{p}}{q} = \frac{p}{q}$, and negative if $\frac{\hat{p}}{q} = \frac{p-1}{q}$.

The absolute value $|\Delta|$ of the error is called the *absolute error*.

It is clear that the absolute error does not exceed $\frac{1}{2q}$ (see Fig. 2):

$$|\Delta| \leqslant \frac{1}{2q}.$$

The number $\frac{1}{2q}$ is the *upper bound of absolute error*. The upper bound depends on the choice of approximation. For

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* Decimal fractions became known in Europe at the end of the 16th century, although in the Orient they had been used since the end of the 15th century. They were invented by the Flemish scientist Simon Stevin. Here is what the English writer Jerome K. Jerome had to say on the matter: "From Gent we went to Bruges (where I had the satisfaction of throwing a stone at the statue of Simon Stevin, who added to the miseries of my school-days by inventing decimals), and from Bruges we came on here." (*Diary of a Pilgrimage*, the entry for Monday, June 9.)

instance, *if we agreed* to approximate the number α by the left-end point of the segment $\left[\frac{p-1}{q}, \frac{p}{q}\right]$, then the upper bound would be $\frac{1}{q}$.

1.1.4. Quality of Approximation. The absolute error approaches the upper bound if α is the middle point of the segment $\left[\frac{p-1}{q}, \frac{p}{q}\right]$. This is the most unfavourable case. If, however, α is very close to one of the end points, the actual absolute error may be considerably smaller than the upper bound.

This observation suggests that the evaluation of the *quality* of approximation is required. It is clear that *the approximation of a number α by a fraction with a small denominator is appropriate if the error is small*; or, to be more precise, if the absolute error is substantially less than the upper bound of the error (Fig. 3).

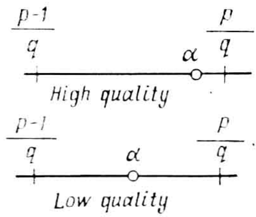

Fig. 3

In order to evaluate the quality of approximation, we have to estimate the ratio of the actual absolute error to the upper bound on the absolute error

$$\frac{\text{absolute error}}{\text{upper bound on absolute error}} = \frac{|\alpha - p/q|}{1/2q} = 2\,|q\alpha - p|.$$

It is convenient to consider one half of this ratio denoted by h and called the *normalized error*,

$$h = |q\alpha - p|. \tag{1}$$

The normalized error h is thus one half of the ratio of the actual absolute error to the maximum possible error. It is obvious that

$$0 < h \leqslant \frac{1}{2}.$$

The quality of approximation is the higher, the less h is.

We call the quantity

$$\lambda = \frac{1}{2h} = \frac{1}{2\,|q\alpha - p|} \tag{2}$$

the *quality factor*. It has a simple and lucid meaning: *The quality factor of approximation is the factor by which the actual absolute error is less than the maximum possible error.* It is obvious that

$$1 \leqslant \lambda < \infty,$$

and the greater λ, the better the approximation.

It would be wrong to expect fractions with greater denominators to be more useful. It could happen that the approximation of the number α by a fraction with the denominator 8 is less accurate than that by a fraction with denominator 7. Let us have a look at the number π, approximated by fractions with denominators from 1 through 10 (see Table 1). We omit the calculations, leaving them to the reader.

Table 1

q	Approximate value of π	Upper bound on absolute error	$\lvert\Delta\rvert$	h	λ
1	$\frac{3}{1}$	$\frac{1}{2}=0.5000$	0.1416	0.1416	3.5
2	$\frac{6}{2}$	$\frac{1}{4}=0.2500$	0.1416	0.2832	1.8
3	$\frac{9}{3}$	$\frac{1}{6}=0.1667$	0.1416	0.4248	1.2
4	$\frac{13}{4}$	$\frac{1}{8}=0.1250$	0.1084	0.4336	1.2
5	$\frac{16}{5}$	$\frac{1}{10}=0.1000$	0.0584	0.2920	1.7
6	$\frac{19}{6}$	$\frac{1}{12}=0.0833$	0.0251	0.1504	3.3
7	$\frac{22}{7}$	$\frac{1}{14}=0.0714$	0.0013	0.0089	56.5 (!)
8	$\frac{25}{8}$	$\frac{1}{16}=0.0625$	0.0166	0.1327	3.8
9	$\frac{28}{9}$	$\frac{1}{18}=0.0556$	0.0305	0.2743	1.8
10	$\frac{31}{10}$	$\frac{1}{20}=0.0500$	0.0416	0.4159	1.2

This table demonstrates that the approximation of π by fractions with denominator 7 is more accurate than that by the other fractions. The actual error is less than its upper bound by a factor of 56.5.

Figure 4 shows the location of π on the number line. Accidentally (but is it indeed accidental?) π happens to be quite close to $3\frac{1}{7}$. If it were prescribed to approximate π with the absolute error less than or equal to 0.0013, how would we proceed? We would write down the condition

$$\frac{1}{2q} \leqslant 0.0013,$$

whence $q \geqslant 385$. Archimedes had achieved the same accuracy using a much smaller denominator. It is worth mentioning here that fractions with denominator 385 make it possible to approximate any real number with an error less than 0.0013, while fractions with denominator 7 are more preferable for approximating the π number.

Archimedes' choice could not therefore be accidental. But how did he make that choice?

Many centuries later (in 1585) a Dutch scientist from Metz, Adriaen Antoniszoon (also known as Adriaen Antonisz) found an approximate value $\frac{355}{113}$ for π.

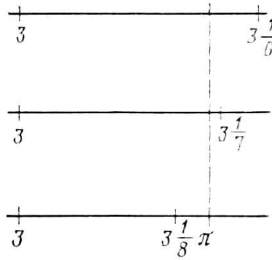

Fig. 4

This result has been published after Antoniszoon's death, by his son Adriaen Metius, so that the value 355/113 is traditionally called *Metius' number*. Metius' number has the same striking property as Archimedes' number: the actual error is less than it could be expected for the denominator 113. We invite the reader to examine Metius' number in the same way as Archimedes' number has been analysed.

There is no doubt that Metius' number was not an accidental discovery. In fact, it was known long before Adriaen Antoniszoon happened to find it (see, e.g. Struik's book in the Bibliography).

1.2. The Puzzle of Pope Gregory XIII

1.2.1. The Mathematical Problem of the Calendar. Pope Gregory XIII was not a mathematician but his name is associated with an important mathematical problem, that of the calendar.

Nature has supplied us with two obvious time units: the year and the day (solar day). We even read in one old text-

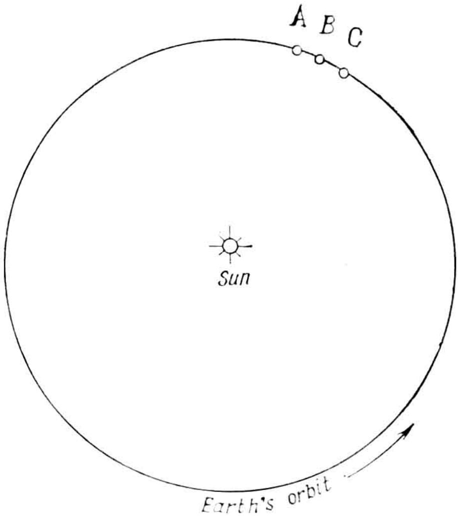

Fig. 5

book on cosmography: "*Unfotunately*, the year does not comprise an integral number of days." We could not but agree with this complaint because the fact does bring a lot of inconvenience. However, it also generates an interesting mathematical problem.

$$1 \text{ year} = 365 \text{ days } 5 \text{ hours } 48 \text{ minutes } 46 \text{ seconds}$$
$$= 365.242199 \text{ days*}.$$

It would be impossible to enact and implement this duration of the year in civil life. But what if the civil year is declared exactly 365 days long? Figure 5 shows the orbit of the Earth. On January 1, 1985, at midnight, the Earth was at point A. On January 1, 1986, at midnight, it will be at point B, and next January 1 it will be at point C; and so forth. As a result, if we mark on the orbit the position of the Earth corresponding to a fixed date, this position will not be the same each year but will retard by nearly six hours.

* Neither the astronomical aspects of the calendar (such as variation in the length of the year) not its history are analysed here in detail; we concentrate only on one mathematical problem connected with the calendar. We recommend that the reader interested in these details look them up elsewhere.