OPERATIONAL CALCULUS

BASED ON THE TWO-SIDED LAPLACE INTEGRAL

BY

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'La langue de l'analyse, la plus parfaite de toutes les langues, étant par elle-même un puissant instrument de découvertes; ses notations, lorsqu' elles sont nécessaires et heureusement imaginées, sont les germes de nouveaux calculs.'

LAPLACE, *Théorie analytique des probabilités*, Paris, 1812, p. 7

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PREFACE

This book was developed from a series of lectures [by van der Pol] given at the 'Technische Hogeschool' of Delft during 1938 and following years, and from a second series given during the first half of 1940 at the 'Philips Research Laboratories', Eindhoven.

The second author [Bremmer] made extensive lecture reports on the latter series; subsequently the material was jointly extended during the German occupation of the Netherlands. In this period the original manuscript, in Dutch, was practically completed, while several problems founded on it were published from 1940 on in *Wiskundige Opgaven* of the Netherlands 'Wiskundig Genootschap'. The English translation of the original manuscript was edited by Dr C. J. Bouwkamp, of this Laboratory.

Primarily this book is intended for application of the operational calculus in its modern form to mathematics, physics and technical problems. We have therefore given not only the basic principles, ideas and theorems as clearly as possible (and rather extensively), but also many worked-out problems from purely mathematical and physical as well as from technical fields. In order to limit the size of the book, proofs of some of the deeper theorems have been omitted, and for these the reader is referred to the mathematical literature.

None the less, it is believed that the purely mathematical treatment is more advanced than is usual in books devoted primarily to practical applications. The Abel and Tauber theorems, for example, are extensively considered, with many examples taken from pure mathematics as well as from technical problems.

It is therefore hoped that the book may be of value to those pure mathematicians who are interested in a rapid and simple derivation of complicated and unexpected relations between various mathematical functions, as well as to the engineer in search, for example, of a very simple treatment of transient phenomena in electrical networks, such as filters. In both cases the operational calculus appears to its best advantage.

Furthermore, several applications of the operational calculus to the theory of numbers are to be found in this book, a field of application which appears to be of the greatest heuristic value and which at present seems to be far from exhausted.

We have endeavoured to give the operational calculus a rigorous mathematical basis; on the other hand, we have tried to give the subject-matter such a form that it can be applied simply to practical problems.

This led us to treat the operational calculus *ab initio*, by means of the bilateral or two-sided Laplace integral, contrary to the usual practice based on the one-sided Laplace integral. This procedure was greatly stimulated by extensive discussions with Dr Ph. le Corbeiller, now at Harvard University, Cambridge, Mass.

The foundation of the theory on the two-sided Laplace transform caused us to introduce at an early stage:

(*a*) the Heaviside unit function, $U(t)$, defined by $U(t) = 0$ for $t<0$, $\frac{1}{2}$ for $t = 0$, and 1 for $t>0$,

(*b*) the Dirac δ-function, $\delta(t)$.

Further, the use of the two-sided Laplace transform requires, for each operational relation, the stipulation of the band of $\text{Re}\, p$ within which this relation is valid. It is felt, however, that the latter complication is more than compensated for by the following advantages:

(i) the class of functions suited to an operational treatment becomes much larger,

(ii) the 'transformation rules' are considerably simplified,

(iii) the entire treatment becomes more rigorous than the usual presentation of the one sided integral in technical books.

The rapid way in which solutions of complicated problems can be found with operational calculus is often astounding. This is mainly due to the fact that discontinuous functions $h(t)$ of a real variable t, which frequently occur in the treatment of electrical and mechanical transients as well as in the theory of numbers, have an operational 'image' $f(p)$ that is analytic in some band of finite or infinite width of the complex p-plane. Simple transformations of these smooth, analytic, functions $f(p)$ then correspond uniquely to operations on the discontinuous functions $h(t)$, and so the complicated handling of these discontinuous functions can be replaced by extremely simple transformations of the corresponding analytic functions.

The treatment in this book is not limited to the Laplace transform of functions of one single variable; an extensive chapter is also devoted to the multidimensional Laplace transforms. This 'simultaneous operational calculus' enabled us, for example, to treat Green's functions in potential and wave problems; this part was mainly developed by the second author. Thus familiar solutions of the Maxwellian equations are obtained by a few extremely simple algebraic transformations in the p-field.

In the Introduction the reader will find a summary of the subjects treated in the various chapters.

The authors wish to express their thanks to several friends who read parts or the whole of the manuscript. In the first place we wish to thank Dr C. J. Bouwkamp for many remarks, which, we believe, have improved

the clarity of the exposition, and for the translation of the original manuscript. Further we wish to thank Prof. N. G. de Bruÿn, whose remarks have contributed materially to the rigour of the treatment.

In conclusion, we would add that we shall be most grateful for remarks and criticism from readers. We feel that this first treatment of the practical operational calculus on the basis of the two-sided Laplace transform, with so many applications to both pure and applied mathematics, and a very great part of which we believe to be essentially new, is bound to show marks of immaturity typical of young scientists as well as of young sciences.

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November 1947

PREFACE TO THE SECOND EDITION

The necessity of a second edition, four years after the appearance of the first one, gave us the opportunity to insert a number of corrections and improvements, scattered throughout the book. The greater part of the corrections are due to suggestions by several correspondents. In connexion herewith we wish to thank particularly Prof. S. Colombo (Paris); Prof. A. Erdelyi (Pasadena); Dr J. H. Pearce (London); and Mr H. van der Weg (Eindhoven).

We have also inserted the following new paragraphs:

Rules for the treatment of *correlation functions*,
A note on the theory of *distributions*,
A note on the *Wiener-Hopf technique*,

as all three, modern, subjects lend themselves well to a concise treatment in the 'language' of operational calculus.

Finally we wish to thank the Cambridge University Press for the care again shown in the preparation of this second edition.

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September 1954

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CHAPTER I

GENERAL INTRODUCTION

1. History of the operational calculus

The operational calculus, a modern treatment of which is aimed at in the present book, can be traced back as far as the work of Oliver Heaviside (1850–1925). Though many scientists (Leibniz, Lagrange, Cauchy, Laplace, Boole, Riemann, and others) preceded Heaviside in introducing operational methods into analysis†, a systematic use of it in physical and technical problems was stimulated only by Heaviside's work.

Heaviside‡ was a 'self-made man', deprived of regular study at the university or the engineering college. Nevertheless, his curious methods, created by himself as they often were, led him to results in technics and theoretical physics that are undoubtedly among the most important ever reached. In this connexion let us remember that Heaviside's work § already contains Maxwell's equations of the electromagnetic field in the modern, now current, vector notation. Also due to him is the conception of the 'Heaviside Layer', which is of the greatest importance in present-day radio communication. Moreover, independently of Lorentz, Heaviside enunciated the theory of the electronic motion in a magnetic field; he further introduced into Maxwell's theory that part of the total current which is due to convection. His concept of impedance, defined independently of Kennelly, is much more general than that of the conventional alternating-current technique. The notion of 'negative resistance', now common property in electrical engineering (e.g. arc lamp, radio valve), is often put forward in his papers, and for the first time in 1895.

But it may be stated that even to-day Heaviside's papers, difficult to read as perhaps they are, still contain a great many views and hidden things, of both mathematical and physical interest, which are not yet very well known and which, therefore, have not met with proper appreciation. Certainly this is largely due to the strange manner in which Heaviside often derives and announces his results. Moreover, the fact that Heaviside was not a university man raised a barrier, a certain antagonism, between him and his contemporaries. The latter reproached him, rightly, with his great

† Compare, for instance, H. T. Davis, *The Theory of Linear Operators*, Bloomington, Indiana, 1936.

‡ For a survey of the life and work of Heaviside the reader is referred to E. T. Whittaker, *Bull. Calcutta Math. Soc.* xx, 216, 1928–9; Balth. van der Pol, *Ned. Tijdschr. Natuurkunde* v, 269, 1938.

§ O. Heaviside, *Electrical Papers*, vols. I and II, Macmillan, London (New York), 1892; *Electromagnetic Theory*, vols. I, II and III (1893–1912), reissued 1922 by Benn Brothers, London.

lack of mathematical rigour. Yet Heaviside did develop an abundance of mathematical and physical methods and results which afterwards, on critical elaboration by various scientists, proved to be substantially true and have been approved as such. Though perhaps reasonable, it is regrettable that such a barrier existed between Heaviside and his fellow-mathematicians. Equally regrettable, but certainly unreasonable, is the point of view occasionally taken by modern mathematicians with regard to Heaviside's work; in many respects it is far superior to the later contributions to this part of science, both for the methods as well as for the results arrived at†.

Fortunately, there are other records too. For instance, Whittaker (loc. cit.) wrote, after discussing the difference in views on mathematics between Heaviside and the pure mathematician:

'Looking back on the controversy after thirty years, we should now place the Operational Calculus with Poincaré's discovery of automorphic functions and Ricci's discovery of the Tensor Calculus as the three most important mathematical advances of the last quarter of the nineteenth century. Applications, extensions and justifications of it constitute a considerable part of the mathematical activity of to-day.'

It is this Operational Calculus to which the present book is devoted.

2. The operational calculus based on the Laplace transform

Heaviside's ideas concerning the operational calculus may perhaps best be interpreted as follows‡. Imagine a linear electrical network originally at rest. Let an electromotive force $E(t)$ be applied to it, where $E(t)$ is an arbitrary function of the time t. The response current, $i(t)$, is then determined by

$$i(t) = Y(D_t)\,E(t), \tag{1}$$

in which $D_t = d/dt$. The function $Y(D_t)$ is an *operator function* applied to the *operand* $E(t)$, to give the current $i(t)$. If $E(t)$ is constant with time, $i(t)$ will be constant too; under these circumstances $Y(D_t)$ degenerates into the reciprocal of an ohmic resistance.

The question arises at once of how we are to interpret the operator function when, for instance, it is of the following form:

$$Y(D_t) = \frac{1}{1+D_t}.$$

† Heaviside was more than 'ein englischer Elektroingenieur', in spite of his (and his successor's) methods being 'mathematisch sehr unzulänglich' and 'allerdings mathematisch unzureichend'. Quotations from G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Berlin, 1937, and New York, 1943, pp. 337, 421.

‡ *Proc. Roy. Soc.* LII, 504, 1892–3; LIV, 105, 1893.

Should we take it in the sense of

$$\frac{1}{1+D_t} = 1-D_t+D_t^2-D_t^3+\ldots, \tag{2}$$

or rather in the sense of

$$\frac{1}{1+D_t} = \frac{1}{D_t}-\frac{1}{D_t^2}+\frac{1}{D_t^3}-\ldots? \tag{3}$$

In the first case it is reasonable to interpret D_t^n as d^n/dt^n. Similarly, when applying (3) we would take $1/D_t$ to mean $\int_a^t d\tau$; and in so doing a second question has presented itself: What value of the constant of integration, a, is required? Guided by practical experience, Heaviside came to the conclusion that, if possible, the form (3) should be chosen rather than (2). He further concluded that in discussing switch-on phenomena in electrical networks (when the electromotive force does not come into action before $t = 0$; that is, $E(t) = 0$ when $t < 0$), the lower limit of integration, a, has to be equated to zero. However, in Heaviside's work, we have not been able to find any rigorous statements concerning this question in general.

A modern treatment of the operational calculus requires, therefore, a much more rigorous base. This is furnished by the Laplace transform, as was already pointed out by Heaviside himself†, though he did not use it extensively. The same Laplace transform was the starting-point of later writers such as Carson‡, Bush§, Humbert||, Doetsch¶, Wagner††, Droste‡‡, McLachlan§§, and Widder||||. When $h(t)$ is supposed to be given, then the Laplace transform of $h(t)$ is the function $f(p)$, defined by the following integral:

$$f(p) = p\int_0^\infty e^{-pt}\,h(t)\,dt. \tag{4}$$

By so doing we let the function f of the variable p correspond to the function h of the variable t. Conversely, as particularly discussed by Carson, the formula (4) may be considered as an integral equation for the unknown function $h(t)$, when $f(p)$ is supposed to be given.

† *Electromagnetic Theory*, III, 236.

‡ John R. Carson, *Electric Circuit Theory and the Operational Calculus*, New York, 1926.

§ V. Bush, *Operational Circuit Analysis*, New York, 1929.

|| P. Humbert, *Le calcul symbolique*, Paris, 1934.

¶ G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Berlin, 1937, and New York, 1943.

†† K. W. Wagner, *Operatoren Rechnung*, Leipzig, 1940.

‡‡ H. W. Droste, *Die Lösung angewandter Differentialgleichungen mittels Laplacescher Transformation*, Berlin, 1939.

§§ N. W. McLachlan, *Complex Variable and Operational Calculus*, Cambridge, 1939.

|||| D. V. Widder, *The Laplace Transform*, Princeton, 1941.

A somewhat different point of view is taken by Bromwich†, who started from the complex integral

$$h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\frac{f(p)}{p}\,dp; \tag{5}$$

this integral, it may be noted in passing, was known to Riemann‡ as early as 1859. A complete survey of Bromwich's work is to be found in Jeffreys' book§. Wagner|| also based his contribution upon the integral (5). Further impact to the calculus is owed to Lévy¶, who pointed out that the solution of (4), considered as an integral equation for $h(t)$, is given by (5), and vice versa. Thus by Lévy's work the two different points of view came together in one consistent theory.

Also based on the Laplace transform (4), with zero as lower limit of integration, are the former investigations of Van der Pol†† and of Van der Pol and Niessen‡‡.

Henceforth the transformation (4) will be called the unilateral or *one-sided* Laplace transform. Contrary to the earlier investigations, this book will be based on the *two-sided* Laplace transform

$$f(p) = p\int_{-\infty}^{\infty} e^{-pt}h(t)\,dt, \tag{6}$$

to obtain a wider base for the operational calculus, as will be discussed in detail in the next chapter. The two-sided Laplace integral has its lower limit of integration equal to $-\infty$ instead of 0. This generalization proves very advantageous, and includes the earlier calculus as a special case. In the first place, the operational rules are considerably simplified by the generalization and, secondly, a much larger class of functions (and phenomena) becomes accessible.

It is worth while to remark that, whether we use (4), (5) or (6) as the basis of the operational calculus, the indefinite concepts of operator and operand wholly disappear. Instead of the vague formulation of the early operational calculus there comes the functional transform (6), by which there corresponds to any given function $h(t)$ a new function $f(p)$ of the complex variable p. In the Volterra sense, $f(p)$ is a 'fonction de ligne' or 'fonctionnelle', indicating that the form of the function $f(p)$ depends on the

† T. J. I'a. Bromwich, *Proc. Lond. Math. Soc.* xv, 401, 1916.

‡ The integral occurs in Riemann's classical paper of only eight pages: 'Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse', *Monatsber. Berl. Akad.* Nov. 1859; see also *Gesammelte Werke*, Leipzig, 1876, p. 136.

§ H. Jeffreys, *Operational Methods in Mathematical Physics*, Cambridge, 1927.

|| K. W. Wagner, *Arch. Elektrotech.* IV, 159, 1916.

¶ P. Lévy, *Le calcul symbolique d'Heaviside*, Paris, 1926.

†† Balth. van der Pol, *Phil. Mag.* VII, 1153, 1929; VIII, 861, 1929; XXVI, 921, 1938; *Physica, 's-Grav.*, IV, 585, 1937.

‡‡ Balth. van der Pol and K. F. Niessen, *Phil. Mag.* XI, 368, 1931; XIII, 537, 1932.

whole set of values which $h(t)$ assumes on the complete real axis of t, $-\infty < t < \infty$.

It is to be emphasized that (6) is essentially a *linear* functional transform, since in the integrand of (6) the function $h(t)$ occurs linearly. As a consequence, the operational calculus is applicable only to linear problems such as switch-on phenomena in linear networks, problems of small vibrations, heat diffusion, potential theory, and electrical cables.

As far as the general outlines of the theory are concerned, this book is restricted to giving an extensive survey; the more complicated theorems underlying the theory are usually stated without proof. For proofs the reader is always referred to existing literature, cited in the text. Our main aim is to demonstrate the vigour of the operational calculus in its applications, by giving many examples. The discussion will not be confined to applications in physics and technics; many problems of pure mathematics will be included too.

If the reader has made himself familiar with the fundamental principles of the calculus presented here, he will certainly become aware of the strength of this mathematical tool of almost unrestricted heuristic-analytic value; he will be guided by many examples illustrating the general theory; he will often be able to construct new analytic relations by quite simple means.

3. Survey of the subject-matter

The starting-point of chapter II is the Fourier integral, on which the foundation of the operational calculus is built. We are then led back to the fundamental expressions (5) and (6). In chapter III some elementary 'operational relations' are derived which prove useful in the course of the subsequent investigations. In chapter IV we shall establish elementary 'operational rules', indicating how certain changes of the p-function correspond to others of the t-function. Chapter V is devoted to a detailed discussion of the unit function, $U(t)$, and the delta or impulse function, $\delta(t)$. The latter was introduced by Dirac in quantum mechanics; but Heaviside had already used it extensively before him. The impulse function is particularly important in relation to the Green function of differential equations. It is formulated in terms of the general concept of the Stieltjes integral. Chapter VI should be considered as a deepening and extension of chapter II. It contains a detailed investigation of questions of convergence, particularly in connexion with the summing of series and integrals by the well-known methods of Abel and Cesàro. In chapter VII, especially the asymptotic expressions for 'image' and 'original' are outlined, as well as related topics. Further, chapter VIII concerns the operational treatment of differential equations having constant coefficients. This matter is extended to a system of equations in chapter IX. These two chapters also include the theory of linear electrical networks, together with the corresponding transient phenomena. Differential equations with variable coefficients are treated in chapter x. Applications are made to Legendre polynomials, Bessel functions, etc. The matter of chapter xi must be considered as a generalization of that given in chapter iv; general rules of more complicated character are discussed. Chapter xii is devoted to the study of step functions, with applications, amongst others, to number-theoretic functions. In chapters xiii and xiv we consider the operational calculus applied to difference equations and integral equations respectively. Chapters xv and xvi concern applications of the theory to problems in several independent variables, particularly with respect to linear partial differential equations. Chapter xvi is thereby based on the simultaneous transposition of more than one variable, which leads to the *simultaneous operational calculus*.

It is clear from the survey given above that the subject-matter in any chapter is determined by some specific part of mathematics to which the operational calculus is successfully applicable in one way or another. It may thus happen that closely interrelated 'operational rules', on the one hand, and 'operational relations' concerning some definite type of function, on the other, are discussed at several places scattered through the book. This may hamper further applications, and the material presented must therefore be made more readily available. We have done this by listing the most important results in an appendix at the end of the book. We have thus an opportunity to supply the reader with some additional results which have not been given explicitly in the course of the work. The first list contains the 'operational rules'; it forms the 'grammar' of the operational calculus. The second list is the 'dictionary', helpful in translating the language of t into that of p and vice versa. The 'operational relations' are ordered so that those which concern related functions are grouped together.

The division indicated is such that some chapters and sections may be omitted by those who are interested in the applications of the operational calculus to technical problems only. Similarly, other chapters may be omitted by the pure mathematician.

The practical man will find in the following chapters and sections an almost complete course for his purpose:

ii, §§ 5, 6, 7; iii, §§ 1, 2, 3, 5, 6; iv, except § 7; v, except §§ 4, 5, 7; vi, § 3; vii, except §§ 5, 12; viii; ix; x, §§ 1, 2, 4; xi, except § 5; xii, §§ 1, 2, 3; xiv, §§ 1, 2, 3, 5; xv; xvi.

On the other hand, the mathematician will find a survey of the subjects of interest to him by reading only the following chapters and sections:

ii; iii; iv; v; vi; vii; viii, §§ 1, 2, 3, 4; x; xi; xii; xiii; xiv; xv, except § 4; xvi.

Both may well find it helpful to read the complete text, since the parts otherwise omitted will serve to throw light on the recommended selection.