Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

1454

F. Baldassarri S. Bosch
B. Dwork (Eds.)

p-adic Analysis

Proceedings, Trento 1989

Springer-Verlag

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Proceedings of the International Conference
held in Trento, Italy, May 29–June 2, 1989

Springer-Verlag
Berlin Heidelberg New York London
Paris Tokyo Hong Kong Barcelona

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Mathematics Subject Classification (1980): 12Jxx, 14Fxx, 12Hxx, 46PO5

ISBN 3-540-53477-6 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-53477-6 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210 – Printed on acid-free paper

INTRODUCTION

The present volume contains the Proceedings of the Congress on «p-adic Analysis» held at Trento from May 28 to June 3, 1989.

The idea of organizing a meeting on this subject in Italy was first promoted by Philippe Robba, whose visits to Italy were always welcomed by his Italian colleagues for both the warmth and the illumination which he brought with him. He died prematurely on October 12, 1988, leaving a profound sense of loss in the world of p-adic analysis. We believe we have expressed the feelings of that whole community by dedicating this Meeting to him. At the opening of the Conference, Elhanan Motzkin commemorated Robba's exceptional character in a touching reminiscence, that will appear in the Seminars of the Groupe d'Etude d'Analyse Ultramétrique, of which Robba was one of the founders.

The conference was organized by the Centro Internazionale per la Ricerca Matematica (CIRM), of Trento, and was also sponsored by the Dipartimento di Matematica Pura e Applicata of the University of Padova. We are grateful to both these institutions.

We wish to express our gratitude in particular to Mr. Augusto Micheletti for his indefatigable efforts on behalf of the conference.

F. Baldassarri, S. Bosch, B. Dwork

Philippe Robba

March 18, 1941 - October 12, 1988

CONTENTS

Work of Philippe Robba

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Among the subjects studied by Philippe Robba were:

1. domains of analyticity
2. p–adic Mittag–Leffler
3. index of differential operators
4. factorization of differential operators corresponding to radii of convergence and to order of logarithmic growth
5. effective estimates for logarithmic growth
6. weak frobenius (dimension one, precursor of work of Christol)
7. L–functions and exponential sums
8. application of p–adic methods to questions of irrationality and transcendence.

Robba's work was so involved with the p–adic theory of ordinary differential equations that it may be useful in an article devoted to his work to give a survey of the present status of this subject.

Let K be a field of characteristic zero, complete under a rank one valuation extending the ordinary p–adic valuation of $\mathbf{Q}$.

Let E be the completion of $K(X)$ under the gauss norm. Elements of E are admissible (resp: superadmissible) if they are analytic elements on the complement of a finite set of residue classes (resp: a finite set of disks of local radius strictly less than unity).

A p–adic Liouville number is an element $\alpha \in \mathbf{Z}_p$ (necessarily transcendental over $\mathbf{Q}$) such that either

$$\liminf_{m \to +\infty} |\alpha - m|^{1/m} < 1$$

or

$$\liminf_{m \to \infty} |\alpha + m|^{1/m} < 1.$$

These conditions are not equivalent and for the operator $L = x\frac{d}{dx} - \alpha$ the first condition gives difficulties at $x = 0$ while the second gives difficulties at $x = \infty$.

We recall the notion of a generic point t in a universal domain Ω which is algebraically closed and complete relative to a valuation extending that of K. We insist that $|t| = 1$ and the residue class of t be transcendental over the residue class field of K. The disks and annuli appearing in our theory involve subsets of Ω.

Let

$$\mathcal{A}_0 = \{\xi \in K[[x]] \mid \xi \text{ converges in } D(0,1^-)\}$$
$$\mathcal{B}_0 = \{\xi \in \mathcal{A}_0 \mid \xi \text{ is bounded on } D(0,1^-)\}$$
$$\mathcal{A}_t(r) = \{\xi \in \overline{K(t)}[[x-t]] \mid \xi \text{ converges in } D(0,r^-)\}$$
$$W_a^{r,\beta} = \{\xi = \Sigma A_j(x-a)^j \in \overline{K(a)}[[x-a]] \mid \operatorname{Sup}_j A_j r^j/(1+j)^\beta < \infty\}.$$

The theory started in 1937 with Lutz's solution of the Cauchy problem:

Let $f(x,\vec{y})$ be an element of $K[[X,Y_1,\ldots,Y_n]]^n$ converging on a polydisk in $n+1$ space. Then the equation

$$\frac{d\vec{y}}{dx} = f(x,\vec{y})$$
$$\vec{y}(0) = 0$$

has a unique solution in $(xK[[x]])^n$ converging on a non-trivial disk about the origin.

Lutz estimated the radius of convergence and applied it to the study of rational points on elliptic curves.

Our own interest in ${}_2F_1(\frac{1}{2},\frac{1}{2},1,x)$ dates to the late 1950's and involved the (long unpublished) calculation of Tate's constant (cf. Dwork 1987). Our interest in the general theory of linear equations goes back to our study of the variation of cohomology of hypersurfaces (Dwork 1964, 66). Clark's work on linear equations at a singular point appeared in 1966. It was here that the question of p-adic Liouville exponents was first discussed. Adolphson [1976a, 1976b] investigated symmetric powers of ${}_2F_1(\frac{1}{2},\frac{1}{2},1,x)$ and studied index in the early 1970's. Robba's work started in 1974.

We will restrict our attention to linear equations but I cannot refrain from mentioning the splendid result of Sibuya and Sperber [1981].

THEOREM. *Let $y_0 \in K[[X]]$ be a formal solution of a non-linear polynomial differential equation, $P(x,y,y',\ldots,y^{(n)}) = 0$, where P is a polynomial in $n+2$ variables with coefficients in K. Substituting $y = y_0 + u$ we obtain the tangent linear operator*

$$L(u) = \frac{\partial P}{\partial y}(x,\vec{y}_0)u + \frac{\partial P}{\partial y'}(x,\vec{y}_0)Du + \cdots + \frac{\partial P}{\partial y^{(n)}}(x,\vec{y}_0)D^n u,$$

defined over $K[[X]]$. If the exponents of L at $x=0$ are p-adically non-Liouville then y_0 has a nontrivial p-adic radius of convergence.

The theory of ordinary p-adic differential equations addresses such questions as:

1. What are the radii of formal local solutions?
2. How do solutions grow as the boundary of the circle of convergence is reached?
3. What are the filtrations of the solution spaces relative to the growth and radii of convergence?
4. Index.

I. Order of growth

The main result of Robba on this question does not directly refer to differential equations.

THEOREM. (Robba 1980b) Let $u_1, \dots u_n \in \mathcal{A}_0$ and let the wronskian

$$w = \begin{vmatrix} u_1 & \dots & u_n \\ u_1' & \dots & u_n' \\ u_1^{(n-1)} & \dots & u_n^{(n-1)} \end{vmatrix}$$

never vanish in $D(0,1^-)$. Then each element $v = \Sigma a_s x^s$ in the K space spanned by $u_1, \dots u_n$ satisfies the condition

$$|a_s| \leq \operatorname*{Sup}_{0 \leq i < n-1} |a_i| \cdot \{s, n-1\}$$

where

$$\{s, n-1\} = 1/Inf\,|z_1 z_2 \cdots z_{n-1}| \quad (\leq s^{n-1}),$$

the inf being over all $1 \leq z_1 < z_2 < \cdots < z_{n-1} \leq s$.

This type of logarithmic estimate first appeared in the study of eigenvectors in our dual theory [Dwork 1964].

This type of estimate is a natural consequence of a strong Frobenius structure [Dwork, 1969]. For example if $\vec{y} = \Sigma \vec{a}_s x^s \in K[[X]]^n$ and if for some $\lambda \in K^\times$, we have $A\vec{y}(x^p) = \lambda\vec{y}(x)$ where $A \in \mathcal{M}_n(B_0)$ and is bounded by unity on $D(0,1^-)$ then $\vec{y}$ must converge in $D(0,1^-)$ and for $s \geq 1$

$$|\vec{a}_s| \leq |\vec{a}_0||\lambda|^{-\log s/\log p]}.$$

More recently effective bounds for solutions at a regular singular point have been found by Adolphson, et al. 1982 and by Christol,Dwork 1990, the former if the local monodromy is nilpotent of maximal rank, the second without the restriction of maximality of rank.

II. Filtration by growth and radius of convergence

For the second type of filtration we have

THEOREM. (Robba 1977a). Let L be a differential operator of order n with coefficients in $K(X)$ (or more generally with superadmissible coefficients). For $r \in (0,1]$, $\mathrm{Ker}(L, \mathcal{A}_t(r))$ defines a monic factor, L_r, of L in $E[D]$ whose coefficients are indeed superadmisible.

In the case of a Frobenius structure, filtration by growth should correspond to filtration by magnitude of eigenvalue.

THEOREM. (Robba 1975a) *Let $L \in E[D]$, $\dim Ker(L, \mathcal{A}_t) = orderL = n$. Then this kernel lies in $W_t^{1,n-1}$ and for each $\beta \in [0, n-1]$, $Ker(L, W_t^{1,\beta})$ determines a monic factor L_β of L in $E[D]$.*

The great contribution of Robba to these questions was to view $\mathcal{R} = E[D]$ as a subspace of the Banach space $\mathcal{L}(W_t^{r,\beta}, W_t^{r,\beta})$. Thus he considered $\overline{\mathcal{R}L}$, the completion in $\mathcal{R}$ of the ideal

$\mathcal{R}L$ under the Banach space norm. This ideal has a generator shown by Robba to be the factor of L corresponding to $\mathrm{Ker}(L, W_t^{r,\beta})$.

There are a number of unanswered questions.

1. If the coefficients of L (in this last theorem) are admissible (or even superadmissible) then the coefficients of L_β need not be superadmissible. But are they admissible? Conjecture: Yes.

2. Let L have coefficients which are analytic elements on $D(0, 1^-)$. We may construct a Newton polygon for L at t whose slopes are the exceptional values β such that $\mathrm{Ker}(L, W_t^{1,\beta-\varepsilon}) \subsetneq \mathrm{Ker}(L, W_t^{1,\beta+\varepsilon})$ for an infinite sequence of $\varepsilon \to 0$, and whose vertices have abscissas given by $\lim_{\varepsilon\to 0} \dim \mathrm{Ker}(L, W_t^{1,\beta-\varepsilon})$ for β exceptional. We may construct a similar polygon at $x = 0$.

CONJECTURE. *The polygon at $x = 0$ lies above the polygon at $x = t$.*

It is known (Dwork 1973, Robba 1975a)

$$\dim \mathrm{Ker}(L, W_t^{1,0)}) \geq \dim \mathrm{Ker}(L, W_0^{1,0})$$
$$\dim \mathrm{Ker}(L, \mathcal{A}_t(1)) \geq 1 \ \text{ implies} \dim \ker(L, W_t^{1,0}) \geq 1.$$

A geometric example of the filtration by growth is given by ${}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, x)$. This was analyzed (Dwork 1969, 1971) in two ways:

(a) by directly demonstrating the admissibility of $F(X)/F(X)^b$ via congruences associated with the p–adic gamma function

(b) by constructing a unit root crystal from the given superadmissible two dimensional crystal.

For ${}_2F_1(a, b, 1, x)$ Robba [1976(b)] gave a treatment based on a weak form of the Hahn Banach theorem. He avoided all references to Frobenius structure.

Dwork (1983) discussed ${}_2F_1(a, b, c, x)$ on the basis of Frobenius structure.

The nature of the factorization subject to geometric type hypotheses have been investigated for hypersurfaces, (Dwork 1973) for kloosterman sums (Adolphson, Sperber 1984) and hyperkloosterman sums (Sperber 1980, Sibuya, Sperber 1985).

Subject to geometric type hypotheses, Sperber and the author [Dwork, Sperber 1990] have found the coefficients of the factor corresponding to the bounded solutions to have mittag–leffler decompositions in which the components are of the form $\Sigma A_j/(x-\alpha)^j$ with $\mathrm{ord}\, A_j > k \log(1+j)$ for some $k > 0$. This has played a role in investigating the unit root zeta function.

III. Index

This question had great interest for Robba. At least four of his articles mention index in the title while others are devoted to applications of index. In his early work (1975, 76) there were no indications of applications but these appeared subsequently (1982c). His 1984 Asterisque article was dominated by the application to one dimensional cohomology and by 1986 he began studying symmetric powers of the Bessel differential equation.

Both Robba and Adolphson used patching arguments to reduce the question of index to the case of $L \in K[X][D]$ and to the calculation of either $\mathcal{A}_0/L\mathcal{A}_0$ or $\mathcal{B}_0/L\mathcal{B}_0$. For the applications it made no difference which one was finite. We consider only this elementary form.

For his application Adolphson was able to reduce to the case of order one and more explicitly to $X\frac{d}{dx} - a$, $a \in \mathbf{Q}$.

Robba [1975a] showed

If $\ker(L, \mathcal{A}_t(1)) = 0$ then $\chi(L, \mathcal{A}_0(1)) = \chi(L, \mathcal{B}_0(1))$.

This result is of interest as it seems to capture the essential point of dagger type cohomology involving over convergent series. Unfortunately this has not been extended to the case of several variables.

Of course if order $L = \dim \operatorname{Ker}(L, \mathcal{A}_0(1))$ then L has index as operator on $\mathcal{A}_0$ (but not on $\mathcal{B}_0$). In particular, by the transfer principle:

If L has no singularity on $D(0, 1^-)$ and if order $L = \dim(\operatorname{Ker} L, \mathcal{A}_t(1))$ then L has index on $\mathcal{A}_0(1)$.

By means of Christol's transfer theorem [Christol 1984] we may extend this last result to the case in which L has just one regular singularity in $D(0, 1^-)$ with non–Liouville exponents.

The operator (Robba 1977a) $L = p(1-x)D^2 - xD - a$ where $\liminf |a - m|^{1/m} = +1$, $\liminf |a + m|^{1/m} < 1$ is an example of an operator with no singularity in $D(0, 1^-)$ but without index in $\mathcal{A}_0$.

For operators of the first order $L = aD + b$, $a, b \in K[X]$, Robba (1985a) gave a beautiful formula

$$\left(\frac{d}{d\log r} \log \rho(L, r)\right)^- = \chi^-(L, r) + \operatorname{ord}^-(a, r)$$

where

$$\begin{aligned}
\chi^- &= \dim \operatorname{Ker} - \dim \operatorname{cokernel} \text{ for } L \text{ as operator on } H(D(0, 1^-)) \\
\rho(L, r) &= \text{radius of convergence of the solution at } t_r, \\
\operatorname{ord}^-(a, r) &= \text{abcissa of point of contact of the Newton polygon of a with the} \\
&\qquad \text{line of support of slope } -\log r/\log p,
\end{aligned}$$

i.e. if $a = \Sigma A_n X^n$, ν minimal such that $|a|_0(r) = |A_\nu| r^\nu$ then $\nu = \operatorname{ord}^-(a, r)$.

This formula showed how the Turritin form may be used to compute the index if the origin is an irregular singular point. It gives further motivation for extending Christol's transfer theorem to the case of irregular singular points. This index need not be equal to the algebraic index.

In view of the failure of crystalline cohomology to provide a proof finiteness of cohomology, the question of finiteness of index in the sense of this section must still be viewed as pertinent. It is our opinion that the critical case is that in which $D(0, 1^-)$ contains more than one regular singularity and order $L = \dim(\operatorname{Ker} L, \mathcal{A}_t(1))$.

We mention a few aspects of Robba's mathematical personality.

He was a very clear expositor and did much to popularize p–adic analysis. Together with Amice and Escassut he organized the GEAU. He gave a total of 24 written exposès, three in the first year, five in the second year.

He had many beautiful ideas. One was his abstract construction of the generic disk (1977c), a second was his explanation of Turrittin's theorem by means of the valuation polygon (1980a), a third was his method for removal of apparent singularities (cf. Christol 1981, Theorem 8.3), a fourth was his construction of a transcendental π, ord$\pi = 1/(p-1)$ which had the property that ord $\pi(x - x^p)$ converges for ord$x > -1/p$.

Bibliography of Ph. Robba

[1966] "Dérivée en moyenne de fonction moyennable", Communication an Congrés des Mathematiciens, Moscow, 2p.

[1967] "Solutions presque-periodiques d'équations differentielles aux différences", Seminaire Lions–Schwartz, 25p.

[1969a] (with E. Motzkin), "Ensembles satisfaisant au principe du prolongement analytique en analyse p–adique" C.R. Acad. Sc. Paris **269**, p. 126–129.

[1969b] __________, "Ensembles d'analyticité en analyse p–adique", C.R. Acad. Sc. Paris **269**, p. 450–453.

[1970] "Decomposition en élements simple d'une fonction analytique sur un corps valué ultramétrique. Application au prolongement analytique", Sem. Delange–Pisot–Poitou, (1969/70), No. 22, 14p.

[1971] "Prolongement analytique pour les fonction de plusieurs variables sur un corps valué complet", Sem. Delange–Pisot–Poitou, (1971/72).

[1972a] "Decomposition en élements simple d'un elément analytique sur un corps ultramétrique", C.R. Acad. Sc. Paris **274**, p. 532–535.

[1972b] "Fonctions analytiques sur un corps ultramétrique", C.R. Acad. Sc. Paris **274**, p. 721–723.

[1972c] "Prolongement analytique en analyse ultramétrique", C.R. Acad. Sci. Paris **274**, p. 830–833.

[1973a] "Fonctions analytiques sur les corps valués ultramétriques complets", *Asterisque*, **10**, p. 109–220.

[1973b] "Prolongement analytique pour les fonctions de plusieurs variables sur un corps valué complet", *Bull. Soc. Math. France* **101**, pp. 193–217.

[1974a] "Croissance des solutions d'une équation differentille homogéne", GEAU 73/74, exp. 1, 15p.

[1974b] "Indice d'un opérateur différentiel", GEAU 73/74, exp. 2, 14p.

[1974c] "Prolongement des solutions d'un équation différentielle p–adique", GEAU 73/74 exp 11, 3p. (also C.R. Acad. Sc. **279**, p. 153–154).

[1974d] "Quelques propriétes des éléments et fonctions analytiques au sens de Krasner", *Bull. Soc. Math. France* Mémoire 39–40, p. 341–349.

[1974e] "Un classe d'ensembles analytiques á plusieurs dimensions", *Bull. Soc. Math France*, Mémoire 39–40, p. 351–357.

[1975a] "On the index of p–adic differential operators I", *Annals of Math.* **101** p. 280–316.

[1975b] "Equations différentielles p–adiques", Sem. Delange–Pisot–Pitou, 74/75, 15p.

[1975c] "Factorisation d'un opérateur différentiel", GEAU 74/75 exp. 2, 16p.

[1975d] "Factorisation d'un opérateur diffèrentiel", Applications, GEAU 74/75 exp. 10, 6p.

[1975e] "Caracterisation des dérivées logarithmiques", GEAU 74/75 exp. 12, 6p.

[1975f] "Lemma de Hensel pour les opérateurs différentielles" GEAU 74/75 exp. 16, 11p.

[1975g] "Structure de Frobenius faible pour les équations différentielles de premier ordre", GEAU 74/75 exp. 20, 11p.

[1976a] "On the index of p–adic differential operators II", *Duke Math. J.* **43**, p. 19–31.

[1976b] "Solutions bornées des systems différentiels linéaires, Application aux fonctions hypergéométriques", GEAU 75/76 exp. 5, 16p.

[1976c] "Factorisation d'un opérateur différentiel II", GEAU 75/76 exp. 6, 6p.

[1976d] "Prolongement algébrique et equations différentielles", Proc. Conference on p–adic analysis, Nijmegen (1978), p. 172–184. Math. Inst. Katholieke Univ. Toernooiveld.

[1976e] "Théorie de Galois p–adique", GEAU 75/76 exp. 14, 8p.

[1976f] "Lemme de Schwarz p–adique pour plusieurs variables" GEAU 75/76 exp. J9, 12p.

[1977a] (with B. Dwork) "On ordinary linear p–adic differential equations", *Trans. Amer. Math. Soc.* **231**, p. 1–46.

[1977b] "Nouveau point de vue sur le prolongement algébrique", GEAU 76/77 exp. 5, 14p.

[1977c] "Disque générique et équations différentielles", GEAU 76/77 exp. 8, 7p.

[1977d] "Nombre de zéros des fonctions exponentielles–polynômes", GEAU 76/77 exp. 9, 3p.

[1978a] "Lemmes de Schwarz of lemmes d'approximation p–adiques en plusieurs variables", *Invent. Math.* **48**, 245–277.

[1978b] "Zéros de suites récurrentes linéaires", GEAU 77/78 exp. 13, 5p.

[1979a] (with B. Dwork) "On natural radii of p–adic convergence", *Trans. Amer. Math. Soc.* **256**, p. 199–213.

[1979b] "Points singuliers d'equations différentielles, d'apres F. Baldassarri", GEAU 78/79 exp. 5, 2p.

[1979c] (with B. Dwork) "Majorations effective", GEAU 78/79 exp. 18, 2p.

[1979d] "Clôture algébrique relative de certain anneaux de séries entières", GEAU 78/79 exp. 21, 2p.

[1979e] "Lemmes de Hensel pour les opérateurs differéntielles", GEAU 78/79 exp. 23, 8p.

[1980a] "Lemmes de Hensel pour les opérateurs différentielles, Application a la reduction formelle des equations différentielles", *Ens. Math.* **26**, p. 279–311.

[1980b] (with B. Dwork) "Effective p–adic bounds for solutions of homogeneous linear differential equations", *Trans. Amer. Math. Soc.* **259**, p. 559–577.

[1980c] (with S. Bosch and B. Dwork) "Un theorem de prolongement pour les fonctions analytiques", *Math. Ann.* **242**, p. 165–173.

[1980d] "Factorisation d'opérateurs différentiel á coefficients rationnels", GEAU 79/81 exp. 12, 3p.

[1981] "Une propriété de specialisation continue, d'après Artin, Bosch, Lang, Van den Dries", GEAU 79/81 exp. 26, 11p.

[1982a] "Sur les equations différentielles linéaires p–adiques II", *Pacific J. Math.* **98**, p. 393–418.

[1982b] "Calcul des residus en analyse p–adique, d'apres Gerritzen et Van der Put", GEAU 81/82 exp. 28, 8p.

[1982c] "Indice d'un opérateur différentiel linéaire p–adique d'ordre 1 et cohomologies p–adiques", GEAU 81/82 exp. J15, 10p.

[1983] "Cohomologie de Dwork, I", GEAU 82/83 exp. 3, 10p.

[1984a] "Une Introduction naive aux cohomologies de Dwork", *Bull. Soc. Math. France* **114** (Supp) Memoire **23**, p. 61–105.

[1984b] "Index of p–adic differential operators III, Application to twisted exponential sums", *Astérisque*, **119–120**, p. 191–266.

[1984c] "Quelques remarques sur les opérateurs différentiels d'order 1", GEAU 83/84 exp. 12, 8p.

[1985a] "Indice d'un opérateur différentiel p–adique IV, Cas des systèms. Mesure d'irregularité dans un disque", *Ann. Inst. Fourier* **35**, p. 13–35.

[1985b] "Conjecture sur les equations différentielles p–adiques linéaires", GEAU exp. 2, 8p.

[1985c] "Product symmetrique de l'equation de Bessel", GEAU exp. 7, 3p.

[1986a] "Symmetric powers of the p–adic Bessel Function", *Crelle* **366**, p. 194–220.

[1986b] (with J. P. Bezivin) "Rational solutions of linear differential equations", Proc. of conference on p–adic analysis, Hengelhoef, Belg. p. 11–18.

[1987] "Propriete d'approximation pour les elements algébriques", *Comp. Math.* **63**, p. 3–14.

[1989a] (with J. P. Bezivin) "A new p–adic method for proving irrationality and transcendence results", *Ann. of Math.* **129**, p. 151–160.

[1989b] "Rational solutions of linear differential equations", *J. Australian Math. Soc.* **46**, p. 184–196.

General Bibliography

[1976a] Adolphson, A., "An index theorem for p– adic differential operators", *Trans. Amer. Math. Soc.* **216**, 279–293.

[1976b] ————————, "A p–adic theory of Hecke polynomials", *Duke Math. J.* **43**, 115–145.

[1982] ————————, Dwork, B., Sperber, S., "Growth of solutions of linear differential equations at a logarithmic singularity", *Trans. Amer. Math. Soc.* **271**, 245–252.

[1984] ————————, Sperber, S., "Twisted kloosterman sums and p–adic Bessel functions", *Amer. J. Math.* **105**, 549–591.

[1981] Christol, "Systémes différentiels linéaires p–adiques", Structure de Frobenius Faible, *Bull. Soc. Math. France* **109**, 83–122.

[1984] __________, "Un theorem de transfert pour les disques singuliers réguliers", *Astérisque* **119–120**, 151–168.

[1990 __________, Dwork, B. "Effective p–adic bounds at regular singular points", *Duke Math. J.* (submitted).

[1966] Clark, D. "A note on the p–adic convergence of solutions of linear differential equations", *Proc. Amer. Math. Soc.* **17**, 262–269.

[1964] Dwork, B. "On the zeta function of a hypersurface II", *Ann. Math.* **80**, 227–299.

[1966] __________, "On the zeta function of a hypersurface III", *Ann. Math.* **83**, 457–519.

[1969] __________, "p–adic cycles", *Pub. Math. IHES* **37**, 27–116.

[1971] __________, "Normalized period matrices I", *Ann. Math.* **93**, 337–388.

[1973] __________, "Normalized period matrices II", *Ann. Math.* **98**, 1–57.

[1982] __________, *Lectures on p–adic differential equations*, Springer–Verlag.

[1987] __________, "On the Tate constant", *Comp. Math.* **61**, 43–59.

[1990] __________, Sperber, S., "Logarithmic decay and over convergence of the unit root and associated zeta functions", *Ann. Sc. ENS*, to appear.

[1973] Katz, N.,*Travaux de Dwork*, Sem. Bourbuki 71/72 exp. 409 (Lecture Notes in Math., Springer–Verlag).

[1937] Lutz, E., "Sur l'equation $y^2 = x^3 - Ax - B$ dans les corps p–adiques", *J. Reine Angew. Math.* **177**, 238–247.

[1981] Sibuya, Y., Sperber, S., "Arithmetic properties of power series solutions of algebraic differential equations", *Ann. Math.* **113**, 111–157.

[1985] __________, On the p–adic continuation of the logarithmic derivative of certain hypergeometric functions, GEAU 84/85 exp. 4, 5p.

[1980] Sperber, S., "Congruence properties of the hyperkloosterman sum", *Comp. Math.* **40**, 3–32.

p-Adic estimates for exponential sums

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1 Introduction

Let k be the finite field of $q = p^a$ elements, $f = \sum_{j\in J} a_j x^j \in k[x_1, \ldots, x_n, (x_1 \cdots x_r)^{-1}]$ where $0 \le r \le n$, $\Psi : k \to \mathbb{Q}(\zeta_p)^\times$ be a nontrivial additive character, and $\chi_1, \ldots, \chi_r : k^\times \to \mathbb{Q}(\zeta_{q-1})^\times$ be multiplicative characters. Define

$$S_1(\chi_1, \ldots, \chi_r, f) = \sum_{x\in(k^\times)^r \times k^{n-r}} \chi_1(x_1) \cdots \chi_r(x_r) \Psi(f(x)). \tag{1}$$

The first problem we consider in this article is:

<u>**Problem 1:**</u> Find a p-adic estimate for S_1.

Let k_m be the extension of k of degree m. We can define for each m an exponential sum related to (1):

$$S_m(\chi_1, \ldots, \chi_r, f) = \sum_{x\in(k^\times)^r \times k^{n-r}} \left(\prod_{i=1}^{r} \chi_i(N_m(x_i))\right) \Psi(Tr_m(f(x))), \tag{2}$$

where $Tr_m : k_m \to k$ is the trace map and $N_m : k_m \to k$ is the norm map. This data can be encapsulated in an L-function:

$$L(\chi_1, \ldots, \chi_r, f; t) = \exp\left(\sum_{m=1}^{\infty} S_m(\chi_1, \ldots, \chi_r, f)\frac{t^m}{m}\right). \tag{3}$$

The following result is well-known:

Theorem 1 *(Dwork, Grothendieck) $L(\chi_1, \ldots, \chi_r, f; t)$ is a rational function, i. e.,*

$$L(\chi_1, \ldots, \chi_r, f; t) = \frac{\prod_{\text{finite}}(1 - \alpha_i t)}{\prod_{\text{finite}}(1 - \beta_j t)}. \tag{4}$$

*Partially supported by NSF Grant No. DMS-8803085.
†Partially supported by NSF Grant No. DMS-8601461.