Iterations of Differential Operators

by

A. V. Babin

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FOREWORD

The basic theme of this book is the examination of the connections between solutions of differential equations and the iterations $A^j f$ of the differential operator A, which occurs in an equation, and which are used in the given function f, occurring in that equation. The first chapter discusses the Kotake-Narasimhan theorem, which establishes a connection between the rate of increase of the norms $\|A^f f\|$ of iterations of the elliptic operator A and the analyticity of the function u - the solution of the equation $Au=f$. Basically, however, this book describes the construction and examination of explicit formulae which express the solution u of different problems, containing the operator A, in terms of $A^j f$, $j = 0,1,...$. The construction of these formulae is based on the systematic use of the methods of the theory of weighting approximations of the analytic functions of one complex variable using polynomials.

The formulae obtained are used to construct solutions of differential equations, and also to examine the properties of solutions of degenerating equations. There are also other applications to the problems of functional analysis and the theory of functions.

We shall describe the contents of the book in more detail.

The solutions of different problems for linear differential equations can be represented in the form of functions of a differential operator occurring in that equation. For example, the solution of the equation $Au-\zeta u=f$ is represented in the form $u=(A-\zeta I)^{-1}f$. The solution of the Cauchy problem $\partial_t u(t) = -Au(t)$, $u(0) = f$ can be written using the formula $u(t) = e^{-tA}f$. As is well known, we can use various methods to make sense of these formulae. For this we usually use either a spectral expansion of the operator A, or a construction of the function $g(A)$ using the resolvent of the operator A. In any case, the expression $g(A)f$ has a fully defined meaning for wide classes of differential operators A, functions $g(\lambda)$ and vectors f.

At the same time, the question of how to use $g(\lambda)$, A and f to construct $u=g(A)f$ is extremely important. This problem is solved in an elementary way if $g(A)=P(A)$ is a polynomial. In this case, to obtain $g(A)f$ it is sufficient to know how to obtain the iteration A^jf. If A is a differential operator with analytic coefficients, and f is an analytic function, it is not difficult, in theory, to obtain A^jf. Often (if the coefficients of the operator A and the function f are polynomials, for example), the corresponding calculations are quite simple. Moreover, in these cases it is also very difficult to calculate the functions $g(A)f$, for example of the form $(A-\zeta I)^{-1}f$ or $e^{-tA}f$, and we need to use finite-dimensional approximations of a different type to obtain $u=g(A)f$ (for example, finite-difference approximations or Galerkin approximations of the initial equation).

In this book we discuss another approach to calculating $g(A)f$. It consists of $g(A)f$ being represented in the form of a limit as $n\to\infty$ of the polynomials $P_n(A)f$ of the n-th degree from the operator A, applied to f. We were able to obtain similar representations for wide classes of differential operators of the first and second order with partial derivatives with analytic coefficients. The problem of constructing the polynomials $P_n(\lambda)$ turned out to be closely connected with the theory of weighting approximations of functions using polynomials on a half-line and on a line. This theory was established by Bernshtein (see his book [1]), and was then developed by a number of mathematicians (see the review articles by Akhiezer [1] and Mergelyan [1], and Mandel'broit's book [1]).

The theory of weighting approximations is developed in a new direction in the third chapter of this book. Namely, we basically consider explicitly constructing - for a specified analytic function $g(\lambda)$, a polynomial $P_n(\lambda)$, which approximates it with weight well on a straight line, and obtaining a fairly exact estimate of the error as $n \to \infty$.

In the fourth and sixth chapters we use the polynomials constructed in the third chapter to construct functions of operators in Hilbert and Banach spaces. We obtain polynomial representations of the form

$$g(A)f = \lim_{n\to\infty} P_n(A)f, \tag{1}$$

where P_n is a polynomial explicitly constructed according to the function g. The chapter contains broad classes of partial differential equations with analytic coefficients, to construct which we use (1).

A classic example of representing $g(A)f$ in the form (1) is the representation of the solution of Cauchy's hyperbolic problem $\partial_t^2 u(t) = -Au(t)$, $u(0)=f$, $\partial_t u(0)=0$, which is given in the Cauchy-Kowalewska theorem. In this case $P_n(A)$ is a finite segment of the series expansion $g(A)f = \cos(t\sqrt{A})f$ in powers of t for small $|t|$.

Using its special interpolation polynomials $P_n(\lambda)=P_n(t,\lambda)$ instead of the Taylor series of the function $g(\lambda)=g(t,\lambda)=\cos(t\sqrt{\lambda})$ enabled us explicitly to construct the solution $u(t)$ not only for small t, but also for as large t as desired. Formulae of the form (1) are also obtained for Cauchy's parabolic problem and for stationary equations.

§8 of the fourth chapter represents (1) in the form of an iteration scheme suitable for solving equations of the form $Au=f$ on a computer without recourse to finite-dimensional approximations. The chapter presents appropriate examples, and compares them with difference methods.

The explicit representations $g(A)f$ in terms of the iteration $A^j f$ of the operator using (1) enable us not only to construct $u=g(A)f$, but also to investigate the properties of the function $u=u(x)$.

The fifth chapter of the book thus analyses the properties of the function $u(x)=g(A)f(x)$ when A is a degenerating operator. In particular, estimates of the x-smoothness of the functions $u(x)$ and $u(x,t)$ of solutions of different types of degenerating differential equations are obtained. These equations have been the subject of thorough analysis (see, for example, the papers of Bony and Schapira [1-3], Moser [1], Cohn and Nirenberg [1], Oleinik [1], and Oleinik and Radkevich [1]).

Our use of the methods of the theory of functions enabled us to obtain, in a unique way, accurate estimates of the smoothness of different types of differential equations, and to establish the connection between the smoothness $u(x)=g(A)f(x)$ and the analytic properties $g(\lambda)$. In addition, we managed to include a number of cases that were not covered by previous papers, and to refine the known estimates of smoothness.

The second chapter examines the question of the existence of polynomial representations of the form (1) of solutions of the elliptic equations $Au=f$ with coefficients and right-hand sides from the Carleman classes $C(M(k))$ of infinitely differentiable functions. It turned out that this question was closely connected to the quasi-analyticity of the class $C(M(k))$. Namely, solutions u of equations with coefficients and the right-hand side from $C(M(k))$ are represented in the form (1) when, and only when, the class $C(M(k))$ is quasi-analytic.

In the seventh chapter, we discuss questions connected with expressing the solution of the nonlinear equation $F(u)=f$ in terms of $F^j(f)$, $j=0,1,\ldots$. This expression is treated in a generalised sense. Namely, it is required to obtain an expression of the value $h(u)$ of the functional h in the solution u of the equation $F(u)=f$ in terms of the value $h(F^j f)$ of this functional in iterations of the operator F, whilst it is required to obtain this expression for the complete system $\{h_i\}$ of functionals.

This problem is solved for a wide class of nonlinear elliptic second-order differential operators on the torus T^m. Its solution is based on the infinite-dimensional generalisation of Poincaré's theorem on the normal form of an analytic mapping. Namely, it is proved that we can construct in $W_p^4(T^m)$ a system of coordinates ξj, $j \in \mathbb{N}$, such that the effect of the operator F will take the form

$$F: \xi_j \to \lambda_j \xi_j, \ \xi_j \in \mathbb{R}, \ j \in \mathbb{N} \tag{2}$$

where λ_j are eigenvalues of the differential $F'(0)$ of the operator F in zero for the corresponding nonlinear differential operator F, $F(0)=0$, which acts in Sobolev's real space $W_p^4(T^m)$, $p > m$.

A mapping of the form (2) is the simplest form of mapping in an infinite-dimensional space. Thus, Poincaré's generalised theorem is, on the one hand, an analog of the theorem on the spectral expansion of linear self-adjoint differential operators to the nonlinear case. On the other, we can consider it a stationary analog of the theorem on the full integrability of evolution equations.

Chapters 2-7 basically contain a systematic discussion of the results obtained in [1-16]. This book demands of the reader a mastery of the fundamentals of analysis (the functions of a complex variable and the functions of many real variables), and also of the fundamentals of functional analysis. Those facts necessary for the discussion which exceed the minimum are formulated and provided in the references.

Chapters 3-6 form the core of the discussion, and can be read independently of the other chapters. If the reader is only interested in applications to differential equations, s/he can begin reading from the fourth chapter. The seventh chapter is completely autonomous (with the exception of §9), and the reader interested in nonlinear equations can begin reading from that chapter.

Each chapter contains formulae, theorems, lemmas and so on, which are numbered independently. If the need arises to refer to a formula or theorem from another chapter, the chapter number is added first. For example, a reference to Formula (3.5.7) is a reference to Formula (5.7) in the third chapter, the formula being in the fifth paragraph of that chapter.

The letter *C* with the same index can denote different constants in proofs and formulations of different theorems, lemmas and propositions, and different constants are numbered using different indices within the proof of one statement.

CONTENTS

ITERATIONS OF DIFFERENTIAL OPERATORS

CHAPTER 1. ITERATIONS OF ELLIPTIC OPERATORS WITH ANALYTIC COEFFICIENTS

§1. Estimates of the norms of iterations of differential operators with analytic coefficients.

Suppose $K \subset \mathbb{R}^m$ is a compactum. We shall use $A(K)$ to denote a set of functions that are really analytic on K. Namely, $f \in A(K)$ if f is definite and infinitely differentiable in some neighbourhood Ω of the compactum K, whilst the constants C_0 and C_1 exist, such that $\forall x \in \Omega \subset \mathbb{R}^m$

$$|\partial^\alpha, f(x)| \leqq C_0 C_1^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{Z}_+^m \tag{1.1}$$

Here and below we will use the notation

$$\alpha = (\alpha_1, \ldots, \alpha_m), \quad \partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_m^{\alpha_m}, \quad \partial_i = \partial / \partial x_i,$$
$$|\alpha| = \alpha_1 + \ldots + \alpha_m, \quad \alpha! = \alpha_1! \ldots \alpha_m!, \quad \xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_m^{\alpha_m}, \qquad (\xi \in \mathbb{R}^m, \alpha \in \mathbb{Z}_+^m).$$

$\mathbb{Z}_+$ is a set of nonnegative integers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

We shall use $\mathcal{O}_\delta^{\mathbb{C}}(K)$ to denote complex δ - neighbourhood of the compactum K

$$\mathcal{O}_\delta^{\mathbb{C}}(K) = \{z \in \mathbb{C}^m : \exists\, x \in K : |z - x| < \delta\}, \tag{1.2}$$
$$|z| = \max(|z_1|, \ldots, |z_m|).$$

We emphasise the difference $|\alpha|$ for $\alpha \in \mathbb{Z}_+^m$ from $|z|$ for $Z \in \mathbb{C}^m$ or $Z \in \mathbb{R}^m$.

Lemma 1.1 If K is a compactum and $f \in A(K)$ then the function $f(x)$ extends from K to the complex neighbourhood $\mathcal{O}_\delta^{\mathbb{C}}(K)$, if $0 < \delta < C_1^{-1} m^{-1}$, C_1 is the same as in (1.1). The following estimate holds:

$$\sup_{z\in\mathcal{O}_\varepsilon^C}|f(z)|\leqq\frac{C_0}{(1-mC_1\varepsilon)}\qquad 0<\varepsilon<\frac{1}{mC_1}. \tag{1.3}$$

Proof. According to (1.1), the function $f(x)$ decomposes in the neighbourhood of each point x into a Taylor series

$$f(x+y)=\sum 1/\alpha!\,\partial^\alpha f(x)y^\alpha. \tag{1.4}$$

The series converges to $f(x+y)$, since by virtue of (1.1) the residual term of Taylor's formula approaches 0 for small $|y|$. Using (1.1) and (1.4), we obtain that series (1.4) majorises using the series

$$C_0\sum\frac{(\alpha_1+\cdots+\alpha_m)!}{\alpha_1!\cdots\alpha_m!}(C_1|y_1|)^{\alpha_1}\ldots(C_1|y_m|)^{\alpha_m}$$
$$=C_0(1-C_1|y_1|-\cdots-C_1|y_m|)^{-1}. \tag{1.5}$$

It is obvious that series (1.5) converges, and consequently series (1.4) also converges for $|y|<1/(mC_1)$. Estimate (1.3) follows from the estimate for the majorant (1.5). Series (1.4) also determines $f(x+y)$ for complex $y\in\mathbb{C}^m$. Therefore formula (1.4) determines the continuation $f(x)$ to $\mathcal{O}_\delta^C(K)$.

Henceforth we shall use $A(K, C_1, C_0)$ to denote the set of functions from $A(K)$, for which estimate (1.1) holds when $x\in K$ with fixed C_0 and C_1.

Lemma 1.2. If the function $f(x)$, which is defined on the compactum $K\subset\mathbb{R}^m$, continues to the function $f(z)$, which is analytic in the complex neighbourhood $\mathcal{O}_\delta^C(K)$ of the compactum K, whilst

$$|f(z)|\leqq C_0(f)\qquad\text{when}\qquad z\in\mathcal{O}_\delta^C(K) \tag{1.6}$$

estimate (1.1) holds, where $C_0 = C_0(f)$, $C_1 = 2/\delta$.

Proof. When $x \in \mathcal{O}_{\delta/2}(K) = \mathcal{O}^c_{\delta/2}(K) \cap \mathbb{R}^m$ from Cauchy's integral formula we obtain

$$\frac{\partial^\alpha f(x)}{\alpha!} = \frac{1}{(2\pi i)^m} \int_{|x_1 - \zeta_1| = r} \cdots \int_{|x_m - \zeta_m| = r} \frac{f(\zeta)\, d\zeta_1 \ldots d\zeta_m}{(\zeta - x)^\alpha (\zeta_1 - x_1) \ldots (\zeta_m - x_m)}$$

where $r = \delta/2$. Estimating the integral on the right-hand side in absolute value using formula (1.6), we obtain that

$$|\partial^\alpha f(x)| \leqq \alpha! C_0(f) r^{-|\alpha|} \leqq |\alpha|! C_0(f)(2/\delta)^{|\alpha|}$$

whence follows (1.1).

Let us now consider, in the bounded domain $\Omega \subset \mathbb{R}^m$, the differential operators B of order p of the form

$$Bu = \sum_{|\beta| \leqq p} a_\beta \partial^\beta u \tag{1.7}$$

where $a_\beta \in A(\overline{\Omega})$.

Note 1.1. All the results formulated and proved in this chapter hold when a_β are matrices of order $\varkappa \times \varkappa$, $\varkappa > 1$, and u is a vector, $u = (u_1, \ldots, u_\varkappa)$ in (1.7).

Since the case $\varkappa > 1$ does not fundamentally differ from the case $\varkappa = 1$, but is more cumbersome, all the proofs will be carried out for the scalar case $\varkappa = 1$.

The following lemma estimates the increase in the norms of iterations of the operator B.

Lemma 1.3. Suppose the compactum $K_0 \subset \Omega$, $q \in \mathbb{N}$, $p \in \mathbb{N}$, $N \in \mathbb{N}$, the function $f \in C^{pN}(\Omega)$ and the derivatives f satisfy estimate

$$|\partial^\alpha f(x)| \leqq C_2 C_3^{|\alpha|} N^{|\alpha| + q} \qquad \text{when} \qquad |\alpha| \leqq Np, \quad x \in K_0 \tag{1.8}$$

Suppose B_j, $j = 1, \ldots, J$ are differential operators of the form (1.7) of order p_j, $p_j < p$ and the coefficients $a_{\beta j}$ of these operators belong to $A(K_0, C_1, C_0)$ where C_1 and C_0 do not depend on β and j. Suppose $0 \le j(i) \le J$ when $i = 1, \ldots, k$, $\sigma_k = p(j(1)) + \ldots + p(j(k))$. Then when $kp + |\alpha| \le Np$, $x \in K_0$ the following inequality holds:

$$|\partial^\alpha B_{j(1)} \cdots B_{j(k)} f(x)| \leqq C_2 C_5^k (C_6 N)^{|\alpha| + \sigma_k + q} \tag{1.9}$$

where we shall take $\max(2pC_1, C_3, 1)$, as the constant C_6, $C_5 = 2C_4 C_0$ and C_4 is the number of terms in formula (1.7).

Proof. We shall carry out induction using k. When $k = 0$, inequality (1.9) follows from (1.8) if we assume $C_6 > C_3$. Suppose (1.9) holds when $k \le n-1$. We shall prove (1.9) when $k = n$. We will assume $x_{n-1} = B_{j(1)} \cdots B_{j(n-1)} f$. It follows from (1.9) that

$$|\partial^\alpha \partial^\beta \chi_{n-1}| \leqq C_2 C_5^{n-1} (C_6 N)^{|\alpha| + |\beta| + \sigma_{n-1} + q} \tag{1.10}$$

when $(n-1)p + |\beta| + |\alpha| < pN$. We shall now take $|\beta| < p(j(n))$ and shall consider the term $\alpha_{\beta j(n)} \partial^\beta$ from formula (1.7), which determines $B = B_j$. Since $|\beta| < p$, inequality (1.10) holds when $|\alpha| + np < pN$, whilst $|\beta| + \sigma_{n-1} < \sigma_n$. Using the fact that $a = a_{\beta j(n)}$ belongs to $A(K_0, C_1, C_0)$ and to estimate (1.10), by differentiating the product $a \partial^\beta x_{n-1}$ we obtain the estimate:

$$|\partial^\alpha (a \partial^\beta \chi_{n-1})| \leqq C_0 \sum_{l=0}^{|\alpha|} \frac{|\alpha|!}{l!(|\alpha| - l)!} C_1^l l! C_2 C_5^{n-1} (C_6 N)^{\sigma_n + |\alpha| - l + q}$$

$$= C_0 C_2 C_5^{n-1} (C_6 N)^{\sigma_n + |\alpha| + q} \sum_{l=0}^{|\alpha|} (C_1/(C_6 N))^l \frac{|\alpha|!}{(|\alpha| - l)!}$$

Since $|\alpha|!/(|\alpha| - l)! < |\alpha|^l$, and $|\alpha| < pN$, hence we obtain the estimate

$$|\partial^\alpha(a\,\partial^\beta\chi_{n-1})| \leqq C_0C_2C_5^{n-1}(C_6N)^{\sigma_n+|\alpha|+q}\sum_{l=0}^{|\alpha|}\left(\frac{pC_1}{C_6}\right)^l \tag{1.11}$$

Since $C_6 > 2pC_1$, the sum on the right-hand side of (1.11) does not exceed 2. Summing estimate (1.11), where $a = a_\beta$, $B = B_{j(n)}$, over β, and bearing in mind that the number of terms in (1.7) equals C_4, from (1.11) we obtain:

$$|\partial^\alpha B_{j(n)}\chi_{n-1}(x)| \leqq 2C_4C_0C_2C_5^{-1}C_5^n(C_6N)^{|\alpha|+\sigma_n+q}$$

Since $C_5 = 2C_4C_0$, hence we obtain inequality (1.9) when $k=n$, and the lemma is proved.

Theorem 1.1. Suppose $f\in A(K)$, K is a compactum, the operator B is determined using formula (1.7) and its coefficients belong to $A(K)$. Then R and C exist, such that

$$|\partial^\alpha B^kf(x)| \leqq CR^{-(pk+|\alpha|}(pk+|\alpha|!) \qquad \forall k\in\mathbb{Z}_+,\quad x\in K \tag{1.12}$$

where $C=C_0(f)$ is the constant C_0 in estimate (1.1) for f, the number R depends on the constants C_0 and C_1 in estimate (1.1) where $f=a_\beta$, on C_4 – the number of terms in (1.7), on p and on the constant C_1 in estimate (1.1) for f.

Proof. According to Lemma 1.1 inequality (1.1) holds. Since $|\alpha|! < |\alpha|^{|\alpha|}$, the following estimate follows from (1.1) when $|\alpha| < pN$

$$|\partial^\alpha f| \leqq C_0C_1^{|\alpha|}|\alpha|! \leqq C_0C_1^{|\alpha|}(pN)^{|\alpha|} = C_0(pC_1)^{|\alpha|}N^{|\alpha|} \tag{1.13}$$

Hence it follows that inequality (1.8) holds, where $C_2 = C_0$, $q = 0$ for any $N\in\mathbb{N}$. We shall also use Lemma 1.3 when $B_j=B$. We will use $N = [k + |\alpha|/p] + 1$, [ ] is the integer part. Since kp +