Lecture Notes in Mathematics 1671

Serge Bouc

Green Functors and *G*-sets

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- a subject index: as a rule this is genuinely helpful for the reader.

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Introduction

The theory of Mackey functors has been developed during the last 25 years in a series of papers by various authors (J.A. Green [8], A. Dress [5], T. Yoshida [17], J. Thévenaz and P. Webb [13],[15],[14], G. Lewis [6]). It is an attempt to give a single framework for the different theories of representations of a finite group and its subgroups.

The notion of Mackey functor for a group G can be essentially approached from three points of view: the first one ([8]), which I call "naive", relies on the poset of subgroups of G. The second one ([5],[17]) is more "categoric", and relies on the category of G-sets. The third one ([15]) is "algebraic", and defines Mackey functors as modules over the Mackey algebra.

Each of these points of view induces its own natural definitions, and the reason why this subject is so rich is probably the possibility of translation between them. For instance, the notion of minimal subgroup for a Mackey functor comes from the first definition, the notion of induction of Mackey functors is quite natural with the second, and the notion of projective Mackey functor is closely related to the third one.

The various rings of representations of a group (linear, permutation, p-permutation...), and cohomology rings, are important examples of Mackey functors, having moreover a product (tensor product or cup-product). This situation has been axiomatized, and those functors have been generally called G-functors in the literature, or Green functors.

This definition of a Green functor for a group G is a complement to the "naive" definition of a Mackey functor: to each subgroup of G corresponds a ring, and the various rings are connected by operations of transfer and restriction, which are compatible with the product through Frobenius relations.

The object of this work is to give a definition of Green functors in terms of G-sets, and to study various questions raised by this new definition. From that point of view, a Green functor is a generalized ring, in the sense that the theory of Green functors for the trivial group is the theory of ordinary rings. Now ring theory gives a series of directions for possible generalizations, and I will treat some cases here (tensor product, bimodules, Morita theory, commutants, simple modules, centres).

The first chapter deals only with Mackey functors: my purpose was not to give a full exposition of the theory, and I just recall the possible equivalent definitions, as one can find for instance in the article of Thevenaz and Webb ([15]). I show next how to build Mackey functors "with values in the Mackey functors", leading to the functors $\mathcal{H}(M, N)$ and $M\hat{\otimes}N$, which will be an essential tool: they are analogous to the homomorphisms modules and tensor products for ordinary modules. Those constructions already appear in Sasaki ([12]) and Lewis ([6]). The notion of n-linear map can be generalized in the form of n-linear morphism of Mackey functors. The reader may find that this part is a bit long: this is because I have tried here to give complete proofs, and as the subject is rather technical, this requires many details.

Chapter 2 is devoted to the definition of Green functors in terms of G-sets, and to the proof of the equivalence between this definition and the classical one. It is then possible to define a module over a Green functor in terms of G-sets. I treat next the fundamental case of the Burnside functor, which plays for Green functors the role of the ring **Z** of integers.

In chapter 3, I build a category $\mathcal{C}_A$ associated to a Green functor A, and show that the category of A-modules is equivalent to the category of representations of $\mathcal{C}_A$. This category is a generalization of a construction of Lindner ([9]) for Mackey functors, and of the category of permutation modules studied by Yoshida ([17]) for cohomological Mackey functors.

Chapter 4 describes the algebra associated to a Green functor: this algebra enters the scene if one looks for G-sets Ω such that the evaluation functor at Ω is an equivalence of categories between the category of representations of $\mathcal{C}_A$ and the category of $\mathrm{End}_{\mathcal{C}_A}(\Omega)$-modules. This algebra generalizes the Mackey algebra defined by Thevenaz and Webb ([15]) and the Hecke algebra of Yoshida ([17]). It is possible to give a definition of this algebra by generators and relations.

This algebra depends on the set Ω, but only up to Morita equivalence. Chapter 5 is devoted to the relation between those Morita equivalences and the classical notion of relative projectivity of a Green functor with respect to a G-set (see for instance the article of Webb [16]). More generally, I will deduce some progenerators for the category of A-modules.

Chapter 6 introduces some tools giving new Green functors from known ones: after a neat description of the Green functors $\mathcal{H}(M, M)$, I define the opposite functor of a Green functor, which leads to the notion of right module over a Green functor. A natural example is the dual of a left module. The notion of tensor product of Green functors leads naturally to the definition of bimodule, and the notion of commutant to a definition of the Mackey functors $\mathcal{H}_A(M, N)$ and $M\hat{\otimes}_A N$.

Those constructions are the natural framework for Morita contexts, in chapter 7. The usual Morita theory can be generalized without difficulty to the case of Green functors for a given group G.

The chapters 8,9, and 10 examine the relations between Green functors and bisets: this notion provides a single framework for induction, restriction, inflation, and coinflation of Mackey functors (see [2]).

In chapter 8, I show how the composition with U, if U is a G-set-H, gives a Green functor $A \circ U$ for the group H starting with a Green functor A for the group G. This construction passes down to the associated categories, so there is a corresponding functor from $\mathcal{C}_{A\circ U}$ to $\mathcal{C}_A$. This gives a functor between the categories of representations, which can also be obtained by composition with U. I study next the functoriality of these constructions with respect to U, and give the example of induction and restriction.

Chapter 9 is devoted to the construction of the associated adjoint functors: I build a left and a right adjoint to the functors of composition with a biset $M \mapsto M \circ U$ for Mackey functors, and I give the classical examples of induction, restriction and inflation, and also the less well-known example of coinflation.

Chapter 10 is the most technical of this work: I show how the previous left adjoint functors give rise to Green functors, and I study the associated functors and their adjoints between the corresponding categories of modules. An important consequence of this is the compatibility of left adjoints of composition with tensor products, which proves that if there is a surjective Morita context for two Green functors A and B for the group G, then there is one for all the residual rings $\overline{A}(H)$ and $\overline{B}(H)$, for any subgroup H of G.

In chapter 11, I classify the simple modules over a Green functor, and describe their structure. Applying those results to the Green functor $A\hat{\otimes}A^{op}$, I obtain a new proof of the theorem of Thévenaz classifying the simple Green functors. Finally, I study how the simple modules (or similarly defined modules) behave with respect to the constructions $\mathcal{H}(-,-)$ and $-\hat{\otimes}-$.

Chapter 12 gives two possible generalizations of the notion of centre of a ring, one in terms of commutants, the other in terms of natural transformations of functors. The first one gives a decomposition of any Green functor using the idempotents of the Burnside ring, and shows that up to (usual) Morita equivalence, it is possible to consider only the case of Green functors which are projective relative to certain sets of solvable π-subgroups. The second one keeps track of the blocks of the associated algebras. Then I give the example of the fixed points functors, and recover the isomorphism between the center of Yoshida algebra and the center of the group algebra. Next, the example of the Burnside ring leads to the natural bijection between the p-blocks of the group algebra and the blocks of the p-part of the Mackey algebra.

Chapter 1

Mackey functors

All the groups and sets with group action considered in this book will be finite.

1.1 Equivalent definitions

Throughout this section, I denote by G a (finite) group and R a ring, that may be non-commutative. First I will recall briefly the three possible definitions of Mackey functors: the first one is due to Green ([8]), the second to Dress ([5]), and the third to Thévenaz and Webb ([15]).

1.1.1 Definition in terms of subgroups

One of the possible definitions of Mackey functors is the following:

A Mackey functor for the group G, with values in the category R-**Mod** of R-modules, consists of a collection of R-modules $M(H)$, indexed by the subgroups H of G, together with maps $t_K^H : M(K) \to M(H)$ and $r_K^H : M(H) \to M(K)$ whenever K is a subgroup of H, and maps $c_{x,H} : M(H) \to M({}^xH)$ for $x \in G$, such that:

- If $L \subseteq K \subseteq H$, then $t_K^H t_L^K = t_L^H$ and $r_L^K r_K^H = r_L^H$.
- If $x, y \in G$ and $H \subseteq G$, then $c_{y,{}^xH} c_{x,H} = c_{yx,H}$.
- If $x \in G$ and $H \subseteq G$, then $c_{x,H} t_K^H = t_{{}^xK}^{{}^xH} c_{x,K}$ and $c_{x,K} r_K^H = r_{{}^xK}^{{}^xH} c_{x,H}$. Moreover $c_{x,H} = Id$ if $x \in H$.
- (Mackey axiom) If $L \subseteq H \supseteq K$, then

$$r_L^H t_K^H = \sum_{x \in L\backslash H/K} t_{L\cap {}^xK}^L c_{x,L^x\cap K} r_{L^x\cap K}^K$$

The maps t_K^H are called *transfers* or *traces*, and the maps r_K^H are called *restrictions*.

A morphism θ from a Mackey functor M to a Mackey functor N consists of a collection of morphisms of R-modules $\theta_H : M(H) \to N(H)$, for $H \subseteq G$, such that if

$K \subseteq H$ and $x \in G$, the squares

$$\begin{array}{ccc} M(K) & \xrightarrow{\theta_K} & N(K) \\ {\scriptstyle t_K^H}\downarrow & & \downarrow{\scriptstyle t_K^H} \\ M(H) & \xrightarrow[\theta_H]{} & N(H) \end{array} \qquad \begin{array}{ccc} M(K) & \xrightarrow{\theta_K} & N(K) \\ {\scriptstyle r_K^H}\uparrow & & \uparrow{\scriptstyle r_K^H} \\ M(H) & \xrightarrow[\theta_H]{} & N(H) \end{array} \qquad \begin{array}{ccc} M(H) & \xrightarrow{\theta_H} & N(H) \\ {\scriptstyle c_{x,H}}\downarrow & & \downarrow{\scriptstyle c_{x,H}} \\ M({}^xH) & \xrightarrow[\theta_{{}^xH}]{} & N({}^xH) \end{array}$$

are commutative.

1.1.2 Definition in terms of G-sets

If K and H are subgroups of G, then the morphisms of G-sets from G/K to G/H are in one to one correspondence with the classes xH, where $x \in G$ is such that $K^x \subseteq H$. This observation provides a way to extend a Mackey functor M to any G-set X, by choosing a system of representatives of orbits $G\backslash X$, and defining

$$M(X) = \bigoplus_{x \in G\backslash X} M(G_x)$$

There is a way to make this equality functorial in X, and this leads to the following definition:

Definition: *Let R be a ring. If G is a (finite) group, let G-**set** be the category of finite sets with a left G action. A Mackey functor for the group G, with values in R-**Mod**, is a bifunctor from G-**set** to R-**Mod**, i.e. a couple of functors (M^*, M_*), with M^* contravariant and M_* covariant, which coincide on objects (i.e. $M^*(X) = M_*(X) = M(X)$ for any G-set X). This bifunctor is supposed to have the two following properties:*

- *(M1) If X and Y are G-sets, let i_X and i_Y be the respective injections from X and Y into $X \coprod Y$, then the maps $M^*(i_X) \oplus M^*(i_Y)$ and $M_*(i_X) \oplus M_*(i_Y)$ are mutual inverse R-module isomorphisms between $M(X \coprod Y)$ and $M(X) \oplus M(Y)$.*
- *(M2) If*

$$\begin{array}{ccc} T & \xrightarrow{\gamma} & Y \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Z & \xrightarrow[\beta]{} & X \end{array}$$

is a cartesian (or pull-back) square of G-sets, then $M^*(\beta).M_*(\alpha) = M_*(\delta).M^*(\gamma)$.

A morphism θ from the Mackey functor M to the Mackey functor N is a natural transformation of bifunctors, consisting of a morphism $\theta_X : M(X) \to N(X)$ for any G-set X, such that for any morphism of G-sets $f : X \to Y$, the squares

$$\begin{array}{ccc} M(X) & \xrightarrow{\theta_X} & N(X) \\ {\scriptstyle M_*(f)}\downarrow & & \downarrow{\scriptstyle N_*(f)} \\ M(Y) & \xrightarrow[\theta_Y]{} & N(Y) \end{array} \qquad \begin{array}{ccc} M(X) & \xrightarrow{\theta_X} & N(X) \\ {\scriptstyle M^*(f)}\uparrow & & \uparrow{\scriptstyle N^*(f)} \\ M(Y) & \xrightarrow[\theta_Y]{} & N(Y) \end{array}$$

are commutative.

I will denote by $Mack_R(G)$ or $Mack(G)$ the category of Mackey functors for G over R.

Conversely, if M is a Mackey functor in the sense of this second definition, then one can build a Mackey functor M_1 in the first sense by setting

$$M_1(H) = M(G/H) \qquad t_K^H = M_*(p_K^H) \qquad r_K^H = M^*(p_K^H) \qquad c_{x,H} = M_*(\gamma_{x,H})$$

where $p_K^H : G/K \to G/H$ is the natural projection, and $\gamma_{x,H} : G/H \to G/{}^xH$ is the map $gH \mapsto gx^{-1}.{}^xH = gHx^{-1}$

1.1.3 Definition as modules over the Mackey algebra

There is a third definition of a Mackey functor, using the Mackey algebra $\mu_R(G)$: consider first the algebra $\mu(G)$ over $\mathbf{Z}$: it is the algebra generated by the elements t_K^H, r_K^H, and $c_{x,H}$, where H and K are subgroups of G such that $K \subseteq H$, and $x \in G$, with the following relations:

$$t_K^H t_L^K = t_L^H \;\; \forall\; L \subseteq K \subseteq H$$

$$r_L^K r_K^H = r_L^H \;\; \forall\; L \subseteq K \subseteq H$$

$$c_{y,{}^xH} c_{x,H} = c_{yx,H} \;\; \forall\; x, y, H$$

$$t_H^H = r_H^H = c_{h,H} \;\; \forall\; h, H \; h \in H$$

$$c_{x,H} t_K^H = t_{{}^xK}^{{}^xH} c_{x,K} \;\; \forall\; x, K, H$$

$$c_{x,K} r_K^H = r_{{}^xK}^{{}^xH} c_{x,H} \;\; \forall\; x, K, H$$

$$\sum_H t_H^H = \sum_H r_H^H = 1$$

$$r_K^H t_L^H = \sum_{x \in K\backslash H/L} t_{K\cap {}^xL}^K c_{x,K^x\cap L} r_{K^x\cap L}^L \;\; \forall\; K \subseteq H \supseteq L$$

any other product of r_H^K, t_H^K and $c_{g,H}$ being zero.

A Mackey functor M for the first definition gives a module $\widetilde{M}$ for the "algebra" $\mu_R(G) = R \otimes_Z \mu(G)$ (which is not really an algebra if R is not commutative) defined by

$$\widetilde{M} = \bigoplus_{H \subseteq G} M(H)$$

and a morphism $f : M \to N$ of Mackey functors gives a morphism of $\mu_R(G)$-modules $\tilde{f} : \widetilde{M} \to \widetilde{N}$.

It it then possible to define a Mackey functor as a $\mu_R(G)$-module, and a morphism of Mackey functors as a morphism of $\mu_R(G)$-modules: if M is a $\mu_R(G)$-module, then M corresponds to a Mackey functor M_1 in the first sense, defined by $M_1(H) = t_H^H M$, the maps t_K^H, r_K^H and $c_{x,H}$ being defined as the multiplications by the corresponding elements of the Mackey algebra.

1.2 The Mackey functors $M \mapsto M_Y$

If Y is a G-set, and M is a Mackey functor for G, then let M_Y be the Mackey functor defined by

$$M_Y(X) = M(X \times Y)$$

and for a map of G-sets $f : X \to X'$, by

$$(M_Y)^*(f) = M^*(f \times Id)$$

$$(M_Y)_*(f) = M_*(f \times Id)$$

This construction is functorial in Y: if Y' is another G-set, and if g is a morphism of G-sets from Y to Y', then there is a morphism of Mackey functors $M_g : M_Y \to M_{Y'}$, defined over the G-set X by

$$M_{g,X} = M_*(Id \times g) : \; M_Y(X) \to M_{Y'}(X)$$

To see this, let $f : X \to X'$ be a map of G-sets. Then the square

$$\begin{array}{ccc} M_Y(X) & \xrightarrow{M_{g,X}} & M_{Y'}(X) \\ {\scriptstyle (M_Y)_*(f)}\downarrow & & \downarrow{\scriptstyle (M_{Y'})_*(f)} \\ M_Y(X') & \xrightarrow[M_{g,X'}]{} & M_{Y'}(X') \end{array}$$

is commutative, since $M_*(f \times Id) \circ M_*(Id \times g) = M_*(f \times g) = M_*(Id \times g) \circ M_*(f \times Id)$. The square

$$\begin{array}{ccc} M_Y(X) & \xrightarrow{M_{g,X}} & M_{Y'}(X) \\ {\scriptstyle (M_Y)^*(f)}\uparrow & & \uparrow{\scriptstyle (M_{Y'})^*(f)} \\ M_Y(X') & \xrightarrow[M_{g,X'}]{} & M_{Y'}(X') \end{array}$$

is also commutative, because the square

$$\begin{array}{ccc} X \times Y & \xrightarrow{X \times g} & X \times Y' \\ {\scriptstyle f \times Id}\downarrow & & \downarrow{\scriptstyle f \times Id} \\ X' \times Y & \xrightarrow[Id \times g]{} & X' \times Y' \end{array}$$

is cartesian.

There is also a morphism M^g from $M_{Y'}$ to M_Y defined over X by

$$M^g_X = M^*(Id \times g) : \; M_{Y'}(X) \to M_Y(X)$$

In other words, I have defined a bifunctor from G-**set** to the category $Mack_R(G)$ of Mackey functors for G over R, which is equivalent to $\mu_R(G)$-**Mod**. I will check the conditions (M1) and (M2) for this bifunctor, proving that $Y \mapsto M_Y$ is a Mackey functor with values in the category of Mackey functors.