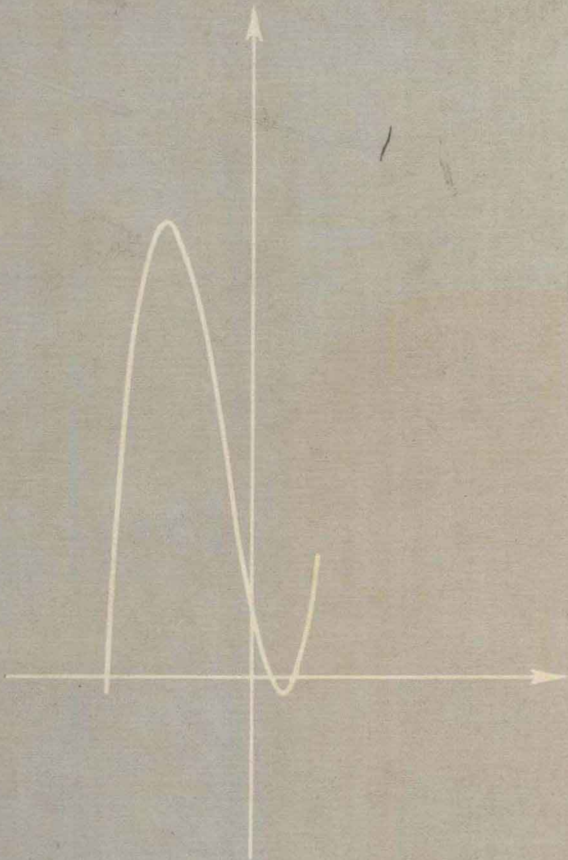

INTRODUCTION TO CALCULUS
with analytic geometry

Introduction to *CALCULUS*
with analytic geometry

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Introduction to CALCULUS

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Introduction to Calculus

Preface

This book is designed for the mature student who wishes to acquire the vital concepts of calculus. The emphasis is on basic ideas rather than on drill in manipulative techniques and tricks. Generalization is stressed throughout. The student is led to see that, for example, work under a variable force is a reasonable generalization of work under a constant force. It is this emphasis on fundamental ideas rather than on "formula juggling" which provides the background needed by a mature student who wishes to learn calculus as an aid to research in the social sciences. Secondary school teachers preparing students for the twelfth grade polynomial calculus course recommended by the Commission on Mathematics of the College Entrance Examination Board and by the Commission on the Undergraduate Program in Mathematics also need emphasis on theory, rather than "juggling."

Much of the text has been used in National Science Foundation Institutes for Secondary School Teachers, and in classes of Mathematics for Social Scientists. It has proved highly successful with both groups.

A social scientist who has a research problem which is stated in mathematical language can usually find a mathematician who is willing to help him solve it. However, very few mathematicians know enough social science to *formulate* the problem. The social scientist *must* learn enough mathematics to formulate his problems in reasonable mathematical terms before he seeks aid. It can be done. Examine the literature in the field—it *is* being done.

This book is designed to provide the fundamentals of calculus needed to understand and formulate new problems.

It has proved reasonable to expect students to complete the text in four semester hours, especially if Chapter 1 can be covered rapidly. It seems inadvisable to skip Chapter 1 unless the student has had an adequate freshman course in mathematics during the last five years.

Richard V. Andree

Acknowledgements

Acknowledgements in a text traditionally thank people for assistance and absolve them from any possible shortcomings of the book. This book has developed so gradually that it would be impossible to name all of the friends, colleagues, and students who assisted in its preparation. However, it would be unfair not to acknowledge two outstanding contributors. Professor John C. Brixey, The University of Oklahoma, Norman, whose thought-provoking discussions during our coauthorship of several texts over the past ten years have helped formulate many of the ideas, and the majority of the problems, in this volume. The author expresses sincere thanks to Dr. Brixey for granting permission to borrow freely from earlier joint efforts, and even more for his penetrating analysis of many problems, both mathematical and pedagogical. Our second bouquet goes to the author's wife, Josephine Peet Andree, who possesses the outstanding qualities of patience and understanding which are so essential in an author's spouse. Since she, herself, is a competent mathematician and an excellent teacher, her contributions have been invaluable, not only on the thankless and arduous task of reading proof, but even more important in the actual formulation of the ideas and the expression of the text itself.

For any shortcomings which may appear, the author hereby blames the publisher, and the publisher blames the author.

R. V. A.

Contents

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Basic Algebraic Theory

1-1. Introduction

This book is designed for a mature reader who wishes a concise course in the fundamental ideas of the calculus. The applications and examples have been designed, specifically, to meet his needs. In addition to the calculus, certain other mathematical concepts which are vital to modern science and mathematics will be discussed. Preliminary concepts are presented in Chap. 1 for the benefit of students who may be unprepared in essential portions of elementary and intermediate algebra. Chapter 2 provides a similar preparation in analytic geometry. Those who feel familiar with these fundamentals may read these chapters rapidly but should not skip them entirely.

1-2. Functions and Relations

function

Possibly the most fundamental mathematical notion is that of a *function*. In brief, the statement "y is a function of x" means that if a suitable value of x is given, then one corresponding value of y is determined in some fashion. The word *function* is used to describe the *correspondence*.

Let X be a set of objects (numbers, points, ideas, or any other specific set). Let Y also be a set of objects. The objects in set Y need not be similar to objects in set X. For example, set X might consist of all the girls living in dormitories at Bryn Mawr College this semester, and set Y might consist of the 10 telephone numbers (51000, 59142, 59143, 59138, 51473, 59176, 52801, 59145, 53544, 59158).

To say a functional relationship exists from set X into set Y means that for each element x in the set X (written $x \in X$),† a corresponding

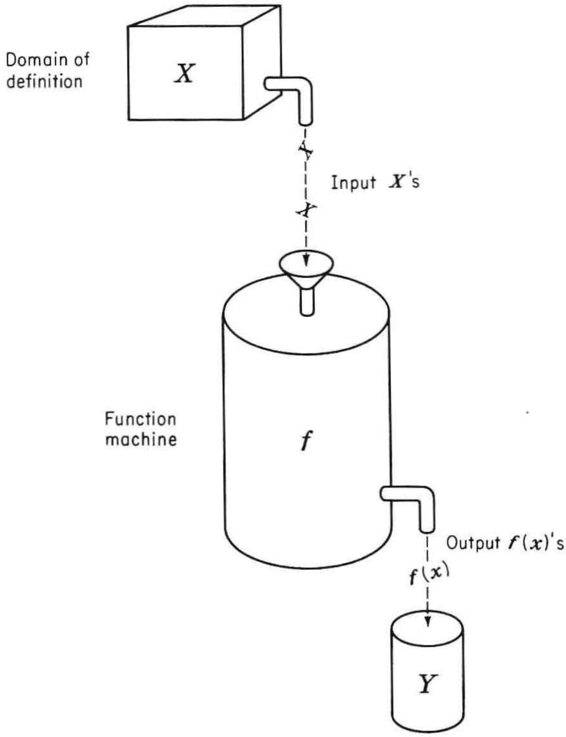

Figure 1-1

value $f(x)$ from the set Y (written $f(x) \in Y$) is determined. In more detail, a function consists of three things (see Fig. 1-1).

function

1. A set X of elements. This set may be called the *domain of definition* of the function.
2. A set Y of elements.
3. A rule f which assigns to each element x of X a unique element $f(x)$ of Y.

† The symbol ϵ is used in set theory. The statement $a \in S$ means that S is a set of elements, one of which is the specific element a. It is common practice to use small letters to represent elements and capital letters to represent a set of elements: thus $x \in X$.

Another way of stating this is that a function is a set of *ordered pairs* $[x, f(x)]$ such that whenever two sets of pairs have the same first element they also have the same second element.

Note that neither the elements of X nor of Y need be numbers.

Illustration

Let X be the set of girls who attended the last major dance at your school and let Y be the set of dominant colors of their dresses. For each girl x in X there is a corresponding dominant dress color $f(x)$ in Y. Some possible examples are f(Suzanne) = white, f(Josie) = blue, f(Phyllis) = red, and f(Lois) = white.

If X is the collection of all names in the University of Oklahoma telephone directory and Y is the collection of telephone numbers with f being the obvious correspondence, then some of the busier offices have several telephone numbers which correspond to them. For example, if x = Psychology Department, then $f(x) = \frac{318}{319}$. The word *relation* is currently used to describe such a "multiple-valued function."

relation

The most serious difficulty students have is the feeling that this definition says more than it does. It is quite possible, for example, that the same element of Y may correspond to several elements of X (that is, change in x does not necessarily imply a change in y). There is no hint that the correspondence needs to be determined by an equation or formula. Indeed, none need exist. If $f(x)$ is given, there need not be a method of determining the x to which the $f(x)$ corresponds. All that the function concept states is that any time an object x is selected from a given set of objects X (the *domain of definition* of the function), then a corresponding object $f(x)$ or y from a set of objects Y is determined. The *subset* of Y (consisting of those values actually assumed) is sometimes known as the *range* of the function, but we shall not make this distinction and shall refer to Y as the range. A function, therefore, is a matching of the objects in set X with objects in set Y such that every element in X is represented.

domain of definition

Letters such as p, g, or L may be used to represent the function in place of f, for example, $p(x)$, $g(x)$, or $L(x)$.

Example 1

Let the domain of definition X consist of those names listed in the current Los Angeles telephone directory. Let the range Y be the telephone numbers listed in the same book. Let the function f be the correspondence (pairing) of the x's (names) and $f(x)$'s (telephone numbers) given in this book. For any name x in X, there is a corresponding telephone number $f(x)$ in Y, determined by the function f. It is consistent to use the notation f(John Magee) to denote the telephone number which the function f associates with the name John Magee.

Example 2

Let X be the set of integers (whole numbers) given below:

$$X = \{x\} = \{4, 5, 6, 7, 8, 9, 10, 11\}.$$

Y will also consist of a set of integers, although neither you nor the author knows exactly what integers. The value $B(x)$ of the function B is the number of scheduled buses which left the main New York City bus terminal in the 35-minute period immediately following x P.M. on September 25, 1961. Thus $B(7)$ is the number of scheduled buses leaving that terminal between 7:00 and 7:35 P.M., on the given date. The correspondence (function) B is determined by consulting the dispatcher's log for the given day.

Example 3

Let X be the set of all possible lengths of radii, that is, the positive real numbers. Let Y be the set of all possible areas of circles, that is, the positive real numbers. Let g be the function consisting of the correspondence between the radius and the area of a circle. In this case there exists a *formula* which also determines the function, namely, $y = \pi x^2$ and if $x = 5$, the corresponding y is 25π. However, the presence of such a formula is in no way essential to the existence of the function, as is illustrated by Examples 1 and 2. When a simple formula exists, it simplifies the description of the function considerably and will be used.

1-3. More General Functions

Single-valued functions of *one variable* have exactly *one* $f(x)$ value which is determined for each permissible x value. The phrase "relation" or "multiple-valued function" is used to describe a correspondence in which a given x (input) may determine more than one corresponding $f(x)$ value (output).

relation

Example 1

The relation C determined by $x^2 + y^2 = 25$ is multiple-valued since some of the permissible x values determine more than one corresponding y value. $C(3) = +4$ or -4, for example.

It is also possible to have a function of several variables in which the x's which are fed into the function "machine" are composites of several variables. The parcel post rate chart provides a familiar illustration of such a function. Both weight and zone must be given to determine the postage.

Example 2

For a given weight W and destination d (from the permissible values of W and d) a postal rate chart gives the corresponding cost of sending a package of weight W from Nashville to destination d for the given values of W and d. It is convenient to think of the x's in this case as consisting of pairs of values (W,d). The $f(x)$'s are the corresponding costs.

A Word of Caution: Note that there is nothing in the definition of function which asserts it is possible to "reason backward." If x is given, $f(x)$ is determined, but it does *not* say that if $f(x)$ is given, then x can be determined. If I tell you that the cost of sending a package is $1.36, it is not possible for you to tell me its weight or its destination.

1-4. The Function Notation

The notation bz means "the quantity b multiplied by the quantity z." Similar meanings are attached to combinations such as Ay, $x(x + 7)$, etc. However, by special agreement, when a scientist, a mathematician, or an engineer writes the Greek letter Δ (delta) next to a letter, as for example Δx, it does *not* mean Δ times that letter x. The symbol Δx (read "delta x") should be thought of as *one single element*, not as a product. In $\frac{\Delta y}{\Delta x}$ we may *not* divide out the deltas. The notation may seem peculiar at this point, but the powerful mathematical tools developed later using this notation will make you appreciate the need for its introduction. At present, just remember that Δx, Δy, Δt, and Δz are elements, not products. The symbol Δx is used to indicate a change in the value of x. For example, if x changes from 13 to 15, then $\Delta x = 2$. More generally, if the initial x value is $x = a$ and the terminal value is $x = b$, then the change in x is $\Delta x = b - a$.

The functional notations $f(x)$, $g(x)$, and $h(t)$ (read "f of x, g of x, and h of t") may also look like products but are not used as such. They represent the element $f(x)$ of Y which the correspondence (function) f associates with a given value of x in X. If there exists a formula which determines the function, such as $f(x) = (4x + 1)(x - 2)$ or $f(x) = 4x^2 - 7x - 2$, then $f(3)$ is determined by simply substituting 3 for x in the formula and combining terms to obtain $f(3) = (13) \cdot (1) = 13$ or $f(3) = 4(3^2) - 7(3) - 2 = 13$. If the function f is determined by means of a table, then $f(3)$ is found from the table.

Example 1

If $f(t) = t^2 - 7t + 2$, then $f(x)$ represents the same formula with t replaced by x. Thus $f(x) = x^2 - 7x + 2$, while

$$f\left(\frac{1}{y}\right) = \left(\frac{1}{y}\right)^2 - 7\left(\frac{1}{y}\right) + 2 = \frac{1 - 7y + 2y^2}{y^2}.$$

The symbol in parentheses need not be a letter; for example,

$$\begin{aligned} f(3) &= (3)^2 - 7(3) + 2 = -10, \\ f(8) &= (8)^2 - 7(8) + 2 = 10, \\ f(x+3) &= (x+3)^2 - 7(x+3) + 2 = x^2 - x - 10, \\ f(y + \Delta y) &= (y + \Delta y)^2 - 7(y + \Delta y) + 2 = y^2 + 2y\,\Delta y \\ &\qquad + (\Delta y)^2 - 7y - 7\Delta y + 2. \end{aligned}$$

Example 2

Using $f(t) = t^2 - 7t + 2$ as before, find $f(y + \Delta y) - f(y)$.

$$\begin{aligned} f(y + \Delta y) - f(y) &= [(y + \Delta y)^2 - 7(y + \Delta y) + 2] \\ &\qquad - (y^2 - 7y + 2) \\ &= 2y\,\Delta y + (\Delta y)^2 - 7\Delta y. \end{aligned}$$

Example 3

If $f(t) = t/(t - 3)$, determine $f(2)$, $f(2 + \Delta y) - f(2)$, and $f(2 + .01) - f(2)$. Clearly,

$$f(2) = \frac{2}{2-3} = -2.$$

$$f(2 + \Delta y) - f(2) = \frac{2 + \Delta y}{2 + \Delta y - 3} - \frac{2}{2-3}$$
$$= \frac{2 + \Delta y}{\Delta y - 1} + 2 = \frac{2 + \Delta y + 2\Delta y - 2}{\Delta y - 1}$$
$$= \frac{3\Delta y}{\Delta y - 1}.$$

The reader should note that both $\frac{3\Delta y}{\Delta y - 1}$ and $f(2 + \Delta y)$ are meaningless if $\Delta y = 1$. In a similar fashion, one obtains $f(2 + 0.01) - f(2) = \frac{3(0.01)}{0.01 - 1} = \frac{0.03}{-0.99} = \frac{-1}{33}$, or approximately $f(2 + 0.01) = -0.03$. This may be obtained by setting $\Delta y = 0.01$ in $f(2 + \Delta y) - f(2)$, which was just computed. Before going further, show that if $g(z) = 3z^2 - 2z + 5$, then $g(5) = 70$ and $g(5 + \Delta x) - g(5) = 28\Delta x + 3(\Delta x)^2$.

Example 4

If $f(x) = 3/x$, find $f(x + \Delta x) - f(x)$.

$$f(x + \Delta x) - f(x) = \frac{3}{x + \Delta x} - \frac{3}{x} = \frac{3}{(x + \Delta x)} \cdot \frac{x}{x} - \frac{3}{x} \cdot \frac{(x + \Delta x)}{(x + \Delta x)}$$
$$= \frac{3x - 3x - 3\Delta x}{x(x + \Delta x)} = \frac{-3\Delta x}{x(x + \Delta x)}$$

Example 5

If $g(t) = 5/t^2$, find $\frac{g(t + \Delta t) - g(t)}{\Delta t}$.

$$\frac{g(t + \Delta t) - g(t)}{\Delta t} = \frac{\frac{5}{(t + \Delta t)^2} - \frac{5}{t^2}}{\Delta t} = \frac{5}{(t + \Delta t)^2\,\Delta t} - \frac{5}{(t)^2\,\Delta t}$$
$$= \cdots$$
$$= \frac{5t^2 - 5[t^2 + 2t\,\Delta t + (\Delta t)^2]}{(t + \Delta t)^2 t^2\,\Delta t}$$
$$= \frac{-5(2t + \Delta t)\,\Delta t}{(t + \Delta t)^2 t^2\,\Delta t} = \frac{-5(2t + \Delta t)}{(t + \Delta t)^2 t^2} \quad \text{if } \Delta t \neq 0.\dagger$$

† The symbols $\Delta t \neq 0$ means "Δt is not equal to zero."