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Boju Jiang (Ed.)

Topological Fixed Point Theory and Applications

Proceedings, Tianjin 1988

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Proceedings of a Conference
held at the Nankai Institute of Mathematics
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FOREWORD

The Conference on Topological Fixed Point Theory and Applications was held at Nankai Institute of Mathematics, Tianjin, China, during April 5–8, 1988. Its aim was to bring together topologists working in various areas of fixed point theory as well as analysts interested in the applications, to discuss recent progress and current trends in research.

This conference was sponsored by the Nankai Institute of Mathematics and supported by a grant from the Chinese Ministry of Education.

We would like to thank all the participants for their enthusiasm. We gratefully acknowledge the assistance of many people who helped make the conference a success. In particular, Prof. Zixin Hou who was one of the organizers of this meeting and Mr. Xiuhua Dong who looked after the logistical details.

Boju Jiang

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BIFURCATION THEORY FOR METRIC PARAMETER SPACES

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Introduction

In [6] we introduced an index BI for bifurcation of fixed points. $BI(f)$ is defined for maps $f : \mathcal{O} \to X$ where

- X is a Banach space, $\mathcal{O} \subset \mathbb{R}^k \times X$ is open,
- f is completely continuous,
- $f(p, 0) = 0$ for all $p \in \mathcal{T} = \mathcal{T}(f) := \mathcal{O} \cap (\mathbb{R}^k \times \{0\})$,
- the set $\mathcal{B} = \mathcal{B}(f) \subset \mathcal{T}$ of bifurcation points is compact. Here $\mathcal{B} = \mathcal{T} \cap clos(\mathcal{F})$, $\mathcal{F} = \mathcal{F}(f) := \{(p, x) \in \mathcal{O}: f(p, x) = x, x \neq 0\}$.

$BI(f)$ is an element of π^s_{k-1} , the stable (k-1)-stem. It has properties analogous to the fixed point index. In this note we shall first show how to define BI if the parameter space is not $\mathbb{R}^k$ but any metric space P. For simplicity we consider only the finite-dimensional case $X = \mathbb{R}^n$. The bifurcation index will then lie in $\widetilde{\omega}^1(P)$ where $\widetilde{\omega}^*$ is reduced stable cohomotopy theory. If $P = S^k = \mathbb{R}^k \cup \{\infty\}$ then $\widetilde{\omega}^1(S^k) \cong \pi^s_{k-1}$ and the new definition reduces to the old one. In addition to the properties of BI proved in [6] we show that BI is natural in P and is commutative. This last property allows to replace X by an ENR or even $P \times X$ by an ENR_P $E \to P$ where the trivial fixed points are given by a section $P \to E$.

Obviously BI has a lot in common with the fixed point index. This similarity is not purely formal as will be shown by a formula relating BI and Dold's fixed point index for fibre-preserving maps (cf. [9]). This formula gives a new way of calculating BI.

1. Construction and properties of BI

We first define $BI(f)$ in the following situation. P is a metric space, $E := P \times \mathbb{R}^n$. We consider a continuous map $f : \mathcal{O} \to E$ over P, i. e. $f(p, x) = (p, f_2(p, x))$; $\mathcal{O} \subset E$ open. Assume $f(p, 0) = (p, 0)$ for all $p \in \mathcal{T} := \mathcal{O} \cap P$. (We identify P and $P \times \{0\}$.)

This paper is in final form. No version of it will be submitted for publication elsewhere.

We are interested in the set

$$\mathcal{F} = \mathcal{F}(f) := \{(p,x) \in \mathcal{O} : f(p,x) = (p,x), x \neq 0\}$$

of nontrivial fixed points and the set

$$\mathcal{B} = \mathcal{B}(f) := \mathcal{T} \cap clos(\mathcal{F})$$

of bifurcation points. Of course, $\mathcal{B}$ is a closed subset of $\mathcal{T}$.

If $\mathcal{B}$ is a closed subset of P we define $BI(f)$ as follows. Let d denote the metric of E given by the sum of the metrics of P and $\mathbb{R}^n$. Then

$$\begin{aligned}
&\varphi : \mathcal{T} \to \mathbb{R}_0^+,\ \varphi(p) := min\Big\{1, \frac{1}{2} \cdot d(\{p\}, clos(\mathcal{F}) \cup \partial\mathcal{O})\Big\},\\
(1.1)\qquad &A_0 := \{(p,x) \in \mathcal{O} :\ p \in \mathcal{T}, \|x\| = \varphi(p)\} \subset \mathcal{O} \setminus \mathcal{F},\\
&A_1 := \{(p,x) \in A_0 :\ p \in \mathcal{T} \setminus \mathcal{B}\} \subset \mathcal{O} \setminus (\mathcal{T} \cup \mathcal{F}),\\
&\rho : (\mathcal{T}, \mathcal{T} \setminus \mathcal{B}) \times S^{n-1} \to (A_0, A_1),\ (p,x) \mapsto (p, \varphi(p) \cdot x).
\end{aligned}$$

(1.2) Definition. $BI(f) \in \widetilde{\omega}^1(P)$ is the image of $1 \in \omega^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$ under the following sequence of homomorphisms ($\iota : P \times \mathbb{R}^n \to \mathbb{R}^n$ the projection).

$$\omega^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \xrightarrow{(\iota - f_2)^*} \omega^n(A_0, A_1) \xrightarrow{\rho^*} \omega^n((\mathcal{T}, \mathcal{T} \setminus \mathcal{B}) \times S^{n-1})$$

$$\xrightarrow{/\Sigma_{n-1}} \omega^1(\mathcal{T}, \mathcal{T} \setminus \mathcal{B}) \xleftarrow{i^*} \omega^1(P, P \setminus \mathcal{B}) \longrightarrow \widetilde{\omega}^1(P)$$

Here $\Sigma_{n-1} \in \omega_{n-1}(S^{n-1})$ is induced by the identity on S^{n-1}.

$$/ : \omega^n((\mathcal{T}, \mathcal{T} \setminus \mathcal{B}) \times S^{n-1}) \otimes \omega_{n-1}(S^{n-1}) \longrightarrow \omega^1(\mathcal{T}, \mathcal{T} \setminus \mathcal{B})$$

is the slant product (cf. Switzer [15], Chapter 13). i^* is an excision isomorphism since $\mathcal{B}$ is closed and $\mathcal{T}$ is open in P.

$BI(f)$ can be thought of as a stable map $P \wedge S^{n-1} \longrightarrow S^n$. If $P = S^k$ this map can be considered both as an element of $\widetilde{\omega}^1(S^k)$ and of π^s_{k-1}. In [6] we chose the latter version. Also observe that in the case $\mathcal{O} \subset \mathbb{R}^k \times \mathbb{R}^n$ $\mathcal{B}$ is closed in $S^k = \mathbb{R}^k \cup \{\infty\}$ iff it is a compact subset of $\mathbb{R}^k$.

(1.3) BI enjoys a number of properties.
Let E^+ denote the space $P \times (\mathbb{R}^n \cup \{\infty\})/P \times \{\infty\}$ and $C^+\mathcal{O} := E^+ \setminus \mathcal{O}$.

Existence. *If $BI(f) \neq 0$ then $\mathcal{B}$ and $C^+\mathcal{O}$ cannot be separated in $\mathcal{B} \cup \mathcal{F} \cup C^+\mathcal{O}$. This means that $\mathcal{B} \cup \mathcal{F} \cup C^+\mathcal{O}$ cannot be written as the disjoint union $U \sqcup V$ of open subsets U and V with $\mathcal{B} \subset U$ and $C^+\mathcal{O} \subset V$. If P is compact this implies the existence of a connected set $S \subset \mathcal{F}$ with $\mathcal{B} \cap clos(S) \neq \emptyset \neq C^+\mathcal{O} \cap clos(S)$ (cf. [3], Proposition 5). $C^+\mathcal{O} \cap clos(S) \neq \emptyset$ is equivalent to the statement that the projection $\mathcal{B} \cap S \subset E \longrightarrow P$ is not proper. In particular, f is not compactly fixed (cf. [9]). Thus $BI(f) \neq 0$ implies that the fixed point index $I(f)$ of f in the sense of [9] is not defined. Even more, there does not exist an open neighbourhood V of $\mathcal{B} \in \mathcal{O}$ such that $I(f|V)$ is defined.*

Localisation. *If* $\mathcal{O}' \subset \mathcal{O}$ *is open and* $\mathcal{B} \subset \mathcal{O}'$ *then* $BI(f) = BI(f|\mathcal{O}')$.

Additivity. *If* $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, $\mathcal{O}_1, \mathcal{O}_2$ *open and* $\mathcal{B} \cap \partial(\mathcal{O}_1 \cap \mathcal{O}_2) = \emptyset$ *then* $BI(f|\mathcal{O}_1)$, $BI(f|\mathcal{O}_2)$ *and* $BI(f|\mathcal{O}_1 \cap \mathcal{O}_2)$ *are defined and*

$$BI(f) = BI(f|\mathcal{O}_1) + BI(f|\mathcal{O}_2) - BI(f|\mathcal{O}_1 \cap \mathcal{O}_2).$$

Homotopy invariance. *If* $\mathcal{P} \subset F = [0,1] \times P \times \mathbb{R}^n$ *is open and* $h: \mathcal{P} \to F$ *is a map over* $[0,1] \times P$ *with* $h(t,p,0) = (t,p,0)$ *and* $\mathcal{B}(h)$ *closed in* $[0,1] \times P$ *then* $BI(h_t)$ *is independent of* $t \in [0,1]$. *Here* $h_t(p,x) = h(t,p,x)$ *is the part of* h *over* t.

Stability. *If* $0: \mathbb{R} \to \mathbb{R}$ *is the constant map then* $BI(f \times 0) = BI(f)$. *Of course,* $(f \times 0)(p,x,y) = (f(p,x),0)$.

Naturality. *Let* Q *be another metric space and* $\psi: Q \to P$ *continuous. Let* $\psi^* f: \psi^* \mathcal{O} \to \psi^* E$ *denote the pullback of* $f: \mathcal{O} \to E$ *(cf. Dold [9], (2.8)). Then*

$$BI(\psi^* f) = \psi^*(BI(f)) \in \tilde{\omega}^1(Q).$$

Commutativity. *Let* $\mathcal{O} \subset E = P \times \mathbb{R}^n$, $\mathcal{P} \subset F = P \times \mathbb{R}^m$ *be open and* $f: \mathcal{O} \to F$, $g: \mathcal{P} \to E$ *be continuous maps over* P *such that* $f(p,0) = (p,0)$, $g(p,0) = (p,0)$. *Then* $\mathcal{B}(f \circ g) = \mathcal{B}(g \circ f) =: \mathcal{B}$ *and if this set is closed in* P $BI(f \circ g) = BI(g \circ f)$.

Only the last two properties have not been proved in [6]. The proofs in [6], though only for the special case $\mathcal{O} \subset \mathbb{R}^k \times \mathbb{R}^n \subset S^k \times \mathbb{R}^n$ and working with stable homotopy ω_* instead of ω^*, can easily be adopted to the situation here. (The statement of the homotopy invariance in [6] is not correct. There it was only assumed that $BI(h_t)$ is defined for all $t \in [0,1]$. This does not imply $BI(h_0) = BI(h_1)$ as can be seen from the example

$$h_t: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R},\ h_t(p,x) = (p, tpx).$$

$BI(h_0) = 0$, $BI(h_t) = 1$ for $t \neq 0$.)

To prove **Naturality** observe that $\mathcal{B}(\psi^* f) = \psi^{-1}(\mathcal{B}(f))$, hence $BI(\psi^* f)$ is defined if $BI(f)$ is defined. Furthermore ψ induces homomorphisms connecting the corresponding terms in the defining sequences of $BI(f)$ and $BI(\psi^* f)$ making everything commute.

The proof of the **Commutativity** property is as in the case of the fixed point index (cf. [11], VII.5.9). For the convenience of the reader we give the necessary deformations and check the bifurcation sets.

$$\alpha: I \times \mathcal{O} \times \mathbb{R}^m \to I \times E \times \mathbb{R}^m = I \times E \times_P F,$$
$$(t,p,x,y) \mapsto (t,p,g_2 \circ f(p,x), t f_2(p,x)),$$

is a homotopy between $\alpha_0 = (g \circ f) \times 0$ and α_1. The fixed point set of α is

$$Fix(\alpha) = \{(t,p,x,y): g \circ f(p,x) = (p,x),\ y = t f_2(p,x)\}.$$

Thus $\mathcal{B}(\alpha) = I \times \mathcal{B}(g \circ f) \subset I \times P$ is closed.

$$\beta: I \times \mathcal{O} \times_P \mathcal{P} \to I \times E \times_P F,$$
$$(t,p,x,y) \mapsto (t,p,(1-t)g_2 \circ f(p,x) + t g_2(p,y), f_2(p,x)),$$

is a homotopy between $\beta_0 = \alpha_1 \mid I \times \mathcal{O} \times_P \mathcal{O}'$ and β_1. The fixed point set of β is

$$\begin{aligned} Fix(\beta) &= \{(t,p,x,y): f(p,x) = (p,y),\ g(p,y) = (p,x)\} \\ &= \{(t,p,x,y): (p,x) \in Fix\ g \circ f,\ y = f_2(p,x)\}. \end{aligned}$$

Again $\mathcal{B}(\beta) = I \times \mathcal{B}(g \circ f) \subset I \times P$ is closed. Applying **Stability**, **Homotopy invariance** and **Localisation** yields $BI(g \circ f) = BI(\beta_1)$. Symmetric deformations yield $BI(f \circ g) = BI(\beta_1)$.

We use **Commutativity** to define BI in more general situations. We remind the reader that an ENR_P is a space over P, $\pi: E \to P$, such that there exists a euclidean space $\mathbb{R}^n$, an open subset U of $P \times \mathbb{R}^n$ and (continuous) maps $E \xrightarrow{i} U \xrightarrow{r} E$ over P with $r \circ i = id_E$.

Let E be an ENR_P, $\mathcal{O} \subset E$ open and $f: \mathcal{O} \to E$ a map over P. Suppose there exists a section $\sigma: P \to E$ such that $f \circ \sigma = \sigma$ on $\mathcal{T} = \sigma^{-1}(\mathcal{O})$. $\mathcal{T}$ is the set of trivial fixed points of f (we identify P and $\sigma(P)$). As usual we set $\mathcal{F} = Fix(f) \setminus \mathcal{T}$ and $\mathcal{B} = \mathcal{T} \cap clos(\mathcal{F})$. If $\mathcal{B}$ is a closed subset of P we define $BI(f)$ as in the case of the fixed point index.

(1.4) Definition. Choose $E \xrightarrow{i} U \xrightarrow{r} E$ as in the definition of an ENR_P. We may assume $i \circ \sigma = i_0: P \to P \times \{0\} \subset U$. This can always be achieved by a translation in each fibre $\{p\} \times \mathbb{R}^n$. Then

$$BI(f) := BI(i \circ f \circ r | r^{-1}(\mathcal{O}))$$

is the bifurcation index of f. It is independent of the choice of i and r (**Commutativity**).

The properties of $BI(f)$ continue to hold in the general situation. The space E^+ needed to formulate the **Existence** property is $E^+ := E \cup \{\infty\}$. A neighbourhood basis of ∞ consists of the complements $E^+ \setminus A$ of subsets $A \subset E$ such that $\pi|A$ is proper. As a consequence of the **Existence** property $BI(f)$ has to be 0 if $\mathcal{O} = E$ and $\pi: E \to P$ is proper.

2. Relation to the fixed point index

As in Part 1 we consider an ENR_P $\pi: E \to P$ with a section $\sigma: P \to E$ and a map $f: \mathcal{O} \to E$ over P, $\mathcal{O} \subset E$ open, with $f \circ \sigma = \sigma$ on $\mathcal{T} = \sigma^{-1}(\mathcal{O})$.

If the parameter space is one-dimensional, $P = \mathbb{R} \cup \{\infty\}$, and if $\mathcal{B}$ is contained in an interval, $\mathcal{B} \subset]p_0, p_1[\subset [p_0, p_1] \subset \mathcal{T}$, then $BI(f)$ is simply the difference of the local fixed point indices $I(f_{p_1}, 0) - I(f_{p_0}, 0)$. Here $f_p: \mathcal{O}_p \to E_p$ is the part of f over p. For $p \notin \mathcal{B}$ 0 is an isolated fixed point, hence $I(f_p, 0)$ is well defined. We shall generalize this formula to the multiparameter situation considered here.

(2.1) Definition. If there is no bifurcation from $\mathcal{T}$, i. e. $\mathcal{B} = \emptyset$, then the local fixed point index of f at $\mathcal{T}$ is defined as

$$I(f, \mathcal{T}) := I(f|W) \in \omega^0(P)$$

where $W \subset \mathcal{O}$ is an open neighbourhood of $\mathcal{T}$ in $\mathcal{O}$ with $W \cap Fix(f) = \mathcal{T}$ and $I(f|W)$ is the fixed point index of $f|W$ in the sense of Dold [9] or [10].

(2.2) Theorem. *If $\mathcal{B} \subset \mathcal{T}$ is closed in P consider open neighbourhoods U, V of $\mathcal{B}$ in P such that $clos(V) \subset U \subset \mathcal{T}$. (Such neighbourhoods exist since P is normal as a metric space.) Then*

$$BI(f) = -i^* \circ (j^*)^{-1} \circ \delta\big(\mathrm{I}(f|\pi^{-1}(U \setminus V), U \setminus V)\big)$$

where

$$\omega^0(U \setminus V) \xrightarrow{\ \delta\ } \omega^1(U, U \setminus V) \xleftarrow{\ j^*\ } \omega^1(P, P \setminus V) \xrightarrow{\ i^*\ } \widetilde{\omega}^1(P).$$

j^* is an excision isomorphism.

PROOF: Since both I and BI are commutative we may assume that $E = P \times \mathbb{R}^n$ and $\mathcal{T} = \mathcal{O} \cap P$.

Claim: $BI(f)$ is the image of $1 \in \omega^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ under the following sequence of homomorphisms.

$$\omega^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \xrightarrow{(\iota - f_2)^*} \omega^n(B_0, B_1) \xrightarrow{\ \rho^*\ } \omega^n\big((U, U \setminus V) \times (D^n \setminus \{0\})\big)$$

$$\xrightarrow{/\Sigma_{n-1}} \omega^1(U, U \setminus V) \longrightarrow \widetilde{\omega}^1(P)$$

Here

$$D^n := \{x \in \mathbb{R}^n \colon \|x\| \le 1\},$$
$$B_0 := \{(p, x) \in \mathcal{O} \colon p \in U,\ 0 < \|x\| \le \varphi(p)\} \subset \mathcal{O} \setminus \mathcal{F},$$
$$B_1 := \{(p, x) \in B_0 \colon p \in U \setminus V\} \subset \mathcal{O} \setminus (\mathcal{T} \cup \mathcal{F}).$$

For later use we set

$$B_2 := \{(p, x) \in \mathcal{O} \colon p \in U \setminus V,\ \|x\| \le \varphi(p)\}.$$

$\varphi \colon \mathcal{T} \to \mathbb{R}_0^+$ and $\rho \colon (U, U \setminus V) \times (D^n \setminus \{0\}) \longrightarrow (B_0, B_1)$ are defined as in **(1.1)**. For notational convenience we do not distinguish $\omega_{n-1}(D^n \setminus \{0\})$ and $\omega_{n-1}(S^{n-1})$. Thus $\Sigma_{n-1} \in \omega_{n-1}(D^n \setminus \{0\})$. The last map in the above sequence is $i^* \circ (j^*)^{-1}$.

The proof of the above claim is an easy excision argument. The theorem is now a consequence of the commutativity of the following diagram. (We write $\dot{D}^n$ for $D^n \setminus \{0\}$.)

$$\begin{array}{ccccc}
\omega^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xleftarrow{\ \delta\ } & \omega^{n-1}(\mathbb{R}^n \setminus \{0\}) & \xrightarrow{\ \delta\ } & \omega^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\
\downarrow (\iota - f_2)^* & & \downarrow (\iota - f_2)^* & & \downarrow (\iota - f_2)^* \\
\omega^n(B_2, B_1) & \xleftarrow{\ \delta\ } & \omega^{n-1}(B_1) & \xrightarrow{\ \delta\ } & \omega^n(B_0, B_1) \\
\downarrow \rho^* & & \downarrow \rho^* & & \downarrow \rho^* \\
\omega^n\big((U \setminus V) \times (D^n, \dot{D}^n)\big) & \xleftarrow{\ \delta\ } & \omega^{n-1}\big((U \setminus V) \times \dot{D}^n\big) & \xrightarrow{\ \delta\ } & \omega^n\big((U, U \setminus V) \times \dot{D}^n\big)
\end{array}$$

$$\begin{array}{ccccc} \Big\downarrow /\Delta_n & & \Big\downarrow /\Sigma_{n-1} & & \Big\downarrow /\Sigma_{n-1} \\ \omega^0(U \setminus V) & \xrightarrow{-1} & \omega^0(U \setminus V) & \xrightarrow{\delta} & \omega^1(U, U \setminus V) \end{array}$$

Here $\Delta_n \in \omega_n(D^n, D^n \setminus \{0\})$ corresponds to Σ_{n-1}, i. e. $\Sigma_{n-1} = \partial\Delta_n$. The last two squares commute according to Switzer [15], 13.56(v), (vi). Now

$$\mathrm{I}(f|\pi^{-1}(U \setminus V), U \setminus V) = (\rho^* \circ (\iota - f_2)^*(1))/\Delta_n$$

by definition (cf. [9], (2.3), (2.14)). Together with the first claim the theorem follows.

(2.3) Corollary. *Let $f: \mathbb{R} \times E \longrightarrow \mathbb{R} \times E \subset S^1 \times E$ be a map over $S^1 \times P$ and assume that the set $\mathcal{B}$ of bifurcation points is contained in $]-R, R[\times P$ and is closed in $S^1 \times P$. Let $f_t: E \to E$ be the part of f over $t \in \mathbb{R}$. Then*

$$I_+ := I(f_t, P) \in \omega^0(P),\ t \geq R,$$

and

$$I_- := I(f_t, P),\ t \leq -R,$$

are well defined. If $I_+ \neq I_-$ then $BI(f) \neq 0 \in \tilde{\omega}^1(S^1 \times P)$.

PROOF: Apply Theorem **(2.2)** to $U = \mathbb{R} \times P$, $V =]-R, R[\times P$ and observe that the kernel of

$$\omega^0(U \setminus V) \xrightarrow{\delta} \omega^1(U, U \setminus V) \xleftarrow{\cong} \omega^1(S^1 \times P, (S^1 \times P) \setminus V)$$

is the diagonal in $\omega^0(U \setminus V) \cong \omega^0(P) \oplus \omega^0(P)$. Thus $\delta(\mathrm{I}_+, \mathrm{I}_-) \neq 0$. Furthermore, the restriction homomorphism $\omega^1(S^1 \times P, S^1 \times P \setminus V) \longrightarrow \tilde{\omega}^1(S^1 \times P)$ is injective since $\omega^1(S^1 \times P, S^1 \times P \setminus V) \cong \omega^1(S^1 \times P, \{\infty\} \times P)$ and $\{\infty\} \times P$ is a retract of $S^1 \times P$ which implies that $\omega^0(S^1 \times P) \longrightarrow \omega^0(\{\infty\} \times P)$ is surjective.

(2.4) Example. If $P = S^1$ consider

$$f_+: S^1 \times \mathbb{C} \longrightarrow S^1 \times \mathbb{C},\ (p, z) \mapsto (p, p - p \cdot z),$$

and

$$f_-: S^1 \times \mathbb{C} \longrightarrow S^1 \times \mathbb{C},\ (p, z) \mapsto (p, 0).$$

Then $I(f_+) = 1$ and $I(f_-) = 0$ in $\tilde{\omega}^0(S^1) \cong \mathbb{Z}/2\mathbb{Z}$. Only the first assertion requires some work since we cannot use the Lefschetz-Hopf formula (6.18) of [10]. Instead we observe that the map

$$\begin{aligned} S^1 &\to SO(2), \\ p &\mapsto (z \mapsto p \cdot z = (\iota - f_{2+})(p, z)) \end{aligned}$$

induces a generator of $\pi_1(SO(2)) \cong \mathbb{Z}$ and hence the nonzero element

$$\gamma_{f_+} = 1 \in \pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z},\ n \geq 3.$$

Now the J-homomorphism

$$J: \pi_1(SO(n)) \longrightarrow \pi_{n+1}(S^n) \cong \tilde{\omega}^0(S^1)$$

is an isomorphism (cf. Adams [1]) and $J(\gamma_{f_+})$ is the fixed point index of f_+. This can be seen by using the Hopf construction.

Now consider a map $f: \mathbb{R} \times S^1 \times \mathbb{C} \longrightarrow \mathbb{R} \times S^1 \times \mathbb{C}$ over $\mathbb{R} \times S^1$ such that $f(t,p,0) = (t,p,0)$. If $f_{\pm t} \simeq f_{\pm}$ for $|t|$ big (controlling the fixed points during the deformation) then **Corollary (2.3)** gives $BI(f) \neq 0 \in \tilde{\omega}^1(S^1 \times S^1)$. Thus there must exist a global branch of fixed points bifurcating from $\mathbb{R} \times S^1$.

In all previous computations of the bifurcation index it was either assumed that there is just one bifurcation point in a disc in the parameter space (compare [2], [8], [12] and the references therein) or a more general situation has been reduced to that case using the properties of the bifurcation index (cf. [6]). **Theorem (2.2)** provides a different way of calculating BI.

3. Problems, Remarks

(3.1) If the domain $\mathcal{O}$ of $f: \mathcal{O} \to E$ contains the whole section P then the defining sequence of homomorphisms in **Definition (1.2)** factors through $\tilde{\omega}^n(\mathbb{R}^n)$. Consequently, $BI(f) = 0$. In that case one can use a refined version as follows. Suppose one knows that the set of bifurcation points is contained in a subset Q of P. Then one can define $BI(f,Q) \in \omega^1(P, P \setminus Q)$. For example, if P is a compact manifold with boundary ∂P the bifurcation points should be restricted to $P \setminus \partial P$, thus $\mathrm{BI}(f) \in \omega^1(P, \partial P)$. Another variation for locally compact P is to replace P by its one point compactification.

(3.2) In [4] and [12] (possibly infinite-dimensional) Banach spaces P occur as parameter spaces. There one assumes the existence of a disc D^l in a finite-dimensional subspace of P with $\mathcal{B} \cap D^l = \{0\}$. Then one considers the bifurcation index of f restricted to the part over D^l. If it is different from 0 the bifurcating branch of solutions has dimension at least $dim\ P\ -\ l$. This type of result has also been proved in [7] where P is a finite-dimensional manifold. In addition, $S^{l-1} = \partial D^l$ and $\mathcal{B}$ are linked, i. e. the inclusion $S^{l-1} \hookrightarrow P \setminus \mathcal{B}$ is topologically nontrivial. It should be easy to replace D^l by more general subspaces Λ of P and look at $\partial\Lambda \hookrightarrow P \setminus \mathcal{B}$. Nonlinear parameter spaces have also been considered in [14] (finite CW-complexes, in particular compact differentiable manifolds) and — in a continuation setting — in [5] (metric spaces).

(3.3) For applications it is necessary to replace $\mathbb{R}^n$ by a normed linear space and ENR_P by ANR_P. One has to approximate f, so one needs some compactness conditions. In [13] Nussbaum uses the fixed point index to prove bifurcation into an ANR X. He considers maps $f: J \times X \to X$ which are strict set contractions on a neighbourhood of the fixed point set. J is an open interval of reals. Of particular importance is the case where X is a cone in a Banach space.

(3.4) Another problem (Dold) is to define $BI(f)$ for maps $f: M \to M$ where M is a manifold and $f|N = id|N$ on a submanifold N. This situation is different from the one considered in this paper since M is not a space over N and f not a map over N.

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A Fixed Point Index Approach to some Differential Equations

Dedicated to Professor Andrzej Granas
on the Occasion of his 60th Birthday

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The aim of this paper is to show that, using the fixed point index method for compact maps as a tool, many types of differential equations with the right side depending on the derivative can be reduced very easily to differential inclusions with right sides not depending on derivative. It is shown in fact, that the fixed point theory for condensing-type maps, which was used to obtain existence results or topological characterizations of the set of solutions in such situations (cf. [14]) is not needed. We apply our method to the following types of differential equations only, but some other applications are also possible:

(i) ordinary differential equations of first or higher order (e.g., the satellite equation, see [15]),

(ii) hyperbolic differential equations,

(iii) elliptic differential equations.

We shall mention these in section 3. General statements needed for applications will be given in section 2. This paper is a continuation of [2] and gives a generalization of respective results in [1,3,13 and 14]. Finally, note that the class of maps for which we are able to obtain an existence result is quite rich (see Theorem (2.7)).

1. Preliminaries. In this paper all topological spaces are assumed to be metric. Let X, Y be two spaces and assume that for every $x \in X$ a nonempty subset $\varphi(x)$ of Y is given; in this case we say that $\varphi : X \to Y$ is a multivalued map. In what follows the symbols φ, ψ, χ are reserved for multivalued maps; singlevalued maps we shall denote by $f, g, h, \cdots$.

A multivalued map $\varphi : X \to Y$ is called ***upper semi continuous*** (***lower semi continuous***) if for each open $U \subset Y$ the set $\{x \in X; \varphi(x) \subset U\}$ ($\{x \in X; \varphi(x) \cap U \neq \emptyset\}$) is an open subset of X. An upper semi continuous (lower semi continuous map) $\varphi : X \to Y$ we shall write shortly as u.s.c. (l.s.c.). A multivalued map $\varphi : X \to Y$ is called ***continuous*** if it is both u.s.c. and l.s.c. It is evident that for $\varphi = f$ a singlevalued map the above three notions coincide. An u.s.c. map $\varphi : X \to Y$ is called ***compact*** provided the closure $\mathrm{cl}(\varphi(X))$ of $\varphi(X) = \bigcup\{\varphi(x); x \in X\}$ in Y is a compact set.

Let $\varphi, \psi : X \to Y$ be two maps. We shall say that ψ is a selector of φ (written $\psi \subset \varphi$), if for each $x \in X$ we have $\psi(x) \subset \varphi(x)$. For a map $\varphi : X \to X$ by $\mathrm{Fix}(\varphi)$ we shall denote the set of all fixed points of φ, i.e.,

$$\mathrm{Fix}(\varphi) = \{x \in X; x \in \varphi(x)\}.$$

We recommend [1,7,10] for details concerning multivalued maps. Let A be a subset of X, then by δA we shall denote the boundary of A in X, by cl$A$ the closure of A in X and by dim$A$ the covering dimension of A (cf. [8]). It is well known (see [9]) that for a compact set A we have: $\dim A = 0$ if and only if A has an open-closed basis.

It immediately implies the following:

(1.1) Proposition. Let A be a compact subset of X such that dim$A = 0$. Then for every $x \in A$ and for every open neighbourhood U of x in X there exists an open neighbourhood $V \subseteq U$ of x in X such that $\delta V \cap A = \emptyset$.

2. General statements. In this section we shall present the topological material needed in our applications to the theory of differential equations. Let X be a metric absolute neighbourhood retract (written $X \in$ ANR) and let $g : X \to X$ be a compact map. Assume further that U is an open subset of X such that $\delta U \cap \text{Fix}(g) = \emptyset$, then (following [11]) we shall denote by $i(g, U)$ the fixed point index of g with respect to U.

We shall start with the following:

(2.1) Proposition. Let $X \in$ ANR and let $g : X \to X$ be a compact map. Assume further that the following two conditions are satisfied:

(2.1.1) $\dim \text{Fix}(g) = 0$,

(2.1.2) there exists an open subset $U \subset X$ such that $\delta U \cap \text{Fix}(g) = \emptyset$ and $i(g, U) \neq 0$.

Then there exists a point $z \in \text{Fix}(g)$ for which we have:

(2.1.3) for every open neighbourhood U_z of z in X there exists an open neighbourhood V_z of z in X such that: $V_z \subset U_z$, $\delta V_z \cap \text{Fix}(g) = \emptyset$ and $i(g, V_z) \neq 0$.

Proof. Let $\Gamma = \{A \subset \text{Fix}(g) \cap U; A$ is compact nonempty and for every open neighbourhood W of A in X there is an open neighbourhood V of A in X which satisfies the following three conditions: $V \subset W$, $\delta V \cap \text{Fix}(g) = \emptyset$ and $i(g, V) \neq 0\}$.
It follows from (2.1.2) that Γ is a nonempty family. We consider in Γ the partial order given by the inclusion between subsets of X. We are going to apply the famous Kuratowski-Zorn Lemma (cf. [6]). To do it let us assume that $\{A_i\}_{i \in I}$ is a chain in Γ. We put $A_0 = \bigcap\{A_i; i \in I\}$. To prove that $A_0 \in \Gamma$ assume that W is an open neighbourhood of A_0 in X. We claim that there is $i \in I$ such that $A_i \subset W$. Indeed, if we assume on the contrary, then we get a family $B_i = (X \setminus W) \cap A_i$, $i \in I$, of compact nonempty sets which has nonempty compact intersection B_0. Then $B_0 \subset X \setminus W$ and $B_0 \subset A_0$ so we obtain a contradiction and hence $A_0 \in \Gamma$. Consequently, in view or Kuratowski-Zorn Lemma, we get a minimal element A_* in Γ. We claim that A_* is a singleton. Let $z \in A_*$. It is sufficient to prove that $\{z\} \in \Gamma$. Since $A_* \in \Gamma$ we obtain an open neighbourhood U_* of A_* in X with the following properties: $U_* \subset U$, $\delta U_* \cap \text{Fix}(g) = \emptyset$ and $i(g, U_*) \neq 0$. Let W be an arbitrary open neighbourhood of z in X. Using (1.1) we can choose an open neighbourhood U_z of z in $U_* \cap W$ such that $\text{Fix}(g) \cap \delta U_z = \emptyset$. Since A_* is a minimal element of Γ the compact set $A_* \setminus U_z$ is not in Γ and hence there exists an open set $V \subset U_*$ such that $(A_* \setminus U_z) \subset V \subset U_*$, $\text{Fix}(g) \cap \delta V = \emptyset$, $V \cap U_z = \emptyset$, $i(g, V) = 0$ and $i(g, U_*) = i(g, V \cup U_z)$. Now from the additivity property of the fixed point index we have

$$i(g, U_*) = i(g, U_z) + i(g, V) \neq 0$$

and consequently $i(g, U_z) \neq 0$. It implies that $\{z\} \in \Gamma$ and the proof is complete.

Now, we are going to consider a more general situation. Namely, let Y be a locally arcwise connected space, $X \in$ ANR and let $f : Y \times X \to X$ be a compact map. In what follows we shall assume that f satisfies the following condition:

(2.2) $\forall y \in Y \exists U_y : U_y$ is open in X and $i(f_y, U_y) \neq 0$,

where $f_y : X \to X$ is given by the formula $f_y(x) = f(y, x)$ for every $x \in X$. Observe that in particular, if X is an absolute retract, then (2.2) holds automatically. We associate with a map $f : Y \times X \to X$ satisfying the above conditions the following multivalued map:

$$\varphi_f : Y \to X, \quad \varphi_f(y) = \mathrm{Fix}(f_y).$$

Then from (2.2) follows that φ_f is well defined. Moreover, we get:

(2.3) Proposition. Under all of the above assumptions the map $\varphi_f : Y \to X$ is u.s.c.

Let us remark that, in general, φ_f is not a l.s.c. map. Below we would like to formulate a sufficient condition which guarantees that φ_f has a l.s.c. selector. To get it we shall add one more assumption. Namely, we assume that f satisfies the following condition:

(2.4) $\forall y \in Y : \dim \mathrm{Fix}(f_y) = 0$.

Now, in view of (2.2) and (2.4), we are able to define the map $\psi_f : Y \to X$ by putting $\psi_f(y) = \mathrm{cl}\{z \in \mathrm{Fix}(f_y)$; for z condition (2.1.3) is satisfied$\}$, for every $y \in Y$.

We prove the following:

(2.5) Theorem. Under all of the above assumptions we have:

(2.5.1) ψ_f is a selector of φ_f,

(2.5.2) ψ_f is a l.s.c. map.

Proof. Since (2.5.1) follows immediately from the definition we shall prove (2.5.2). To do it we let:

$$\eta_f : Y \to X, \quad \eta_f(y) = \{z \in \mathrm{Fix}(f_y); \quad z \text{ satisfies (2.1.3)}\}.$$

For the proof it is sufficient to show that η_f is l.s.c. Let U be an open subset of X and let $y_0 \in Y$ be a point such that $\eta_f(y_0) \cap U \neq \emptyset$. Assume further that $x_0 \in \eta_f(y_0) \cap U$. Then there exists an open neighbourhood V of x_0 in X such that $V \subset U$ and $i(f_{y_0}, V) \neq 0$. Since φ_f is an u.s.c. map and Y is locally arcwise connected we can find an open arcwise connected W in Y such that $y_0 \in W$ and for every $y \in W$ we have:

$$(*) \qquad \mathrm{Fix}(f_y) \cap \delta V = \emptyset.$$

Let $y \in W$ and let $\sigma : [0,1] \to W$ be an arc joining y_0 with y, i.e., $\sigma(0) = y_0$ and $\sigma(1) = y$. We define a homotopy $h : [0,1] \times V \to X$ by putting: $h(t,x) = f(\sigma(t), x)$. Then it follows form (*) that h is a well defined homotopy joining f_{y_0} with f_y and hence we get: $i(f_{y_0}, V) = i(f_y, V) \neq 0$; so $\mathrm{Fix}(f_y) \cap V \neq \emptyset$ and our assertion follows from (2.1).

(2.6) Remark. Let us remark that the above results remain true for admissible multivalued maps (cf. [7] and [10]); proofs are completely analogous.

Observe that condition (2.4) is quite restrictive. Therefore it is interesting to characterize the topological structure of all mappings satisfying (2.4). We shall do it in the