QUANTUM THEORY OF FIELDS

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By GREGOR WENTZEL, PROFESSOR OF PHYSICS,
UNIVERSITY OF CHICAGO

Translated from the German by CHARLOTTE HOUTERMANS and J. M. JAUCH. With an Appendix by J. M. JAUCH.

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Preface

In research in theoretical physics of the past years, work in the quantum theory of fields has held a significant place. The student who wishes to become acquainted, by means of periodicals, with this branch of study will often find access difficult, even though he may be fully conversant with elementary quantum mechanics. Many of my colleagues who have introduced their pupils to the original literature agree with me in this. This book may help make that access easier. Naturally, I have wondered whether the theory is well enough established to be dealt with in a text-book, but I believe I can ignore such considerations. Certainly the theory has a problematic aspect (see "Self-energy" in the index). If, in the near future, important progress in this theory could be foreseen, we might expect some parts of this book to become rapidly obsolete, yet we can hardly hope for so favorable a development at the present time. While waiting for the liberating new ideas, we must depend on our present theory, and so it seems worth while to make the theory more easily accessible to the interested. Only those who know the theory can understand the problems.

As the title suggests, this book is only an introduction, not an all-inclusive account. The didactic purpose precludes a too systematic approach. It appeared appropriate to start with the "canonical" quantization rules of elementary quantum mechanics (Heisenberg's commutation relations), even though further investigation shows that these rules are too narrow and must be generalized, as, for example, in dealing with particles obeying the Pauli exclusion principle. On the other hand, I have not attempted to deal separately and in detail with the classical theory of wave fields. No doubt, it would have been instructive to illustrate certain deductions in quantum theory by pointing out the corresponding classical considerations; however, since the operator technique of quantum mechanics often simplifies the calculations, I have preferably used the quantum version.

In the first chapter the fundamentals of the theory are dealt with in a general manner, that is, without specification of the field equations or the Lagrange function. The remaining chapters are devoted to particular field types which by means of quantization are associated with particles of various spin, charge, and mass values. No effort was made to attain completeness of treatment; the fields considered may be regarded as typical examples. Naturally, the electromagnetic field and the electron-wave field cannot be omitted. In order to spare the reader unnecessary difficulties, I have deferred many questions, which could have been handled earlier in the general part, to the particular chapters, although at the cost of some repetition. Still, I believe the general part is indispensable, as it is here that the inner homogeneity and consistency of the theory is manifested to some extent. In §4 especially, the proof that the field quantization is Lorentz-invariant is prepared so far that it can be completed for the particular fields without much trouble; at the same time it becomes clear why the "invariant D-function" automatically appears in the relativistic commutation relations. The reader who finds Chapter I difficult in places is urged to study the examples of scalar fields (§6 and §8) as illustrations. The sections in small type are devoted to special questions and applications. The reader may omit these if he wishes.

It is assumed that the reader is acquainted with such fundamentals as are to be found in the current text-book literature. This applies not only to elementary quantum mechanics but especially to Dirac's wave mechanics of spin-electrons (from §17 on). The reader will find the necessary preparation for instance in the article by W. Pauli in Geiger-Scheel's *Handbuch der Physik*, Volume 24, I.

Zurich
August, 1942

G. Wentzel

Preface to the English Edition

I hope the reader will welcome the English translation of my book—initiated by Interscience Publishers—as much as I do. A rough translation of the text was made by Mrs. Charlotte Houtermans, and then revised and corrected scientifically and linguistically by Dr. J. M. Jauch, who also contributed the Appendix. Dr. F. Coester was kind enough to read proof of the galleys and to prepare the index.

The demand for an introduction into the quantum theory of fields seems to be even larger today than some years ago when this book was written. New interest in the subject has been awakened by recent experimental discoveries: the fine structure anomalies of the hydrogen levels, and the correction to the magnetic moment of the electron, which have been interpreted as electromagnetic self-energy terms; the observation of several kinds of mesons and their artificial production, marking a new development that promises a much better understanding of mesons and may eventually lead to an improved meson field theory of nuclear interactions.

As to quantum electrodynamics, it was not possible to incorporate any references to the new theoretical developments into the present edition. We must be content to provide the reader with such basic information as will enable him to follow independently the original literature now appearing in the periodicals. In the chapters on meson theory no changes were necessary except in §15, dealing with the applications to problems of nuclear physics; here it was easy to modernize the text and to adapt it to the present state of knowledge. In various sections throughout the book references to more recent publications have been added. The most significant addition to the original is the Appendix, on the general construction of the energy-momentum tensor according to F. J. Belinfante.

I hope that the book proves useful to many readers in the English-speaking world.

Chicago
January, 1949

G. WENTZEL

Contents

Chapter I

General Principles

§ 1. The Canonical Formalism

In classical physics a "field" is described by one or several (real) space-time functions $\psi_\sigma(x, t)$ which satisfy certain partial differential equations, the so-called "field equations." An alternative procedure is to start with a variational principle chosen in such a way that its Euler differential equations are the same field equations. Let L be a function of the $\psi_\sigma(x, t)$ and of their first time and space derivatives:[1]

$$L = L\,(\psi_1, \nabla\,\psi_1, \dot{\psi}_1;\ \psi_2, \nabla\,\psi_2, \dot{\psi}_2;\ \ldots). \tag{I.1}$$

By integration over a volume V and a time interval from t' to t'' we form:

$$\int_{t'}^{t''} dt \int_V dx\, L\,(\psi_1, \ldots) = I.$$

Varying the function ψ_σ for a fixed region of integration:

$$\psi_\sigma\,(x, t) \rightarrow \psi_\sigma\,(x, t) + \delta\psi_\sigma\,(x, t),$$

subject to the restriction that the variations $\delta\psi_\sigma$ vanish at the boundary of the domain of integration (i. e., at the surface of the volume V and for $t = t'$ and $t = t''$), one obtains in the familiar way:

$$\delta I = \int_{t'}^{t''} dt \int_V dx\, \delta L$$

$$= \int_{t'}^{t''} dt \int_V dx \sum_\sigma \left\{ \frac{\partial L}{\partial \psi_\sigma}\,\delta\psi_\sigma + \sum_k \frac{\partial L}{\partial \dfrac{\partial \psi_\sigma}{\partial x_k}}\,\frac{\partial}{\partial x_k}\,\delta\psi_\sigma + \frac{\partial L}{\partial \dot{\psi}_\sigma}\,\frac{\partial}{\partial t}\,\delta\psi_\sigma \right\}$$

$$= \int_{t'}^{t''} dt \int_V dx \sum_\sigma \delta\psi_\sigma \left\{ \frac{\partial L}{\partial \psi_\sigma} - \sum_k \frac{\partial}{\partial x_k}\,\frac{\partial L}{\partial \dfrac{\partial \psi_\sigma}{\partial x_k}} - \frac{\partial}{\partial t}\,\frac{\partial L}{\partial \dot{\psi}_\sigma} \right\}.$$

[1] Partial derivatives with respect to the time will be indicated by dots: $\partial\psi/\partial t = \dot{\psi}$.

We require now that the classical field be determined by the condition that the integral I shall be stationary ($\delta I = 0$) for arbitrary variations $\delta\psi_\sigma$ which satisfy the above-mentioned conditions and for an arbitrary choice of the integration region. From this it follows that for all times and for all positions:

$$\frac{\partial L}{\partial \psi_\sigma} - \sum_k \frac{\partial}{\partial x_k} \frac{\partial L}{\partial \frac{\partial \psi_\sigma}{\partial x_k}} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\psi}_\sigma} = 0 \quad (\sigma = 1, 2, \ldots). \tag{1.2}$$

These equations are by (1.1) partial differential equations of the second order at most for the field functions ψ_σ; they are the field equations.[1]

This variational principle can be connected with Hamilton's least action principle of classical mechanics. The field variables ψ_σ differ from the coordinates q_j of a mechanical system of particles because the latter are only functions of the time, while the former depend on the position vector x as well. It is, however, possible to establish a correspondence between ψ_σ and the generalized coordinates q_j if we let the discrete index j, which numbers the (finite) degrees of freedom of the system, correspond not only to the discrete index σ but also to the continuous variable x of the position vector. By this procedure a field may be interpreted as a mechanical system of infinitely many degrees of freedom.

[1] One calls

$$\frac{\partial L}{\partial \psi_\sigma} - \sum_k \frac{\partial}{\partial x_k} \frac{\partial L}{\partial \frac{\partial \psi_\sigma}{\partial x_k}} \equiv \frac{\delta L}{\delta \psi_\sigma}$$

the "functional derivative" of $L = \int dx\, L$ with respect to ψ_σ. With this notation, (1.2) reduces to:

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\psi}_\sigma} = \frac{\delta L}{\delta \psi_\sigma}.$$

The substitution:

$$L \to L + \sum_k \frac{\partial}{\partial x_k} \Lambda_k (\psi_1, \psi_2, \ldots) + \frac{\partial}{\partial t} \Lambda_0 (\psi_1, \psi_2, \ldots),$$

where $\Lambda_0, \ldots \Lambda_3$ are arbitrary functions of the ψ_σ, leaves the field equations invariant, for the integral:

$$\int_{t'}^{t''} dt \int_V dx \left\{ \sum_k \frac{\partial \Lambda_k}{\partial x_k} + \frac{\partial \Lambda_0}{\partial t} \right\}$$

may be transformed into an integral over the surface of the space-time region and hence its variation vanishes identically. The $\Lambda_0, \ldots \Lambda_3$ may even depend on the derivatives of the ψ_σ, provided L remains independent of the second derivatives.

This analogy may be further expounded by subdividing the space into finite cells $\delta x^{(s)}$ which we distinguish by the upper index s; the value of the field function ψ_σ in the cell (s) shall be denoted by $\psi_\sigma^{(s)}(t)$. The space derivatives of ψ_σ will be replaced by the corresponding differences. In this way it is possible to represent the function $\mathfrak{L}$ (1.1), or rather its value $\mathfrak{L}^{(s)}$ in any cell (s), as a function of the generalized coordinates $q_j = \psi_\sigma^{(s')}$ and of the corresponding velocities $\dot{q}_j = \dot{\psi}_\sigma^{(s')}$. Similarly for the sum over all the cells:

$$L = \sum_s \delta x^{(s)} \, \mathfrak{L}^{(s)}.$$

The variational principle stated above requires then that the time integral:

$$I = \int_{t'}^{t''} dt \, L\,(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots)$$

shall be extremal for the variations $q_j(t) \rightarrow q_j(t) + \delta q_j(t)$ provided δq_j vanishes for $t = t'$ and $t = t''$ as well as in cells (s) outside the region V. Since V is arbitrary, this corresponds precisely to Hamilton's principle in mechanics. The field equations (1.2), which are Euler's differential equations of the variational principle, correspond to Lagrange's equations of motion with the Lagrange function L. In the transition to the limit of infinitesimal cells, one obtains for the Lagrange function the integral extended over all space:

$$L = \int dx \, \mathfrak{L}\,(\psi_1, \nabla \psi_1, \dot{\psi}_1;\; \psi_2, \nabla \psi_2, \dot{\psi}_2;\; \ldots). \tag{1.3}$$

We shall call the integrand $\mathfrak{L}$ "differential Lagrange function."

In order to make the transition to Hamilton's formalism—still in the framework of the classical theory—we must introduce first the momenta $p_j = \partial L / \partial \dot{q}_j$, which are canonically conjugate to the coordinates q_j: with $L = \sum_{s'} \delta x^{(s')} \, \mathfrak{L}^{(s')}$ the quantity canonically conjugate to $q_j = \psi_\sigma^{(s)}$ becomes:[1]

$$p_j = \delta x^{(s)} \cdot \frac{\partial \mathfrak{L}^{(s)}}{\partial \dot{\psi}_\sigma^{(s)}}$$

In a field theory it is customary to denote the space-time function, obtained from $\mathfrak{L}$ (1.1) by partial differentiation with respect to $\dot{\psi}_\sigma$ (for constant ψ_σ and $\nabla\psi_\sigma$), as the field canonically conjugate to ψ_σ.

[1] $\dot{\psi}_\sigma^{(s)}$ is found in $\mathfrak{L}^{(s)}$ only, and not in the remaining terms of the sum in $L(s' \neq s)$, since according to the definition (1.1) $\mathfrak{L}$ at the point x depends only on the value of $\dot{\psi}_\sigma$ at the same point (and not on $\nabla\dot{\psi}_\sigma$ also).

$$\pi_\sigma = \frac{\partial \mathbf{L}}{\partial \dot{\psi}_\sigma}. \tag{1.4}$$

Hence, if its value in the cell (s) is $\pi_\sigma^{(s)}$:

$$p_j = \delta x^{(s)} \,.\, \pi_\sigma^{(s)}.$$

Hamilton's function is then obtained by:

$$H = \sum_j p_j \dot{q}_j - L = \sum_s \delta x^{(s)} \left\{ \sum_\sigma \pi_\sigma^{(s)} \dot{\psi}_\sigma^{(s)} - \mathbf{L}^{(s)} \right\},$$

H is to be considered as a function of the q_j and p_j or of the ${}_\sigma\psi^{(s)}$ and $\pi_\sigma^{(s)}$. In the limit of the continuum we obtain:

$$H = \int dx\, \mathbf{H}, \qquad \mathbf{H} = \sum_\sigma \pi_\sigma \dot{\psi}_\sigma - \mathbf{L}. \tag{1.5}$$

By eliminating $\dot{\psi}_\sigma$ with the help of (1.4) we may consider the "differential Hamiltonian function" $\mathbf{H}$ as a function of the ψ_σ, $\nabla\psi_\sigma$ and π_σ:

$$\mathbf{H} = \mathbf{H}\,(\psi_1, \nabla \psi_1, \pi_1;\ \psi_2, \nabla \psi_2, \pi_2;\ \ldots). \tag{1.6}$$

Introducing the Hamiltonian H makes it possible to replace the field equations (1.2) by the "canonical field equations" which correspond to the canonical (Hamilton) equations of motion in classical particle mechanics:[1]

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

We shall, however, not elaborate on this point here, since it is not essential for obtaining the corresponding equations in quantum theory.

Since our classical ψ-field is, as we have seen, equivalent to a mechanical system of particles, with infinitely many degrees of freedom, it is natural to carry out its quantization according to the standard rules of quantum mechanics. The transition from classical to quantum mechanics can be effected by replacing the canonical variables q_j, p_j, by Hermitian operators which satisfy the commutation rules:[2]

$$[q_j, q_{j'}] = [p_j, p_{j'}] = 0, \quad [p_j, q_{j'}] = \frac{h}{i}\,\delta_{jj'}$$

[1] More details in: Heisenberg and Pauli, *Z. Phys.* 56, 1, 1929, §1.

[2] Our notations are the usual ones: $[a, b] = ab - ba$, h = Planck's constant divided by 2π, i = imaginary unit, $\delta_{jj} = 1$, $\delta_{jj'} = 0$ for $j \neq j'$.

The mechanical properties of the system are determined by its Hamiltonian which is formally taken over from the classical theory, but reinterpreted as Hermitian operator. In a similar way we may consider the properties of a quantized field as defined by its Hamiltonian $H = \int dx\, \mathsf{H}$ or also by its Lagrangian $L = \int dx\, \mathsf{L}$, from which H may be obtained according to (1.4) and (1.5). The $\psi_\sigma(x)$ and $\pi_\sigma(x)$ in H (1.6) stand now for Hermitian operators, with commutation rules which result from those of $q_j = \psi_\sigma^{(s)}$, $p_j = \delta x^{(s)}\, \pi_\sigma^{(s)}$ by the transition to the continuum. This procedure is characteristic of the so-called "canonical field quantization" as it was first formulated for general fields by Heisenberg and Pauli.[1]

Accordingly we postulate the following equations:

$$\left[\psi_\sigma^{(s)}, \psi_{\sigma'}^{(s')}\right] = \left[\pi_\sigma^{(s)}, \pi_{\sigma'}^{(s')}\right] = 0, \qquad \left[\pi_\sigma^{(s)}, \psi_{\sigma'}^{(s')}\right] = \frac{h}{i}\,\delta_{\sigma\sigma'} \cdot \frac{\delta_{ss'}}{\delta x^{(s)}}.$$

Writing $\psi_\sigma(x)$, $\pi_\sigma(x)$, instead of $\psi_\sigma^{(s)}$, $\pi_\sigma^{(s)}$, we find for the commutation rules:

$$(1.7)\quad [\psi_\sigma(x), \psi_{\sigma'}(x')] = [\pi_\sigma(x), \pi_{\sigma'}(x')] = 0, \quad [\pi_\sigma(x), \psi_{\sigma'}(x')] = \frac{h}{i}\,\delta_{\sigma\sigma'} \cdot \delta(x, x');$$

Here $\delta(x, x')$ stands for a function the value of which is $(\delta x^{(s)})^{-1}$ or 0, according to whether the points x and x' lie in the same or in different cells. Integrating for fixed x' with respect to x, we obtain $\int dx\, \delta(x, x') = 1$. Furthermore, $\int dx f(x)\, \delta(x, x')$ is equal to the average value of the function $f(x)$ taken over the cell (s') in which x' is situated. In the limit of the continuum (cell volume $\delta x^{(s)} \to 0$) $\delta(x, x')$ goes over into the (three-dimensional) Dirac δ-function:

$$(1.8)\quad \delta(x, x') \to \delta(x - x') = \begin{cases} 0 \text{ for } x \neq x', \\ \infty \text{ for } x = x', \text{ in such a way that} \end{cases}$$

$$\int dx\, \delta(x - x') = 1.$$

This limiting process has meaning only if $\delta(x, x')$ appears in the integrand of a space integral.

$$(1.9)\quad \int dx\, f(x)\, \delta(x, x') \to \int dx\, f(x)\, \delta(x - x') = f(x').$$

It has become customary to write directly $\delta(x - x')$, instead of $\delta(x, x')$, and we shall adopt the same notation in the following, since there is no doubt as to its meaning.

[1] Cited above.

In the quantum theory of fields ψ_σ, π_σ represent operators which depend on the coordinates x of the position as parameters[1] and satisfy the commutation rules (1.7, .8) Later on we shall introduce particular representations for these operators when we discuss special fields. It should be emphasized that for the time being we consider the operators $\psi_\sigma(x)$, $\pi_\sigma(x)$ as time-independent, the construction of time-dependent operators being reserved for a later discussion in §4 in connection with the question of relativistic invariance. Together with ψ_σ and π_σ, H, too, is a space-dependent operator and the integral Hamiltonian $H = \int dx\, \mathsf{H}$ is then an operator independent of position. It may be necessary to arrange non-commuting factors in a suitable way so as to make H a Hermitian operator (symmetrization).

At this point a particular case should be mentioned which plays a certain role in the quantum theory of fields. It happens with certain Lagrangians that some field equations do not contain the second derivatives $\ddot{\psi}_\sigma$ with respect to time; these equations then represent "subsidiary conditions" which establish a relationship between the variables ψ_σ and $\dot{\psi}_\sigma$.[2] In this case the variables ψ_σ, π_σ are no longer independent of each other, and it follows that the commutation rules (1.7) lead to contradictions. A case like this will be encountered in §§12 and 16 where one of the $\dot{\psi}_\sigma$ does not occur in L and consequently the corresponding $\pi_\sigma = \partial \mathsf{L}/\partial\dot{\psi}_\sigma$ vanishes identically. The commutation rule $[\pi_\sigma(x), \psi_\sigma(x')] = -i\,h\,\delta(x - x')$ is then obviously incorrect. We will see later how the quantization can be carried through in such a case. One possibility is to eliminate the redundant field components and to postulate the commutation rules (1.7) for the remaining independent variables (cf. §12, page 77). By this elimination procedure the Hamiltonian may become dependent on the space derivatives of the π_σ so that we have instead of (1.6):

$$\mathsf{H} = \mathsf{H}\,(\psi_1, \nabla\,\psi_1, \pi_1, \nabla\,\pi_1; \ldots). \tag{1.10}$$

[1] x itself is no operator, but a "c-number."

[2] If the Lagrangian has, for instance, the form:

$$\mathsf{L} = \dot{\psi}_1 \cdot F + G,$$

where F is independent of all $\dot{\psi}_\sigma$ and G of $\dot{\psi}_1$, then the equation (1.2) with $\sigma = 1$ has evidently this above-mentioned character of a subsidiary condition. For then:

$$\pi_1 = \frac{\partial \mathsf{L}}{\partial \dot{\psi}_1} = F,$$

i.e., π_1 depends only upon the ψ_σ and their space derivatives. If the operators ψ_σ all commute with each other, it follows that π_1 also commutes with all ψ_σ, in contradiction to (1.7).

The following considerations hold also for this more general type of Hamiltonians.

In quantum mechanics, as is well known, the canonical equations of motion are valid as operator equations on account of the commutation rules of the q_j and p_j:

$$\dot{q}_j \equiv \frac{i}{h}[H, q_j] = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j \equiv \frac{i}{h}[H, p_j] = -\frac{\partial H}{\partial q_j}.$$

In general the time derivative of any function φ of the q_j and p_j not depending on time explicitly, may be expressed by:[1]

$$\dot{\varphi} \equiv \frac{i}{h}[H, \varphi] \tag{1.11}$$

Since we have taken over the commutation rules of particle mechanics, the same result holds for the field theory: the time derivative of any field quantity, i.e., any functional of the $\psi_\sigma(x)$ and $\pi_\sigma(x)$ and their space derivatives depending not explicitly on time, satisfies an equation (1.11). In particular the operators $\dot{\psi}_\sigma$ and $\dot{\pi}_\sigma$ are defined by:

$$\dot{\psi}_\sigma(x) \equiv \frac{i}{h}[H, \psi_\sigma(x)], \quad \dot{\pi}_\sigma(x) \equiv \frac{i}{h}[H, \pi_\sigma(x)] \tag{1.12}$$

The evaluation of these commutators with the help of the commutation rules (1.7,.8) leads to operator equations which are formally equivalent with the field equations (1.2) in the same way as the canonical equations of motion in particle mechanics ($q_j = \partial H/\partial p_j$, $p_j = -\partial H/\partial q_j$) are equivalent to Lagrange's equations. We forego here a general proof.[2] In the following application of the theory to special types of fields, we shall verify this equivalence in each case. On account of their analogy to the canonical equations of motion, we shall refer

[1] To clarify this it is to be remembered that φ signifies an operator which does not depend on time explicitly. The operator $\dot{\varphi}$ as defined by (1.11) is therefore not simply the partial derivative of φ with respect to time. It derives its significance, however, from the well-known theorem in quantum mechanics that the expectation value of $\dot{\varphi}$ is equal to the time derivative of the expectation value of φ, thus:

$$\bar{\dot{\varphi}} \equiv \frac{i}{h}\overline{[H, \varphi]} = \frac{d\bar{\varphi}}{dt}.$$

[2] Cf. footnote 1, page 2, Heisenberg and Pauli. The commutators (1.12), are equal to the functional derivatives of the Hamiltonian H with respect to π_σ and $-\psi_\sigma$, respectively.

$$\frac{i}{h}[H, \psi_\sigma] = \frac{\delta H}{\delta \pi_\sigma}, \quad \frac{i}{h}[H, \pi_\sigma] = -\frac{\delta H}{\delta \psi_\sigma}$$

[Cf. footnote 1, page 0; from it results the stipulated equivalence with (1.2)].

to the operator equations which stem from the definitions (1.12) as "canonical field equations."

All questions regarding the stationary states of the system, the eigen values of H or any other field quantities, can be answered with the well-known methods of quantum mechanics, notwithstanding the fact that we are dealing with a system of infinitely many degrees of freedom. Examples will follow. Before we go into this, some questions of a more general character shall be discussed which have no analogy in particle mechanics.

§ 2. Conservation Laws for Energy, Momentum, and Angular Momentum

We return to the classical (non-quantized) theory. The conservation of energy is expressed by the equation $dH/dt = 0$, provided that the Lagrangian does not depend explicitly on time. It is to be expected that the application of this conservation law to the integral Hamiltonian function (1.5), which represents the total energy of the field, leads to the interpretation of the differential Hamiltonian function $\boldsymbol{H}$ as an energy density, for which a continuity equation holds.

$$\frac{\partial \boldsymbol{H}}{\partial t} + \nabla \cdot \boldsymbol{S} = 0. \tag{2.1}$$

One obtains in fact from (1.4,.5):

$$\frac{\partial \boldsymbol{H}}{\partial t} = \sum_\sigma \left\{ \dot{\psi}_\sigma \frac{\partial}{\partial t} \frac{\partial \boldsymbol{L}}{\partial \dot{\psi}_\sigma} + \ddot{\psi}_\sigma \frac{\partial \boldsymbol{L}}{\partial \dot{\psi}_\sigma} - \left(\dot{\psi}_\sigma \frac{\partial \boldsymbol{L}}{\partial \psi_\sigma} + \ddot{\psi}_\sigma \frac{\partial \boldsymbol{L}}{\partial \dot{\psi}_\sigma} + \sum_k \frac{\partial \dot{\psi}_\sigma}{\partial x_k} \frac{\partial \boldsymbol{L}}{\partial \dfrac{\partial \psi_\sigma}{\partial x_k}} \right) \right\},$$

and using the field equations (1.2):

$$\frac{\partial \boldsymbol{H}}{\partial t} = -\sum_\sigma \sum_k \left\{ \dot{\psi}_\sigma \frac{\partial}{\partial x_k} \frac{\partial \boldsymbol{L}}{\partial \dfrac{\partial \psi_\sigma}{\partial x_k}} + \frac{\partial \dot{\psi}_\sigma}{\partial x_k} \frac{\partial \boldsymbol{L}}{\partial \dfrac{\partial \psi_\sigma}{\partial x_k}} \right\} =$$

$$= -\sum_k \frac{\partial}{\partial x_k} \sum_\sigma \dot{\psi}_\sigma \frac{\partial \boldsymbol{L}}{\partial \dfrac{\partial \psi_\sigma}{\partial x_k}};$$

hence the continuity equation (2.1) is satisfied by:

$$S_k = \sum_\sigma \dot{\psi}_\sigma \frac{\partial L}{\partial \dfrac{\partial \psi_\sigma}{\partial x_k}}. \tag{2.2}$$

This definition of the energy current density is however not unique, since any source-free field could be added to **S**.

Furthermore we will try to define the momentum of the field:

$$G = \int dx\, \mathbf{G} \tag{2.3}$$

so that momentum is conserved:

$$\frac{\partial \mathbf{G}_k}{\partial t} + \sum_j \frac{\partial T_{jk}}{\partial x_j} = 0, \tag{2.4}$$

Here T is a stress tensor. With the notation:

$$x_4 = i\,c\,t, \tag{2.5}$$

$$T_{44} = -H, \quad T_{k4} = \frac{i}{c} S_k, \quad T_{4k} = i\,c\,G_k \quad (k = 1, 2, 3) \tag{2.6}$$

one can summarize the conservation laws (2.1,.4)

$$\sum_{\mu=1}^{4} \frac{\partial T_{\mu\nu}}{\partial x_\mu} = 0 \quad (\nu = 1\,..\,4). \tag{2.7}$$

An expression for the energy-momentum tensor T, which agrees in the 4,4- and k,4-components with (1.5) and (2.2), is:

$$T_{\mu\nu} = -\sum_\sigma \frac{\partial \psi_\sigma}{\partial x_\nu} \frac{\partial L}{\partial \dfrac{\partial \psi_\sigma}{\partial x_\mu}} + L\ \delta_{\mu\nu}. \tag{2.8}$$

This expression also satisfies the conservation equations (2.7), provided that L does not depend explicitly on the x_ν ($\nu = 1 \ldots 4$), for a short calculation gives:

$$\sum_{\mu=1}^{4} \frac{\partial T_{\mu\nu}}{\partial x_\mu} = -\sum_\sigma \frac{\partial \psi_\sigma}{\partial x_\nu} \left\{ \sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} \frac{\partial L}{\partial \dfrac{\partial \psi_\sigma}{\partial x_\mu}} - \frac{\partial L}{\partial \psi_\sigma} \right\},$$

and this vanishes on account of the field equations (1.2), which with the notation