Lecture Notes in Mathematics 1543

A. L. Dontchev T. Zolezzi

Well-Posed Optimization Problems

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- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

Lecture Notes in Mathematics 1543

To our parents, and
To Dora, Mira, Kiril,
Orietta and Guido

PREFACE

This book aims to present, in a unified way, some basic aspects of the mathematical theory of well-posedness in scalar optimization.

The first fundamental concept in this area is inspired by the classical idea of J. Hadamard, which goes back to the beginning of this century. It requires existence and uniqueness of the optimal solution together with continuous dependence on the problem's data.

In the early sixties A. Tykhonov introduced another concept of well-posedness imposing convergence of every minimizing sequence to the unique minimum point. Its relevance to (and motivation from) the approximate (numerical) solution of optimization problems is clear.

In the book we study both the Tykhonov and the Hadamard concepts of well-posedness, the links between them and also some extensions (e.g. relaxing the uniqueness).

Both the pure and the applied sides of our topic are presented. The first four chapters are devoted to abstract optimization problems. Applications to optimal control, calculus of variations and mathematical programming are the subject matter of the remaining five chapters.

Chapter I contains the basic facts about Tykhonov well-posedness and its generalizations. The main metric, topological and differential characterizations are discussed. The Tykhonov regularization method is outlined.

Chapter II is the key chapter (as we see from its introduction) because it is devoted to a basic issue: the relationships between Tykhonov and Hadamard well-posedness. We emphasize the fundamental links between the two concepts in the framework of best approximation problems, convex functions and variational inequalities.

Chapter III approaches the generic nature of well-posedness (or sometimes ill-posedness) within various topological settings. Parametric optimization problems which are well-posed for a dense, or generic, set of parameters are considered. The relationship with differentiability (sensitivity analysis) is pointed out.

Chapter IV establishes the links between Hadamard well-posedness and variational or epi-convergences. In this way several characterizations of Hadamard well-posedness in optimization are obtained. For convex problems the well-posedness is characterized via the Euler-Lagrange equation. An application to nonsmooth problems is presented, and the role of the convergence in the sense of Mosco is exploited, especially for quadratic problems.

Chapter V is the first one devoted to applications of the theory developed in the first four chapters. Characterizations of well-posedness in optimal control problems for ordinary (or partial) differential equations are discussed. We deal with various forms of well-posedness, including Lipschitz properties of the optimal state and control.

Chapter VI discusses the equivalence between the relaxability of optimal control problems and the continuity of the optimal value (with an abstract generalization). The link with the convergence of discrete-time approximations is presented.

Chapter VII focuses on the study of singular perturbation phenomena in optimal control from the point of view of Hadamard well-posedness. Continuity properties of various mappings appearing in singularly perturbed problems (e.g. the reachable set depending on a small parameter in the derivative) are studied.

Chapter VIII is devoted to characterizations of Tykhonov and Hadamard well-posedness for Lagrange problems with constraints in the calculus of variations, after treating integral functionals without derivatives. We also discuss the classical Ritz method, least squares, and the Lavrentiev phenomenon.

Chapter IX considers first the basic (Berge-type) well-posedness results in a topological setting, for abstract mathematical programming problems depending on a parameter. Then we characterize the stability of the feasible set defined by inequalities, via constraint qualification conditions; Lipschitz properties of solutions to generalized equations are also discussed. Hadamard well-posedness in convex mathematical programming is studied. Quantitative estimates for the optimal solutions are obtained using local Hausdorff distances. Results about Lipschitz continuity of solutions in nonlinear and linear programming end the chapter.

We have made an attempt to unify, simplify and relate many scattered results in the literature. Some new results and new proofs are included. We do not intend to deal with the theory in the most general setting; our goal is to present the main problems, ideas and results in as natural a way as possible.

Each chapter begins with an introduction devoted to examples and motivations or to a simple model problem in order to illustrate the specific topic. The formal statements are often introduced by heuristics, particular cases and examples, while the complete proofs are usually collected at the end of each section and given in full detail, even when elementary. Each chapter contains notes and bibliographical remarks.

The prerequisites for reading this book do not extend in general beyond standard real and functional analysis, general topology and basic optimization theory. Some topics occasionally require more special knowledge that is always either referenced or explicitly recalled when needed.

Some sections of this book are based in part on former lecture notes (by T. Zolezzi) under the title "Perturbations and approximations of minimum problems".

We benefited from the help of many colleagues. We would like to thank especially G. Dal Maso, I. Ekeland, P. Kenderov, D. Klatte, R. Lucchetti, F. Patrone, J. Revalski, K. Tammer, V. Veliov. The support of the Bulgarian Academy of Sciences, Consiglio Nazionale delle Ricerche, MPI and MURST is gratefully acknowledged.

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June 1992

Asen L. Dontchev
Tullio Zolezzi

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Chapter I.

TYKHONOV WELL-POSEDNESS

Section 1. Definition and Examples.

Let X be a set endowed with either a topology or a convergence structure. Let

$$I : X \to (-\infty, +\infty]$$

be a proper extended real - valued function. Consider the problem

$$\text{to minimize } I(x) \text{ subject to } x \in X,$$

which we denote by (X, I).

We are interested in the well-posedness of (X, I).

A natural well - posedness concept arises when we require the following two conditions. First, we impose existence and uniqueness of the global minimum point

$$x_0 = \arg \min (X, I).$$

Second, we require that, whenever we are able to compute approximately the optimal value

$$I(x_0) = \inf I(X),$$

then we automatically do approximate the optimal solution x_0. That is, every method constructing minimizing sequences for (X, I) corresponds to approximately computing x_0.

More precisely, the latter condition means that, if x_n is any sequence from X such that

$$I(x_n) \to \inf I(X)$$

then

$$x_n \to \arg\min (X, I).$$

This condition is clearly of fundamental relevance to the approximate (numerical) solving of (X, I). We shall consider first the case when X is a convergence space, as follows.

Let X be a convergence space, with convergence of sequences denotes by $\longrightarrow$, see e.g. Kuratowski [1, p. 83 - 84]. Let $I : X \to (-\infty, +\infty]$ be a proper extended real-valued function. The problem (X, I) is *Tykhonov well-posed* iff I has a unique global minimum point on X towards which every minimizing sequence converges.

An equivalent definition is the following: there exists exactly one $x_0 \in X$ such that $I(x_0) \leq I(x)$ for all $x \in X$, and

$$I(x_n) \to I(x_0) \text{ implies } x_n \to x_0.$$

Notice that the existence of some x_0 as above implies its uniqueness (if x_0, y_0 do then take the minimizing sequence $x_0, y_0, x_0, y_0, ...$).

Tykhonov well - posedness of (X, I) is often stated equivalently as *strong uniqueness* of arg min (X, I), or *strong solvability* of (X, I). Sometimes well - posedness is translated literally as *correctness*. Problems which are not well - posed will be called *ill - posed.* Sometimes they are referred to as *improperly posed.*

1 Example. Let $X = R^n$ and $I(x) = |x|$ (taking any norm).
Then $0 =$ arg min (X, I) and clearly (X, I) is Tykhonov well - posed.

2 Example. Let $X = R$ and

$$I(x) = x \text{ if } x > 0, \ = |x+1| \text{ if } x \le 0.$$

Then the only minimum point is $x_0 = -1$ but (X, I) is Tykhonov ill - posed since the minimizing sequence $x_n = 1/n$ does not converge to x_0.

Remark. Let A be an open set in R^n, let $x_0 \in A$ and $f \in C^2(A)$. Suppose that $\nabla f(x_0) = 0$ and the Hessian matrix of f at x_0 is positive definite. Then x_0 is a local minimizer of f. By Taylor's formula, there exist some $\alpha > 0$ and a ball $B \subset A$ centered at x_0 such that

$$f(x) \ge f(x_0) + \alpha |x - x_0|^2, x \in B,$$

so that (B, f) is well - posed.

3 Example. Let X be the unit ball in $L^\infty(0,1)$ equipped with the strong convergence. Given $u \in X$ let $x(u)$ be the only absolutely continuous solution to

$$\dot{x} = u \text{ a.e. in } (0,1), x(0) = 0,$$

and let

$$I(u) = \int_0^1 x(u)^2 \, dt.$$

Then arg min (X, I) reduces to the single function $u = 0$ and the optimal value is 0. This (optimal control) problem is not Tykhonov well - posed since

$$u_n(t) = \sin nt$$

is a minimizing sequence because

$$x(u_n)\,(t) = \int_0^t u_n \, dt \to 0 \text{ uniformly on } [0,1],$$

therefore $I(x_n) \to 0$. But u_n does not converge to 0 in X since $||u_n|| = 1$ for any $n \ge 2$.

4 Example. Everything is as in example 3, except that X is now equipped with L^1- convergence . Then X is a compact metric space (by Alaoglu's theorem and separability of $L^1(0,1)$: see Dunford-Schwartz [1, V.4.2 and V.5.1]). Let u_n be any minimizing sequence. Fix any subsequence of u_n, then a further subsequence $v_n \to u_0 \in X$ and $I(v_n) \to 0$. Then $x(v_n) \to x(u_0)$ uniformly in $[0,1]$. So $I(u_0) = 0$, hence $u_0 = 0$. This shows that the original sequence u_n converges towards 0 in X, thus (X, I) is Tykhonov well - posed.

5 Example. Let X be the unit ball of $L^2(0,1)$ equipped with the strong convergence, and let $x(u)$ be as in example 3. Put

$$I(u) = \int_0^1 x(u)^2 \, dt + \varepsilon \int_0^1 u^2 \, dt, \quad \varepsilon \geq 0.$$

If $\varepsilon > 0$ then (X, I) is Tykhonov well - posed, but it is not when $\varepsilon = 0$.

6 Example. Let X be a sequentially compact convergence space, let I be a proper and sequentially lower semicontinuous function on X. Suppose that (X, I) has a unique global minimum point. Then (X, I) is Tykhonov well - posed. So, assuming uniqueness of the minimizer, well - posedness obtains with respect to the natural convergence associated to the direct method (compactness and lower semicontinuity) of the calculus of variations. Another case: $X = R^n$, $I : R^n \to (-\infty, +\infty)$ is strictly convex, and $I(x) \to +\infty$ as $x \to \infty$. Then I is continuous and (X, I) is Tykhonov well - posed. More generally, (X, I) is Tykhonov well - posed with respect to the strong convergence whenever X is a compact convex subset of some Banach space and I is finite-valued, strictly convex and strongly lower semicontinuous on X.

7 Example. The problem (R^k, I) is Tykhonov well - posed if I is convex, lower semicontinuous and there exists an unique $x_0 = \arg\min\ (R^k, I)$. To see this, replace I by $x \to I(x_0 + x) - I(x_0)$. Then we can assume without loss of generality that

$$I(0) = 0 < I(x) \text{ if } x \neq 0.$$

Let x_n be any minimizing sequence. If $|x_n| \to +\infty$ for some subsequence, then by convexity

$$0 \leq I(\frac{x_n}{|x_n|}) \leq \frac{1}{|x_n|} I(x_n) \to 0 \text{ as } n \to +\infty.$$

On the other hand, for a further subsequence

$$\frac{x_n}{|x_n|} \to y \text{ with } |y| = 1.$$

By lower semicontinuity we get $I(y) = 0$, a contradiction. So any cluster point u of x_n fulfills (for some subsequence) $I(x_n) \to I(u) = 0$, yielding $u = 0$ by uniqueness. Then $x_n \to 0$ for the original sequence.

8 Example. Let X be a convergence space and $I : X \to (-\infty, +\infty)$ be lower semicontinuous and coercive, i.e. $y_n \in X$ and $\sup I(y_n) < +\infty$ imply that some subsequence of y_n converges. Then (X, I) is Tykhonov well - posed if it has a unique global minimum point. If I is lower semicontinuous and bounded from below, then (X, I) is Tykhonov well - posed iff every minimizing sequence converges.

9 Remark. If X is a Banach space, I is convex on X and (X, I) is Tykhonov well - posed with respect to the strong convergence on X, then $I(x) \to +\infty$ as $||x|| \to +\infty$. Indeed, let $x_0 =$ arg min (X, I) and arguing by contradiction, assume

$$L = \liminf I(x) < +\infty \text{ as } ||x|| \to +\infty.$$

By well - posedness, $I(x_0) < L$. For some sequence x_n with $||x_n|| \to +\infty$ we have $I(x_n) \to L$.

Put $2y_n = x_n + x_0$. Then $||y_n|| \to +\infty$. By convexity, for some $\varepsilon > 0$

$$I(y_n) \leq \frac{1}{2}I(x_0) + \frac{1}{2}I(x_n) \leq \frac{1}{2}(L - \varepsilon) + \frac{1}{2}(L + 2\frac{\varepsilon}{3})$$

thus lim sup $I(y_n) < L$, a contradiction.

We shall see that the conditions

$$I(x) \to +\infty \text{ as } ||x|| \to +\infty \text{ and arg min } (X, I) \text{ is a singleton,}$$

do not imply Tykhonov well - posedness in the infinite-dimensional setting (modify I in example 18 by imposing $I(x) \to +\infty$ as $||x|| \to +\infty$: for example

$$I(x) = \sum_{n=1}^{\infty} \frac{< x, e_n >^2}{n^2} + (||x|| - 1)^2 \text{ if } ||x|| > 1).$$

10 Example. Let X be a nonempty closed convex subset of the real Hilbert space H. Let $u \in H$ be fixed and let

$$I(x) = ||u - x||, x \in X.$$

By (the proof of) the Riesz projection theorem we see that (X, I) is Tykhonov well - posed with respect to the strong convergence. In fact, let x_n be any minimizing sequence. By the parallelogram law, for every n and k

$$||u - \frac{x_n + x_k}{2}||^2 + ||\frac{x_n - x_k}{2}||^2 = \frac{1}{2}(||u - x_k||^2 + ||u - x_n||^2).$$

Since $(x_n + x_k)/2 \in X$, we get

$$||u - \frac{x_n + x_k}{2}||^2 \geq m^2$$

where $m =$ dist (u, X). Therefore

$$||\frac{x_n - x_k}{2}||^2 \leq \frac{1}{2}(||u - x_n||^2 + ||u - x_k||^2) - m^2,$$

hence $||x_n - x_k|| \to 0$ as $n, k \to +\infty$. Therefore there exists some $z \in X$ such that

$$x_n \to z \text{ and } ||u - x_n|| \to ||u - z|| = m.$$

Uniform convexity of H implies uniqueness of z, whence well - posedness.

Example 10 will be generalized, in a very significant way, in section II.1.

As a final example, let X be a real normed space and $A \subset X$. By the very definition (see Giles [1 p.195]), a point $x \in A$ is *strongly exposed* iff there exists $u \in X^*$ such that (A, u) is Tykhonov well-posed with respect to strong convergence and $x =$ arg min (A, u).

Section 2. Metric Characterizations.

The weaker the convergence on X is, the easier Tykhonov well-posedness obtains. As a matter of fact, as we saw in example 6, under uniqueness of the minimizer Tykhonov well-posedness of (X, I) is a fairly common property, since it can be obtained by the direct method in the calculus of variations. However, in the applications we often need sufficiently strong convergence of minimizing sequences. Therefore the following setting is of interest.

Standing assumptions: X is a metric space with metric d, and

$$I : X \to (-\infty, +\infty).$$

Remark. We can assume I real-valued without loss of generality, since (X, I) is Tykhonov well-posed iff (K, I) is, where

$$K = \text{ effective domain of } I = \{x \in X : I(x) < +\infty\}.$$

We shall consider X as a convergence space equipped by the (natural) convergence structure induced by the metric.

In the sequel we shall use the following conditions:

(1) I is sequentially lower semicontinuous and bounded from below.

(2) X is complete.

The basic idea behind the next fundamental theorem can be roughly explained as follows. If (X, I) is Tykhonov well - posed then $\varepsilon -$ arg min (X, I) shrinks to the unique optimal solution as $\varepsilon \to 0$. Conversely, if diam $[\varepsilon -$ arg min $(X, I)] \to 0$ then every minimizing sequence is Cauchy, therefore it will converge to the unique solution of (X, I), provided that (1) and (2) hold.

11 Theorem *If* (X, I) *is Tykhonov well-posed then*

(3) $$diam\ [\varepsilon - arg\ min\ (X, I)] \to 0 \text{ as } \varepsilon \to 0.$$

Conversely, (3) *implies Tykhonov well-posedness under* (1) *and* (2).

Since Tykhonov well - posedness of (X, I) amounts to the existence of some $x_0 \in$ arg min (X, I) such that

$$I(x_n) \to I(x_0) \Rightarrow d(x_n, x_0) \to 0,$$

then it is reasonable to try to find some estimate from below for $I(x) - I(x_0)$ in terms of $d(x, x_0)$. This aims to quantitative results about Tykhonov well-posedness (e.g. rates of convergence of minimizing sequences).

A function

$$c : D \to [0, +\infty)$$

is called a *forcing function* iff