G. Da Prato P.C. Kunstmann I. Lasiecka
A. Lunardi R. Schnaubelt L. Weis

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Functional Analytic Methods for Evolution Equations

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Editors: M. Iannelli
R. Nagel
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Authors

Giuseppe Da Prato
Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 Pisa, Italy

e-mail: g.daprato@sns.it

Peer C. Kunstmann
Lutz Weis
Mathematisches Institut I
Universität Karlsruhe
Englerstrasse 2
76128 Karlsruhe, Germany

e-mail: peer.kunstmann@math.uni-karlsruhe.de
lutz.weis@math.uni-karlsruhe.de

Irena Lasiecka
Department of Mathematics
University of Virginia
Kerchof Hall, P.O. Box 400137
Charlottesville
VA 22904-4137, U.S.A.

e-mail: il2v@weyl.math.virginia.edu

Alessandra Lunardi
Department of Mathematics
University of Parma
via D'Azeglio 85/A
43100 Parma, Italy

e-mail: alessandra.lunardi@unipr.it

Roland Schnaubelt
FB Mathematik und Informatik
Martin-Luther Universität
Theodor-Lieser-Str. 5
6099 Halle, Germany

e-mail: schnaubelt@mathematik.uni-halle.de

Editors

Mimmo Iannelli
Department of Mathematics
University of Trento
via Sommarive 14
38050 Trento, Italy

e-mail: iannelli@science.unitn.it

Rainer Nagel
Susanna Piazzera
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen, Germany

e-mail: rana@fa.uni-tuebingen.de
supi@fa.uni-tuebingen.de

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Preface

Evolution equations describe time dependent processes as they occur in physics, biology, economy or other sciences. Mathematically, they appear in quite different forms, e.g., as parabolic or hyperbolic partial differential equations, as integrodifferential equations, as delay or difference differential equations or more general functional differential equations. While each class of equations has its own well established theory with specific and sophisticated methods, the need for a unifying view becomes more and more urgent.

To this purpose functional analytic methods have been applied in recent years with increasing success. In particular the concepts of *Abstract Cauchy Problems* and of *Operator Semigroups* on Banach spaces allow a systematic treatment of general evolution equations preparing the ground for a better theory even for special equations.

It is the aim of this volume to make this evident. Five contributions by leading experts present recent research on functional analytic aspects of evolution equations.

In the first contribution, **Giuseppe Da Prato** from the Scuola Normale Superiore in Pisa (Italy) gives an introduction to stochastic processes on infinite dimensional spaces. His approach is based on the concept of Markov semigroups and does not require familiarity with probability theory. The main emphasis is on important qualitative properties of these semigroups and the Ornstein-Uhlenbeck semigroup serves as his major example.

In the second contribution, which is by far the longest, **Peer Kunstmann** and **Lutz Weis** (both from the University of Karlsruhe in Germany) discuss the (maximal) regularity of the solutions of inhomogeneous parabolic Cauchy problems.

Regularity properties are fundamental for a theory of nonlinear parabolic equations. Since 1998 this theory has made enormous progress with spectacular breakthroughs based on new Fourier multiplier theorems with operator-valued functions and "square function estimates" for the holomorphic H^∞-functional calculus (some of these results are due to the authors). This contribution is a unifying and accessible presentation of this theory and some of its applications.

Control theoretic aspects of evolution equations in finite dimensions have been studied for a long time. Thanks to functional analytic tools there is now a well established infinite dimensional theory described in some recent monographs. However, this theory does not cover so-called boundary and point control problems. **Irena Lasiecka** from the University of Virginia (USA) developed (mostly with Roberto Triggiani) a systematic approach to these problems using a beautiful combination of abstract semigroups methods and sharp PDE estimates. In her contribution she explains this approach and discusses illustrating examples such as systems of coupled wave, plate and heat equations.

While in the previous contributions the focus is on the dynamics of the state variable, **Alessandra Lunardi** (University of Parma, Italy) studies moving boundary problems. In order to explain these highly nonlinear problems, she concentrates on the heat equation on a moving domain, first in one and then in higher dimensions. Her work is intended as an introduction to an important new field, to very recent results, and to interesting open problems.

Roland Schnaubelt (University of Halle, Germany) shows in the last contribution how nonautonomous linear evolution equations can be studied by a reduction to an autonomous problem to which semigroup methods apply. In particular, the well developed spectral theory for semigroups allows a systematic characterisation of, e.g., exponential dichotomy of the solutions. He then applies these results to obtain qualitative properties of the solutions to nonlinear equations.

These contributions were the basis of lectures given at the Autumn School on "Evolution Equations and Semigroups" at Levico Terme (Trento, Italy) from October 28 to November 2, 2001, within the program of the CIRM (Centro Internazionale per la Ricerca Matematica). We thank Professor Mario Miranda for the support provided to the School. Thanks are also due to the speakers for their cooperation and the permission to collect the expanded notes of their lectures. We hope that this volume will be valuable for beginners as well as for experts in evolution equations.

Trento and Tübingen,
March, 2004

Mimmo Iannelli
Rainer Nagel
Susanna Piazzera

Contents

An Introduction to Markov Semigroups

Lectures held at the Autumn School in Levico Terme (Trento, Italy), October 28-November 2, 2001

Giuseppe Da Prato

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy.
g.daprato@sns.it

Preface

This paper contains the notes of a short course on Markov semigroups. The main aim was to give an introduction to some important properties as: ergodicity, irreducibility, strong Feller property, invariant measures, relevant to some important Markov semigroups arising in infinite dimensional analysis and in stochastic dynamical systems. We have considered in particular the heat semigroup in infinite dimensions, the Ornstein–Uhlenbeck semigroup, the transition semigroup of a one dimensional dynamical system perturbed by noise.

The lectures were designed for an audience having a basic knowledge of functional analysis and measure theory but not familiar with probability. An effort has been done in order to make the lectures as self–contained as possible. In this spirit, the first part was devoted to collect some basic properties of Gaussian measures in Hilbert spaces including the reproducing kernel and the Cameron–Martin formula, a tool that was systematically employed.

Several concepts and results contained in this course are taken from the the notes [3] and the monographs [4], [5], [6].

1 Gaussian Measures in Hilbert Spaces

In all this section H represents a separable Hilbert space, (inner product $\langle\cdot,\cdot\rangle$, norm $|\cdot|$), and $L(H)$ the set of all linear continuous operators from H into H. We denote by $\Sigma(H)$ the subset of $L(H)$ consisting of all symmetric operators and we set

$$L^+(H) = \{T \in \Sigma(H): \ \langle Tx, x\rangle \geq 0, \quad x, y \in H\}.$$

An important role will be played by symmetric nonnegative *trace class* operators,

$$L_1^+(H) = \{T \in L^+(H) : \text{ Tr } T < +\infty\},$$

where

$$\text{Tr } T = \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$$

and (e_k) is a complete orthonormal system in H.

We recall that Tr T does not depend on the choice of the orthonormal system (e_k). Moreover if T is of trace class then it is compact and Tr T is the sum of its eigenvalues repeated according to their multiplicity, see e. g. [18].

Finally we shall denote by $\mathcal{B}(H)$ the σ–algebra generated by all open (or closed) subsets of H.

1.1 Measures in Hilbert Spaces

Let μ be a probability measure on $(H, \mathcal{B}(H))$. Assume that its first moment is finite,

$$\int_H |x| \mu(dx) < +\infty.$$

Then the linear functional $F : H \to \mathbb{R}$ defined as

$$F(h) = \int_H \langle x, h \rangle \mu(dx), \ h \in H,$$

is continuous since $|F(h)| \le \int_H |x| \mu(dx) \ |h|, \ \ h \in H$. By the Riesz representation theorem there exists $m_u \in H$ such that

$$\langle m_\mu, h \rangle = \int_H \langle x, h \rangle \mu(dx), \ \ h \in H.$$

m_μ is called the *mean* of μ. We shall write $m_\mu = \int_H x \mu(dx)$.

Assume now that the second moment of μ is finite,

$$\int_H |x|^2 \mu(dx) < +\infty.$$

Then we can consider the bilinear form $G : H \times H \to \mathbb{R}$ defined as

$$G(h, k) = \int_H \langle h, x - m_\mu \rangle \langle k, x - m_\mu \rangle \mu(dx), \ \ h, k \in H.$$

G is continuous since

$$|G(h, k)| \le \int_H |x - m_\mu|^2 \mu(dx) \ |h| \ |k|, \ \ h, k \in H.$$

Therefore there is a unique $Q_\mu \in L(H)$ such that

$$\langle Q_\mu h, k\rangle = \int_H \langle h, x - m_\mu\rangle\langle k, x - m_\mu\rangle\mu(dx), \quad h, k \in H.$$

Q_μ is called the *covariance* of μ.

Proposition 1.1. *$Q_\mu \in L_1^+(H)$ that is, is symmetric positive and of trace class.*

Proof. Symmetry and positivity of Q_μ are clear. To prove that Q_μ is of trace class fix a complete orthonormal system (e_k) in H and note that, since

$$\langle Q_\mu e_k, e_k\rangle = \int_H |\langle x - m_\mu, e_k\rangle|^2\mu(dx), \ k \in \mathbb{N},$$

we have, by the monotone convergence theorem and the Parseval identity, that

$$\mathrm{Tr}\ Q_\mu = \sum_{k=1}^{\infty}\int_H |\langle x - m_\mu, e_k\rangle|^2\mu(dx) = \int_H |x - m_\mu|^2\mu(dx) < +\infty.$$

□

If μ is a probability measure on $(H, \mathcal{B}(H))$ we define its *Fourier transform* by setting

$$\hat{\mu}(h) = \int_H e^{i\langle h,x\rangle}\mu(dx), \qquad h \in H.$$

The following result holds, see e.g. [17].

Proposition 1.2. *Let μ and ν be probability measures on $(H, \mathcal{B}(H))$ such that $\hat{\mu}(h) = \hat{\nu}(h)$ for all $h \in H$. Then $\mu = \nu$.*

Let K be another Hilbert space and let $X : H \to K$ be a Borel mapping ([1]). If μ is a probability measure on $(H, \mathcal{B}(H))$ we denote by $\mathcal{L}(X)$ or μ_X the *law* of X. $\mathcal{L}(X)$ is the probability measure on $(K, \mathcal{B}(K))$ defined as

$$\mathcal{L}(X)(I) = \mu_X(I) = \mu(X^{-1}(I)), \ \ I \in \mathcal{B}(K).$$

The following formula of change of variables is basic.

Proposition 1.3. *Let $X : H \to K$ be a Borel mapping and $\varphi : K \to \mathbb{R}$ a bounded Borel mapping. Then we have*

$$\int_H \varphi(X(x))\mu(dx) = \int_K \varphi(y)\mu_X(dy). \tag{1.1}$$

[1] that is $F \in \mathcal{B}(K) \Rightarrow X^{-1}(F) \in \mathcal{B}(H)$.

Proof. It is enough to prove (1.1) when φ is a simple function, see e.g. [15]. Let $\varphi = \chi_A$ where $A \in \mathcal{B}(K)$ ([2]). Then we have $\varphi(X) = \chi_{X^{-1}(A)}$, and so

$$\int_H \varphi(X(x))\mu(dx) = \mu(X^{-1}(A)) = \mu_X(A).$$

On the other hand we have

$$\int_K \varphi(y)\mu_X(dy) = \int_K \chi_A(y)\mu_X(dy) = \mu_X(A).$$

So, the conclusion follows. □

1.2 Gaussian Measures

Let us first consider Gaussian measures on $\mathbb{R}$. For any $m \in \mathbb{R}$ and $\lambda \geq 0$, we define the Gaussian measure $N_{m,\lambda}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as

$$N_{m,\lambda}(d\xi) = \begin{cases} \frac{1}{\sqrt{2\pi\lambda}}\, e^{-\frac{(\xi-m)^2}{2\lambda}} d\xi & \text{if } \lambda > 0, \\ \delta_m(d\xi) & \text{if } \lambda = 0, \end{cases} \tag{1.2}$$

where δ_m is the Dirac measure concentrated at m:

$$\delta_m(I) = \begin{cases} 1 & \text{if } m \in I \\ 0 & \text{if } m \notin I, \end{cases} \qquad I \in \mathcal{B}(H).$$

The following identities are easy to check.

$$\int_{-\infty}^{+\infty} \xi N_{m,\lambda}(d\xi) = m, \tag{1.3}$$

$$\int_{-\infty}^{+\infty} (\xi - m)^2 N_{m,\lambda}(d\xi) = \lambda, \tag{1.4}$$

$$\int_{-\infty}^{+\infty} e^{i\alpha\xi} N_{m,\lambda}(d\xi) = e^{i\alpha m - \frac{\lambda}{2}\alpha^2}. \tag{1.5}$$

We now consider a general separable Hilbert space H. For any $a \in H$ and any $Q \in L(H)$ we want to define a Gaussian measure $N_{a,Q}$ on $(H, \mathcal{B}(H))$, having mean a, covariance operator Q, and Fourier transform given by

$$\widehat{N_{a,Q}}(h) = \exp\left\{ i\langle a, h\rangle - \frac{1}{2}\,\langle Qh, h\rangle \right\}, \quad h \in H. \tag{1.6}$$

[2] For any Borel set A we denote by χ_A the function that holds 1 on A and 0 on the complement of A.

Notice that, in view of Proposition 1.1, the fact that Q is symmetric, positive and of trace class is a necessary requirement.

Let us consider the natural isomorphism γ between H and the Hilbert space ℓ^2 of all sequences (x_k) of real numbers such that

$$\sum_{k=1}^{\infty} |x_k|^2 < +\infty,$$

defined by $\gamma(x) = (x_k)$, $x \in \ell^2$. In all this section we shall identify H with ℓ^2.

Since Q is of trace class it is compact, so there exist a complete orthonormal basis (e_k) on H and a sequence of nonnegative numbers (λ_k) such that

$$Qe_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

For any $x \in H$ we set $x_k = \langle x, e_k \rangle$, $k \in \mathbb{N}$.

We are going to define $N_{a,Q}$ as a product measure

$$N_{a,Q} = \bigtimes_{k=1}^{\infty} N_{a_k,\lambda_k}.$$

For this we first recall, following [10, Section 38], some general results about countable products of measures. Let μ_k be a sequence of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We shall define a product measure

$$\mu = \bigtimes_{k=1}^{\infty} \mu_k,$$

on the space $\mathbb{R}^\infty = \bigtimes_{k=1}^{\infty} \mathbb{R}$, consisting of all sequences of real numbers (endowed with the product topology).

We first define μ on all *cylindrical subsets* $I_{k_1,\dots k_n;A}$ of $\mathbb{R}^\infty$, where $n, k_1 < \dots < k_n$ are positive integers, and $A \in \mathcal{B}(\mathbb{R}^n)$:

$$I_{k_1,\dots k_n;A} = \{(x_j) \in \mathbb{R}^\infty : (x_{k_1}, \dots, x_{k_n}) \in A\}.$$

It is easy to see that the family of all cylindrical subsets of $\mathbb{R}^\infty$ is an algebra, denoted by $\mathcal{C}$ and that μ is additive on $\mathcal{C}$. Moreover the σ–algebra generated by $\mathcal{C}$ coincides with $\mathcal{B}(\mathbb{R}^\infty)$. See e.g. [7, page 9].

We define

$$\mu(I_{k_1,\dots k_n;A}) = (\mu_{k_1} \times \dots \times \mu_{k_n})(A), \ I_{k_1,\dots k_n;A} \in \mathcal{C}$$

and show that μ is σ–additive on $\mathcal{C}$. This will imply, see e.g. [10], that μ can be extended to a probability measure on the product σ–algebra $\bigtimes_{k=1}^{\infty} \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^\infty)$.

Proposition 1.4. *μ is σ–additive on $\mathcal{C}$ and has a unique extension to a probability measure on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$.*

Proof. To prove σ-additivity of μ it is enough to show continuity of μ at 0. This is equivalent to prove that if (E_j) is a decreasing sequence on $\mathcal{C}$ such that $\mu(E_j) \geq \varepsilon$, $j \in \mathbb{N}$, for some $\varepsilon > 0$, then we have

$$\bigcap_{j=1}^{\infty} E_j \neq \emptyset.$$

To prove this fact, let us consider the *sections* of E_j defined as

$$E_j(\alpha) = \{x \in \mathbb{R}_1^\infty : (\alpha, x) \in E_j\}, \ \alpha \in \mathbb{R},$$

where we have used the notation $\mathbb{R}_n^\infty = \times_{k=n+1}^\infty \mathbb{R}$, $n \in \mathbb{N}$. Set

$$F_j^{(1)} = \left\{\alpha \in \mathbb{R} : \mu^{(1)}(E_j(\alpha)) \geq \frac{\varepsilon}{2}\right\}, \ j \in \mathbb{N},$$

where $\mu^{(n)} = \times_{k=n+1}^\infty \mu_k$, $n \in \mathbb{N}$. Then by the Fubini theorem we have

$$\begin{aligned}\mu(E_j) &= \int_{\mathbb{R}} \mu^{(1)}(E_j(\alpha))\mu_1(d\alpha) \\ &= \int_{F_j^{(1)}} \mu^{(1)}(E_j(\alpha))\mu_1(d\alpha) + \int_{[F_j^{(1)}]^c} \mu^{(1)}(E(\alpha))\mu_1(d\alpha) \\ &\leq \mu_1(F_j^{(1)}) + \frac{\varepsilon}{2}.\end{aligned}$$

Therefore $\mu_1(F_j^{(1)}) \geq \frac{\varepsilon}{2}$.

Since μ_1 is a probability measure, it is continuous at 0. Therefore, since $(F_j^{(1)})$ is decreasing, there exists $\overline{\alpha_1} \in \mathbb{R}$ such that

$$\mu^1(E_j(\overline{\alpha_1})) \geq \frac{\varepsilon}{2}, \quad j \in \mathbb{N},$$

and consequently we have

$$E_j(\overline{\alpha_1}) \neq \emptyset. \tag{1.7}$$

Now set

$$E_j(\overline{\alpha_1}, \alpha_2) = \{x_2 \in \mathbb{R}_2^\infty : (\overline{\alpha_1}, \alpha_2, x) \in E_j\}, \ j \in \mathbb{N}, \ \alpha_2 \in \mathbb{R},$$

and

$$F_j^{(2)} = \left\{\alpha_2 \in \mathbb{R} : \mu^{(2)}(E_j(\alpha)) \geq \frac{\varepsilon}{2}\right\}, \ j \in \mathbb{N}.$$

Then by the Fubini theorem we have

$$\begin{aligned}\mu^1(E_j(\overline{\alpha_1})) &= \int_{\mathbb{R}} \mu^{(2)}(E_j(\overline{\alpha_1},\alpha_2))\mu_2(d\alpha_2)\\ &= \int_{F_j^{(2)}} \mu^{(2)}(E_j(\overline{\alpha_1},\alpha_2))\mu_2(d\alpha_2) + \int_{[F_j^{(2)}]^c} \mu^{(2)}(E_j(\overline{\alpha_1},\alpha_2))\mu_2(d\alpha_2)\\ &\le \mu_2(F_j^{(2)}) + \frac{\varepsilon}{4}.\end{aligned}$$

Therefore $\mu_2(F_j^{(2)}) \ge \frac{\varepsilon}{4}$. Since $(F_j^{(2)})$ is decreasing, there exists $\overline{\alpha_2} \in \mathbb{R}$ such that

$$\mu^2(E_j(\overline{\alpha_1},\overline{\alpha_2})) \ge \frac{\varepsilon}{4},\quad j \in \mathbb{N},$$

and consequently we have

$$E_j(\overline{\alpha_1},\overline{\alpha_2}) \ne \emptyset. \tag{1.8}$$

Arguing in a similar way we see that there exists a sequence $(\overline{\alpha_k}) \in \mathbb{R}^\infty$ such that

$$E_j(\overline{\alpha_1},...,\overline{\alpha_n}) \ne \emptyset, \tag{1.9}$$

where

$$E_j(\alpha_1,...,\alpha_n) = \{x \in \mathbb{R}_n^\infty:\ (\ \alpha_1,...,\alpha_n,x) \in E_j\},\ n \in \mathbb{N}.$$

This implies, as easily seen, that

$$(\alpha_n) \in \bigcap_{j=1}^{\infty} E_j.$$

Therefore $\bigcap_{j=1}^\infty E_j$ is not empty as required. Thus we have proved that μ is σ–additive on $\mathcal{C}$. Now the second statement follows since a σ–additive function on an algebra $\mathcal{A}$ can be uniquely extended to a probability measure on the σ–algebra generated by $\mathcal{A}$, see [10]. □

We can now define the Gaussian measure $N_{a,Q}$.

Theorem 1.5. *For any $a \in H$ and any $Q \in L_1^+(H)$ there exists a unique measure μ such that its Fourier transform $\widehat{\mu}$ is given by*

$$\widehat{\mu}(h) = e^{i\langle a,h\rangle}e^{-\frac{1}{2}\langle Qh,h\rangle},\quad h \in H. \tag{1.10}$$

Moreover,

$$\int_H |x|^2 N_{a,Q}(dx) = \operatorname{Tr} Q + |a|^2, \tag{1.11}$$

$$\int_H \langle x,h\rangle \mu(dx) = a,\quad h \in H, \tag{1.12}$$

$$\int_H \langle x-a,h\rangle\langle x-a,k\rangle \mu(dx) = \langle Qh,k\rangle,\quad h,k \in H. \tag{1.13}$$

We call μ the Gaussian measure in H with mean 0 and covariance operator Q and we set $\mu = N_{a,Q}$.
Proof. Let us consider the product measure on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$,

$$\mu = \bigtimes_{k=1}^{\infty} N_{a_k,\lambda_k},$$

defined by Proposition 1.4. We claim that the support of μ is included in ℓ^2 that is that $\mu(\ell^2) = 1$ ([3]). We have in fact, from the monotone convergence theorem

$$\int_{\mathbb{R}^\infty} \sum_{k=1}^{\infty} x_k^2 \mu(dx) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} x_k^2 \mu_k(dx_k) = \sum_{k=1}^{\infty} (\lambda_k + a_k^2), \tag{1.14}$$

where $\mu_k = N_{a_k,\lambda_k}$. Therefore

$$\mu\left(\left\{x \in \mathbb{R}^\infty : |x|_{\ell^2}^2 < \infty\right\}\right) = 1,$$

as claimed.

Now we define the Gaussian measure $N_{a,Q}$ as the restriction of μ to ℓ^2. To check (1.10) it is useful to introduce a sequence (P_n) of projections in H :

$$P_n x = \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \quad x \in H, \ n \in \mathbb{N}.$$

Obviously $\lim_{n\to\infty} P_n x = x, \quad x \in H$. Consequently, by the dominated convergence theorem we have, recalling (1.5),

$$\int_H e^{i\langle h,x\rangle} \mu(dx) = \lim_{n\to\infty} \int_H e^{i\langle P_n h, P_n x\rangle} \mu(dx) = \lim_{n\to\infty} \prod_{k=1}^{n} \int_{\mathbb{R}} e^{ih_n x_n} N_{a_n \lambda_n}(dx)$$

$$= \prod_{k=1}^{\infty} e^{ia_n h_n} e^{-\frac{1}{2}\lambda_n h_n^2} = e^{i\langle a,h\rangle} e^{-\frac{1}{2}\langle Qh,h\rangle},$$

and so (1.10) is proved. (1.11) follows from (1.14).

Let us prove (1.12). Since $|\langle x, h\rangle| \le |x| \; |h|$ and $\int_H |x| \mu(dx)$ is finite by (1.11), we have, by the dominated convergence theorem,

$$\int_H \langle x, h\rangle \mu(dx) = \lim_{n\to\infty} \int_H \langle P_n x, h\rangle \mu(dx).$$

But

[3] It is easy to see that ℓ^2 is a Borel subset of $\mathbb{R}^\infty$.

$$\int_H \langle P_n x, h\rangle \mu(dx) = \sum_{k=1}^{n} \int_H x_k h_k \mu(dx)$$

$$= \sum_{k=1}^{n} h_k \int_{\mathbb{R}} x_k N_{a_k,\lambda_k}(dx_k) = \sum_{k=1}^{n} h_k a_k = \langle P_n a, h\rangle \to \langle a, h\rangle,$$

as $n \to \infty$. Thus (1.12) is proved. The proof of (1.13) is similar. □

The last part of this section is devoted to the computation of some Gaussian integrals, which will be often used in what follows. For the sake of simplicity we assume that Ker $Q = \{0\}$ and that (this is not a restriction), $\lambda_1 \geq \lambda_2 \geq ....\lambda_n \geq ....$

To formulate the next result notice that for any $\varepsilon < \frac{1}{\lambda_1}$, the linear operator $1 - \varepsilon Q$ is invertible and $(1 - \varepsilon Q)^{-1}$ is bounded. We have in fact, as easily checked,

$$(1 - \varepsilon Q)^{-1} x = \sum_{k=1}^{\infty} \frac{1}{1 - \varepsilon \lambda_k} \langle x, e_k\rangle e_k, \ x \in H.$$

In this case we can define the *determinant* of $(1 - \varepsilon Q)$ by setting

$$\det(1 - \varepsilon Q) = \lim_{n\to\infty} \prod_{k=1}^{n} (1 - \varepsilon\lambda_k) := \prod_{k=1}^{\infty} (1 - \varepsilon\lambda_k).$$

It is easy to see that, in view of the assumption $\sum_{k=1}^{\infty} \lambda_k < +\infty$, the product below is finite and positive.

Proposition 1.6. *Let $\mu = N_{a,Q}$ and $\varepsilon \in \mathbb{R}$. Then we have*

$$\int_H e^{\frac{\varepsilon}{2}|x|^2} \mu(dx) = \begin{cases} [\det(1 - \varepsilon Q)]^{-1/2} e^{-\frac{\varepsilon}{2}\langle (1-\varepsilon Q)^{-1} a, a\rangle}, & \text{if } \varepsilon < \frac{1}{\lambda_1}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.15}$$

Proof. For any $n \in \mathbb{N}$ we have

$$\int_H e^{\frac{\varepsilon}{2}|P_n x|^2} \mu(dx) = \prod_{k=1}^{n} \int_{\mathbb{R}} e^{\frac{\varepsilon}{2} x_k^2} N_{a_k,\lambda_k}(dx_k).$$

Since $|P_n x|^2 \uparrow |x|^2$ as $n \to \infty$ and, by an elementary computation,

$$\int_{\mathbb{R}} e^{\frac{\varepsilon}{2} x_k^2} N_{a_k,\lambda_k}(dx_k) = \frac{1}{\sqrt{1 - \varepsilon\lambda_k}} e^{-\frac{\varepsilon}{2} \frac{a_k^2}{1-\varepsilon\lambda_k}},$$

the conclusion follows from the monotone converge theorem. □

Exercise 1.7. Compute the integral

$$J_m = \int_H |x|^{2m} \mu(dx), \quad m \in \mathbb{N}.$$

Hint. Notice that $J_m = 2^m F^{(m)}(0)$, where

$$F(\varepsilon) = \int_H e^{\frac{\varepsilon}{2}|x|^2} \mu(dx).$$

Proposition 1.8. *Let $\mu = N_{a,Q}$. Then we have*

$$\int_H e^{\langle h,x \rangle} \mu(dx) = e^{\langle a,h \rangle} e^{\frac{1}{2}\langle Qh,h \rangle}, \quad h \in H. \tag{1.16}$$

Proof. For any $\varepsilon > 0$ we have

$$e^{\langle h,x \rangle} \le e^{|x|\,|h|} \le e^{\varepsilon |x|^2} \, e^{\frac{1}{\varepsilon}|x|^2}.$$

Choose $\varepsilon < \frac{1}{\lambda_1}$; then we see by Proposition 1.6 that the function $x \to e^{\langle h,x \rangle}$ is integrable with respect to μ. Consequently, by the dominate convergence theorem, it follows that

$$\int_H e^{\langle h,x \rangle} \mu(dx) = \lim_{n \to \infty} \int_H e^{\langle h, P_n x \rangle} \mu(dx),$$

which yields the conclusion. □

2 Gaussian Random Variables

Let H and K be separable Hilbert spaces and let μ be a probability measure on $(H, \mathcal{B}(H))$. A *random variable* X in H with values in K is a Borel mapping $X : H \to K$, that is

$$I \in \mathcal{B}(K) \Rightarrow X^{-1}(I) \in \mathcal{B}(H).$$

When $K = \mathbb{R}$ we call X a *real random variable.*

X is called *Gaussian* if its law $\mathcal{L}(X) = \mu_X$ is a Gaussian measure on K. We say also that X is a *Gaussian random variable* in H taking values in K.

Proposition 2.1. *Let $X : H \to K$ be a random variable on $(H, \mathcal{B}(H), \mu)$ such that $\int_H X(x)\mu(dx) = 0$. Then the covariance Q_X of the law of X is given by*

$$\langle Q_X \alpha, \alpha \rangle_K = \int_H \langle X(x), \alpha \rangle_K^2 \mu(dx), \quad \alpha \in K. \tag{2.1}$$

Proof. We have in fact by the change of variables formula (1.1),

$$\langle Q_X \alpha, \alpha \rangle_K = \int_K \langle y, \alpha \rangle_K^2 \mu_X(dy) = \int_H \langle X(x), \alpha \rangle_K^2 \mu(dx).$$

□