
**PROBABILITY,
RANDOM VARIABLES,
AND RANDOM
SIGNAL PRINCIPLES**

Second Edition

Peyton Z. Peebles, Jr., Ph.D.

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Peyton Z. Peebles, Jr., Ph.D.

*Professor of Electrical Engineering
University of Florida*

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Maida Erlene Denton Dials

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PREFACE TO THE SECOND EDITION

Because the first edition of this book was well received by the academic and engineering community, a special attempt was made in the second edition to include only those changes that seemed to clearly improve the book's use in the classroom. Most of the modifications were included only after obtaining input from several users of the book.

Except for a few minor corrections and additions, just six significant changes were made. Only two, a new section on the central limit theorem and one on gaussian random processes, represent modification of the original text. A third change, a new chapter (10) added at the end of the book, serves to illustrate a number of the book's theoretical principles by applying them to problems encountered in practice. A fourth change is the addition of Appendix F, which is a convenient list of some useful probability densities that are often encountered.

The remaining two changes are probably the most significant, especially for instructors using the book. First, the number of examples that illustrate the topics discussed has been increased by about 30 percent (over 85 examples are now included). These examples were carefully scattered throughout the text in an effort to include at least one in each section where practical to do so. Second, over 220 new student exercises (problems) have been added at the ends of the chapters (a 54 percent increase).

The book now contains 630 problems and a complete solutions manual is available to instructors from the publisher. This addition was in response to instructors that had used most of the exercises in the first edition. For these instructors' convenience in identifying the new problems, they are listed in each chapter as "Additional Problems."

All other aspects of the book, such as its purpose (a textbook), intended audience (juniors, seniors, first-year graduate students), level, and style of presentation, remain as before.

I would like to thank D. I. Starry for her excellent work in typing the manuscript and the University of Florida for making her services available. Finally, I am again indebted to my wife, Barbara, for her selfless efforts in helping me proofread the book. If the number of in-print errors is small, it is greatly due to her work.

Peyton Z. Peebles, Jr.

PREFACE TO THE FIRST EDITION

This book has been written specifically as a textbook with the purpose of introducing the principles of probability, random variables, and random signals to either junior or senior engineering students.

The *level* of material included in the book has been selected to apply to a typical undergraduate program. However, a small amount of more advanced material is scattered throughout to serve as stimulation for the more advanced student, or to fill out course content in schools where students are at a more advanced level. (Such topics are keyed by a star *.) The *amount* of material included has been determined by my desire to fit the text to courses of up to one semester in length. (More is said below about course structure.)

The *need* for the book is easily established. The engineering applications of probability concepts have historically been taught at the graduate level, and many excellent texts exist at that level. In recent times, however, many colleges and universities are introducing these concepts into the undergraduate curricula, especially in electrical engineering. This fact is made possible, in part, by refinements and simplifications in the theory such that it can now be grasped by junior or senior engineering students. Thus, there is a definite need for a text that is clearly written in a manner appealing to such students. I have tried to respond to this need by paying careful attention to the organization of the contents, the development of discussions in simple language, and the inclusion of text examples and many problems at the end of each chapter. The book contains over 400 problems and a solutions manual for all problems is available to instructors from the publisher.

Many of the examples and problems have purposely been made very simple in an effort to instill a sense of accomplishment in the student, which, hopefully,

will provide the encouragement to go on to the more challenging problems. Although emphasis is placed on examples and problems of electrical engineering, the concepts and theory are applicable to all areas of engineering.

The International System of Units (SI) has been used primarily throughout the text. However, because technology is presently in a transitional stage with regard to measurements, some of the more established customary units (gallons, °F, etc.) are also utilized; in such instances, values in SI units follow in parentheses.

The *student background* required to study the book is only that typical of junior or senior engineering students. Specifically, it is assumed the student has been introduced to multivariable calculus, Fourier series, Fourier transforms, impulse functions, and some linear system theory (transfer function concepts, especially). I recognize, however, that students tend to forget a fair amount of what is initially taught in many of these areas, primarily through lack of opportunity to apply the material in later courses. Therefore, I have inserted short reviews of some of these required topics. These reviews are occasionally included in the text, but, for the most part, exist in appendixes at the end of the book.

The *order of the material* is dictated by the main topic. Chapter 1 introduces probability from the axiomatic definition using set theory. In my opinion this approach is more modern and mathematically correct than other definitions. It also has the advantage of creating a better base for students desiring to go on to graduate work. Chapter 2 introduces the theory of a single random variable. Chapter 3 introduces operations on one random variable that are based on statistical expectation. Chapter 4 extends the theory to several random variables, while Chapter 5 defines operations with several variables. Chapters 6 and 7 introduce random processes. Definitions based on temporal characterizations are developed in Chapter 6. Spectral characterizations are included in Chapter 7.

The remainder of the text is concerned with the response of linear systems with random inputs. Chapter 8 contains the general theory, mainly for linear time-invariant systems; while Chapter 9 considers specific optimum systems that either maximize system output signal-to-noise ratio or minimize a suitably defined average error.

Finally, the book closes with a number of appendixes that contain material helpful to the student in working problems, in reviewing background topics, and in the interpretation of the text.

The book can profitably be used in curricula based on either the quarter or the semester system. At the University of Tennessee, a *one-quarter undergraduate course* at the junior level has been successfully taught that covers Chapters 1 through 8, except for omitting Sections 2.6, 3.4, 4.4, 8.7 through 8.9, and all starred material. The class met three hours per week.

A *one-semester undergraduate course* (three hours per week) can readily be structured to cover Chapters 1 through 9, omitting all starred material except that in Sections 3.3, 5.3, 7.4, and 8.6.

Although the text is mainly developed for the undergraduate, I have also

successfully used it in a *one-quarter graduate course* (first-year, three hours per week) that covers Chapters 1 through 7, including all starred material.

It should be possible to cover the entire book, including all starred material, in a *one-semester graduate course* (first-year, three hours per week).

I am indebted to many people who have helped make the book possible. Drs. R. C. Gonzalez and M. O. Pace read portions of the manuscript and suggested a number of improvements. Dr. T. V. Blalock taught from an early version of the manuscript, independently worked a number of the problems, and provided various improvements. I also extend my appreciation to the Advanced Book Program of Addison-Wesley Publishing Company for allowing me to adapt and use several of the figures from my earlier book *Communication System Principles* (1976), and to Dr. J. M. Googe, head of the electrical engineering department of the University of Tennessee, for his support and encouragement of this project. Typing of the bulk of the manuscript was ably done by Ms. Belinda Hudgens; other portions and various corrections were typed by Kymberly Scott, Sandra Wilson, and Denise Smiddy. Finally, I thank my wife, Barbara, for her aid in proofreading the entire book.

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