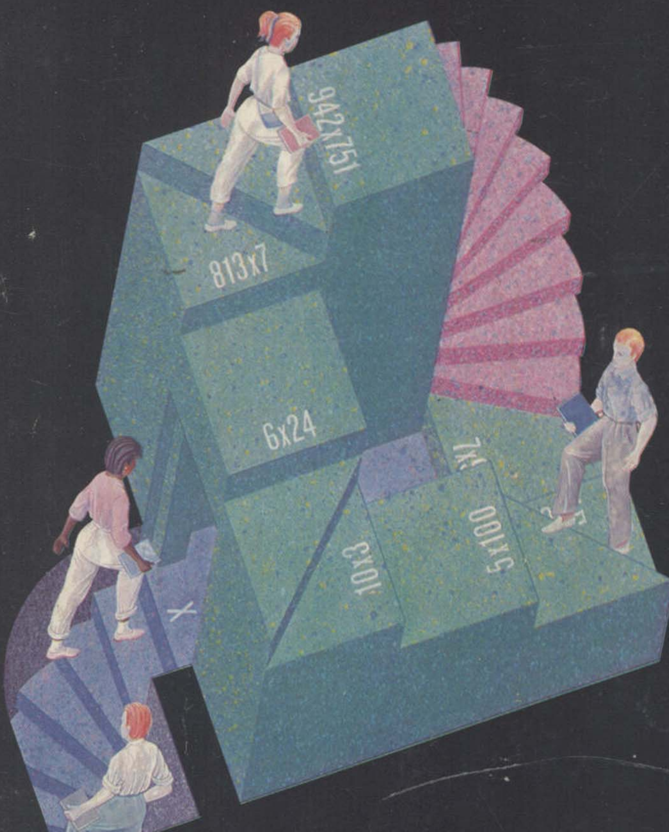
The Psychology of Learning Mathematics

RICHARD R. SKEMP



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Richard Skemp was born at Bristol in 1919. He was a foundation scholar at Wellington College, from which he won an open mathematical scholarship to Hertford College, Oxford, in 1937. From 1939 to 1945 he served with the Royal Signals, returning afterwards to Oxford to complete his mathematical degree. During five years of teaching mathematics at secondary schools, he became convinced of the need for more understanding by teachers of how children learn. He went back once more to Oxford and took a psychology degree in 1955.

From then until 1973 he was in the Psychology Department at Manchester University, taking his Ph.D. in 1958, and in charge of the Child Study Unit from 1965. In 1962 he was one of the two British members at the Unesco International Symposium on School Mathematics held in Budapest, and from 1962 to 1969 he directed the Leicestershire Psychology and Mathematics Project. As well as writing school mathematics texts, he has published a number of papers on the psychology of human learning, and is a Fellow of the British Psychological Society. In 1973 he became Professor of Education at Warwick University. In 1978 his title was changed to Professor of Educational Theory, and he became Director of the Mathematics Education Research Centre at Warwick University. From 1980 to 1982 he was President of the International Group for the Psychology of Mathematics Education.

Richard Skemp is married, with one son, and is interested in music, photography, sailing and travel.

Acknowledgements

Much of the material in Part A of this book was first given at a series of weekend and day courses organized by my friend and colleague Laurie Buxton, at that time Staff Inspector (Mathematics) in the Inner London Education Authority. I am most grateful to him, and to the teachers attending the courses, for this stimulus to my thinking, and for the discussions with teachers which kept me closely in touch with their viewpoints and needs. Laurie Buxton also read the manuscript and made a number of helpful suggestions.

In this, the second, edition, I am glad to have the opportunity to thank the very many persons who have written to me from all over the world – teachers of all kinds, and others who found that, after all, mathematics could and did make sense to them. If I may be forgiven for singling out one from so many, I should like to quote a short extract out of a letter from Beijing which gave me particular pleasure: 'Never didn't I read such good book like this one. It is as though I had found a treasure.'

This and other equally kind letters have been a great enrichment to my life. Thank you all very much.

Editorial Foreword

Mathematics is a curious subject, psychologically. It seems to divide people into two camps, just as there are said to be cat-lovers and dog-lovers; there are those who can do mathematics and there are those who cannot, or who think they cannot, and who 'block' at the first drop of a symbol. Again, mathematicians are said often to be very musical, with the implication that mathematical and musical thinking have something in common. On the other hand, mathematicians are often poor practitioners with words, so supporting the idea that verbal and mathematical intelligence may not go together.

Professor Skemp, the author of this book, is both a mathematician and a psychologist, and, most unusually, he is expert in teaching both subjects. In the first part of the book he looks at the thought processes which people adopt when they do mathematics, and he analyses it psychologically. It is done painlessly so that the reader who knows neither psychology nor mathematics will find himself introduced to a fascinating and important field of inquiry. And those who know something of both subjects will be interested to see the way in which the author has gone beyond previous notions.

In the second part of the book Professor Skemp produces what might almost be thought of as a textbook based on the psychological ideas developed in the first half. It is a bold step, and one which is calculated to give those who already know a little mathematics new insights into their own thinking and into the subject itself. This book will also be a stimulus to all those concerned with improving mathematical teaching.

B. M. FOSS, 1971

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PART A

Introduction

Since the early 1960s there has been much concern about the teaching of mathematics, and much activity. In many parts of the world projects have proliferated, and new topics have been introduced under the name of 'modern mathematics' (though most of the topics so described date from before the turn of the century).

Nevertheless, after two decades of effort by many groups of intelligent and hard-working people, so little progress had been made in the U.K. that it was found necessary to set up a governmental committee to inquire into the problems of mathematical education.[1] (Readers elsewhere must judge for themselves whether a similar situation also applies in their own countries.) So where have things gone astray?

Readers for whom mathematics at school was a collection of unintelligible rules which, if memorized and applied correctly, led to 'the right answer' (the criterion for which was a tick by the teacher) would probably agree that there was need for change. Parents of children whose school mathematics also answers this description may feel that there still is. But change is not necessarily for the better, and the introduction of new topics does not bring about better understanding if these are taught in the same bad old way. In contrast, some of us learnt what is now called 'traditional' mathematics in a meaningful way. New topics are not in themselves a sufficient answer.

Other reformers try to present mathematics as a logical development. This approach is laudable in that it aims to show that mathematics is sensible and not arbitrary, but is mistaken in two ways. First, it confuses the logical and the psychological approaches. The main purpose of a logical presentation is to convince doubters; that of a psychological one is to bring about understanding. Second, it gives only the end-product of mathematical discovery ('This is it: all you have to do is learn it'), and fails to bring about in the learner those processes by which mathematical discoveries are made. It teaches mathematical thought, not mathematical thinking.

Problems of learning and teaching are psychological problems, and we can expect little improvement in the teaching of mathematics until we know more about how it is learnt. This book is offered as a contribution mainly in the latter field. It will therefore, I hope, be of interest not only to teachers of mathematics (present and prospective) but also to those who, in spite of past encounters, would still like to learn to understand a little of this subject, and also to parents.

The first part of the book will be concerned with this most basic problem: what *is* understanding, and by what means can we help to bring it about? We certainly think we know whether we understand something or not; and most of us have a fairly deep-rooted belief that it matters. But just what happens when we do understand that does not happen when we don't understand, most of us have no idea. Sometimes, moreover, we think that we have understood something, only to find afterwards that we did not. So until we have a better understanding of understanding itself, we shall be in a poorer position either to understand mathematics ourselves, or to help other people to do so.

In Part B this knowledge will be applied to some fairly basic parts of mathematics; and at the same time the mathematics will be used to illustrate further the ideas developed in Part A. So the first part is mainly about psychology, with occasional references to mathematics; the second part, mainly mathematics, with occasional references to psychology. The mathematics in Part A is there mainly by way of example of psychological ideas, and since other non-mathematical examples are also given, the mathematics here should be taken lightly at any point where it offers difficulty.

Psychology and human learning

Since this is a book about both psychology and mathematics, written by a mathematician turned psychologist, it will be worth discussing the interaction between these two fields of study. For this a few lines of an autobiographical nature are necessary.

I began my professional career as a teacher of mathematics. As my task shifted from that of learning mathematics myself to that of teaching it to other people, I became increasingly concerned with the problem of those pupils who, though intelligent and hard-working, seemed to have a blockage about mathematics. This did not seem to make sense. Surely the main ability required for mathematics was the ability to form and manipulate abstract ideas; and surely this ability coincided closely with what we mean by intelligence? So there seemed to be a contradiction. Gradually I became more and more interested in problems of learning and teaching. These are psychological problems, and to study them more fully, I eventually went back to college and took a degree in psychology. Afterwards, when I was fortunate enough to get a job with opportunities for research, lecturing in psychology at Manchester University, my natural choice for a field of investigation was the problems of learning mathematics.

Habit learning and intelligent learning

A first thread which emerged from these researches was that there seemed to be a qualitative difference between two kinds of learning which we may call habit learning, or rote-memorizing, and learning involving understanding, which is to say intelligent learning. The former can be replicated in the laboratory rat or pigeon, and for various reasons (such as the greater degree of experimental control which is possible) academic psychologists for many years preferred to study this kind of learning. Since the first edition of this book appeared, however, there has been important progress; and influenced by the pioneering work of Piaget, psychologists in many countries are now studying cognitive processes in children and adults.

It is intelligent learning, in contrast to habit learning, in which human beings most excel and in which they most differ from all other species. Though one must not deny intelligence of a sort to the other animals, the intelligence of our own species is so much greater that it has the effect of a difference of kind * and not just of degree. Now, intelligence is a subject which has long been well to the fore in psychological research. But studies in this field have in the main been directed either towards psychometrics (the measurement of intelligence) or towards the relative

* This 'difference of kind' is discussed further in Chapter 1, page 26.

contributions of heredity and environment on intelligence ('nature v. nurture'), while learning theorists still have little to tell us about the interaction between intelligence and learning.

For the psychologist who is interested in intelligent learning, which is to say the formation of conceptual structures communicated and manipulated by means of symbols,* mathematics offers what is perhaps the clearest and most concentrated example. In studying the learning and understanding of mathematics, we are studying the functioning of intelligence in what is at once a particularly pure, and also a widely available, form.

What is intelligence?

What do we mean by intelligence? Though the psychological meaning of this word corresponds roughly to our everyday usage of it, there is as yet no general agreement among psychologists about how it may best be defined.[2] Of those offered by other psychologists, the one which fits in best with my own approach remains that given by Vernon in 1969: 'Intelligence B is the cumulative total of the schemata or mental plans built up through the individual's interaction with his environment, insofar as his constitutional equipment allows.'[3]

Two terms here require further explanation: 'intelligence B' and 'schemata'. The latter will be discussed at length in Chapter 2; briefly, 'schemata' (or 'schemas') means the same as the conceptual structures referred to above. 'Intelligence B' is a term introduced by Hebb in 1949. In his own words: 'From this point of view it appears that the word "intelligence" has *two* valuable meanings. One is (A), an *innate potential*, the capacity for development, a fully innate property that amounts to the possession of a good brain and a good neural metabolism. The second is (B), the functioning of the brain in which development has gone on, determining an *average level of performance or comprehension* by the partly grown or mature person. Neither, of course, is observed directly; but *intelligence B*, a hypothetical level of development in brain function, is a much more direct inference from behaviour than *intelligence A*, the original potential.'[4] Hebb's discussion, from which this passage is extracted, is written from a neurological point of view. The term 'intelligence

* The meaning of this sentence will be developed in later chapters.

B' has since become widely used also in the context of mental functioning (as distinct from neural activity).

Mathematics and intelligence B

Mathematics is a particularly good example of intelligence B. There are two reasons for this. First (which is to summarize what has been said already), the learning of mathematics affords many and clear examples of the development of the schemas whose total (including, of course, non-mathematical schemas also) constitutes intelligence B as described by Vernon. Second, the application of mathematics to problems of natural sciences, of technology and of commerce is so powerful that mathematics appears as one of the most, perhaps the most, highly developed mental tools available to us for dealing with our physical environment. If intelligence B is intelligence in its function of understanding, predicting and controlling our physical environment, then mathematics exemplifies intelligence B in one of its most successful developments.

Unrealized potential in Homo sapiens

Our own species, *Homo sapiens*, is a new breakthrough in evolution. But the intelligence wherein lies our superiority remains, in many cultures, largely unrealized. It is no exaggeration to say that the differences in material standards between the technologically most advanced cultures and the most primitive are as great as or greater than the differences between the latter and the highest of the other animal species. The former difference results not from differences in intelligence A, about which the evidence is still quite inconclusive, but from differences in intelligence B, since it is this which is effective in controlling the environment.

But even in those cultures where, through schools, colleges, printing, broadcasting and other means, potential intelligence A is more fully realized as functioning intelligence B, the developmental process is still almost entirely based on tradition and opinion rather than on scientific knowledge and research. If we were able consciously and deliberately to foster the growth of intelligence B, who knows what further strides our civilization might make? And if we want to find out how this might be done, it would be hard to think of a better starting point than to study the learning of mathematics, which increasingly I have come to see as one of the most adaptable and powerful mental tools which human intelligence has constructed, cumulatively over the centuries, for its own use.

This is what the rest of the present book is about. For myself, this starting point has indeed led to greater understanding of human intelligence in a wider context.[5] I hope that it may prove to be so for others also.

CHAPTER 1

The Formation of Mathematical Concepts

Two terms which have recurred throughout the introduction are 'concept' and 'schema'. In this chapter we shall examine what we mean by concepts, and how we form, use and communicate these. Then, in Chapter 2, we shall consider how concepts fit together to form conceptual structures, called schemas, and examine some of the results which follow from the organization of our knowledge into these structures.

Abstracting and classifying

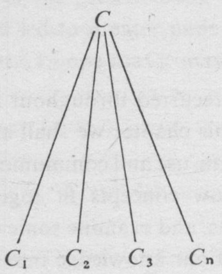
Though the term 'concept' is widely used, it is not easy to define. Nor, for reasons which will appear later, is a direct definition the best way to convey its meaning. So I shall approach it from several directions, and with a variety of examples. Since mathematical concepts are among the most abstract, we shall reach these last.

First, two pre-verbal examples. A baby boy aged twelve months, having finished sucking his bottle, crawled across the floor of the living room to where two empty wine bottles were standing and stood his own empty feeding bottle neatly alongside them. A two-year-old boy, seeing a baby on the floor, reacted to it as he usually did to dogs, patting it on the head and stroking its back. (He had seen plenty of dogs, but had never before seen another baby crawling.)

In both these cases the behaviour of the children concerned implies: one, some kind of classification of their previous experience; two, the fitting of their present experience into one of these classes.

We all behave like this all the time; it is thus that we bring to bear our past experience on the present situation. The activity is so continuous and automatic that it requires some slightly unexpected outcome thereof, such as the above, to call it to our attention.

At a lower level we classify every time we recognize an object as one which we have seen before. On no two occasions are the incoming sense-data likely to be exactly the same, since we see objects at different distances and angles, and also in varying lights. From these varying inputs we abstract certain *in*variant properties, and these properties persist in memory longer than the memory of any particular presentation of the



object. In the diagram, C_1, C_2 . . . represent successive past experiences of the same object, say, a particular chair. From these we abstract certain common properties, represented in the diagram by C. Once this abstraction is formed, any further experience, C_n, evokes C, and the chair is *recognized*: that is, the new experience is classified with C_1, C_2, etc.; C_n and C are now experienced together; and from their combination we experience both the *similarity* (C) of C_n to our previous experiences of seeing this chair and also the particular distance, angle, etc., on this occasion (C_n).

We progress rapidly to further abstractions. From particular chairs, C, C', C'', we abstract further invariant properties, by which we recognize Ch (a new object seen for the first time, say, in a shop window) as a member of this class. It is the second-order abstraction (from the set of abstractions C, C', . . .) to which we give the name 'chair'. The invariant