LINEAR PROGRAMMING IN INFINITE-DIMENSIONAL SPACES

Edward J. Anderson
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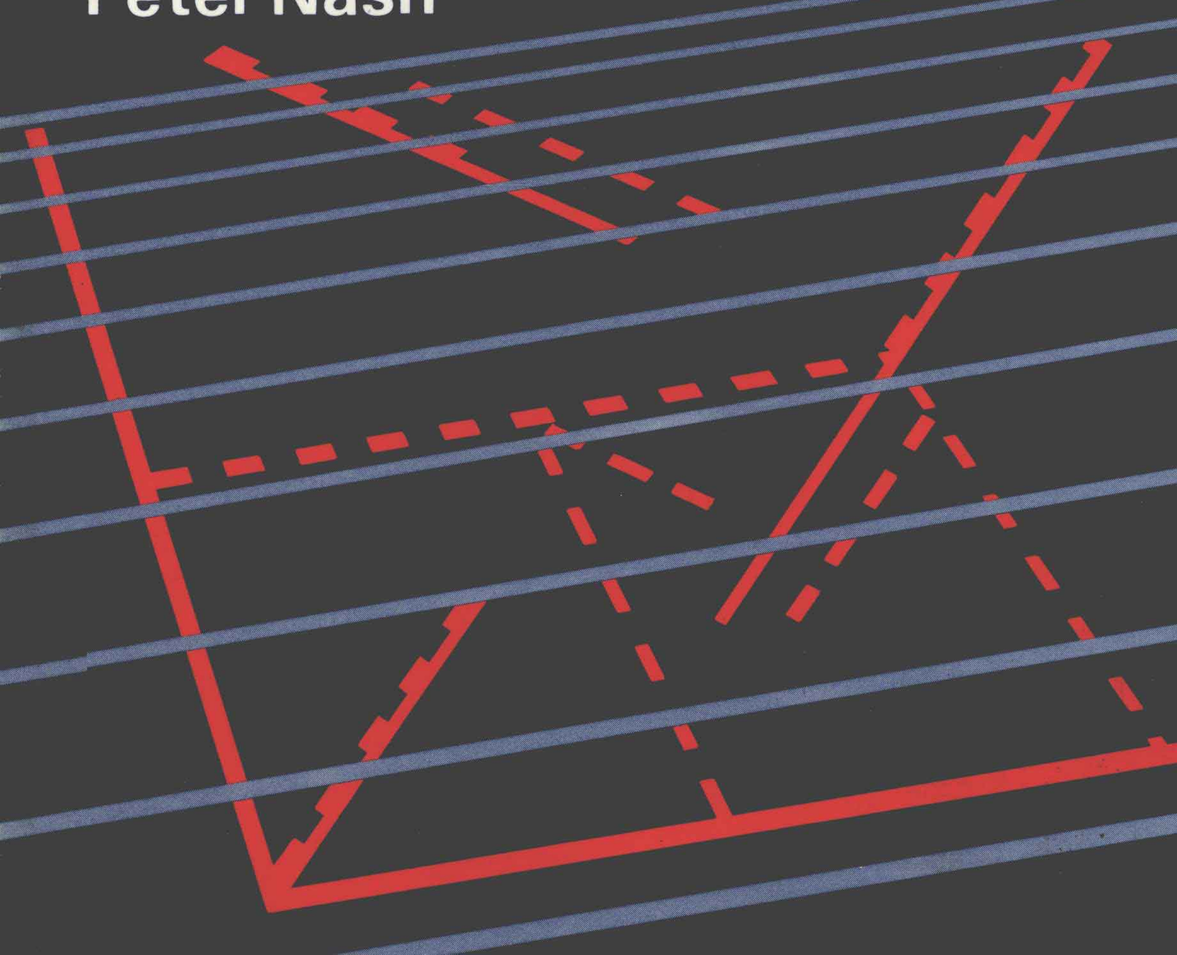

LINEAR PROGRAMMING IN INFINITE-DIMENSIONAL SPACES

Theory and Applications

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Preface

A linear program is an optimization problem with linear objective function and linear constraints. Problems of this type are of central importance in the theory and practice of optimization, for several reasons. Principally, the theory of linear programming in finite-dimensional spaces is both elegant and complete, an appealing embodiment of Lagrangian duality theory. The simplex algorithm and all its variations are very efficient practical solution methods, and provide not just an optimal solution but very useful information about the sensitivity of the solution to variations in the problem data, crucial when these data are known only imprecisely. In consequence, it is often found worthwhile to make linear models of non-linear real-world problems, despite the loss of accuracy that this involves.

In recent years, there has been a movement towards the integration of a number of seemingly different areas of optimization theory through the study of optimization problems posed in an abstract setting, capable of encompassing both finite- and infinite-dimensional problems. A landmark in this area is the book *Optimization by Vector Space Methods*, by D. G. Luenberger, published in 1969. This book is a convincing demonstration of the power of an abstract approach, and the way in which it can deepen our understanding of optimization theory.

It is probably inevitable, then, that even without the impetus of applications, linear programming in an abstract setting would have been a subject of study. In fact, infinite-dimensional linear programs arise in a number of applications, and it was from these that the first attempts to extend the finite-dimensional theory sprang. These early attempts were more or less equally concerned with duality theory and with the development of extensions of the simplex algorithm. Latterly, work has tended to concentrate rather more on the former than the latter, although both remain active research areas.

At the time of writing of this book, a fairly extensive theory exists for so-called semi-infinite programming, which is concerned with problems in which either the number of variables or the number of constraints is infinite, but not both. This theory encompasses both duality and solution algorithms. In the wider setting of infinite-dimensional linear programming, no such complete theory exists, and the work on duality theory and that on simplex extensions have not so far been integrated to the same extent. The main objectives of this book are to survey the theory as it now stands, and to attempt as far as possible to present these two areas of work in a way which brings out the connections between them and their implications for each other. In doing this we treat a number of interesting problems arising from very disparate areas of mathematics.

We are particularly concerned to try to elucidate the way in which the properties of linear programs and the workings of simplex-like algorithms depend on the underlying properties of the spaces in which the program is set. Accordingly, the book begins with an examination of the sorts of infinite-dimensional problem that arise in applications, and examines what happens when we try to extend the concepts of finite linear programming to them in a naïve way. Numerous difficulties appear, and in Chapter 2 we attempt to resolve these by formulating appropriate definitions of the concepts of basis and degeneracy. In doing this, we postulate a bare minimum of structure in the spaces in which our programs are posed: linear-space structure, a partial order, a linear objective functional and a linear constraint map. These suffice for some sort of strong duality result for general problems, though not necessarily of practical value. More useful strong duality results are derived in Chapter 3, where we make assumptions about the topological structures of the problem spaces, and use functional analytic methods to establish the existence of optimal dual solutions.

Chapter 4 contains a brief treatment of semi-infinite programming. This by its nature lies half way between finite LP and the fully infinite problems discussed in the following chapters. In Chapters 5–7, three particular classes of infinite-dimensional linear programs are examined in some detail. For each, the duality theory is developed, illustrating the theory described in Chapters 2 and 3, and used in the formulation of solution algorithms. These classes of programs are infinite-dimensional analogues of the assignment problem, the maximal flow in a network problem, and a problem related to the minimum-cost multi-commodity network flow problem, which we have called the separated continuous linear program. This progression is an attempt to go from problems with a lot of structure, and work towards the more general. The final chapter of the book gives a very brief discussion of four other types of infinite-dimensional linear program.

Our interest in infinite-dimensional linear programs has grown out of work on algorithms for the solution of some special classes of infinite-dimensional linear programs. This has meant that our concern is with linear programming rather than just linear programs. However, we have chosen not to include descriptions of detailed numerical methods.

This book could be used as the text for a course of lectures on infinite-dimensional LP, or in conjunction with other books in a more general course on optimization in general vector spaces. An early draft was in fact used for the first purpose here in Cambridge. The prerequisites for the book are an exposure to optimization theory in R^n (and in particular, linear programming) and elementary convex analysis, linear algebra and functional analysis.

Much of this book, especially the sections dealing with algorithms, has come out of work done in Cambridge over the last four years. In particular, Chapters 5 and 6 owe a great deal to Dr. A. B. Philpott, who proved many of the results given there and was responsible for a large part of the formulation of the algorithms. One of us (P. N.) would like to acknowledge the support provided by the Allen Clark Research Fellowship at Churchill College. Both of us are grateful for the facilities and the stimulating environment provided by the Management Studies Group of Cambridge University Engineering Department.

Eddie Anderson
Peter Nash

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1

Infinite-dimensional Linear Programs

1.1 INTRODUCTION

A linear program is an optimization problem with a linear objective functional and linear constraints. In 1947, G. B. Dantzig discovered the simplex method for the solution of such problems, and since then the method has become perhaps the most widely used of all optimization techniques. Two main factors seem to have contributed to this popularity. On the one hand, the simplex method is observed to be very efficient in practice. On the other hand, the method provides a very complete solution to the problem. As well as the optimal values of the decision variables, the output of the simplex algorithm includes much information about the sensitivity of the optimal solution to changes in the problem data. This information can be very useful when the parameters of the problem are known only imprecisely. These factors have made it often worthwhile to accept the inaccuracies involved in using a linear model of a non-linear real-world problem.

Until comparatively recently, a complete theory of linear programming existed only for problems involving a finite number of decision variables subject to a finite number of constraints. We shall refer to such a problem as a finite or finite-dimensional LP. Formally, it is posed as

$$\begin{aligned} \text{FLP:}\quad &\text{minimize} && c^{\mathrm{T}}x \\ &\text{subject to} && Ax = b, \\ & && x \in R^n, \qquad x \geqslant 0, \end{aligned}$$

where $c \in R^n$, $b \in R^m$ and the $m \times n$ matrix A are given. The positivity constraint $x \geqslant 0$ is interpreted as meaning $x_i \geqslant 0$, $i = 1, 2, \ldots, n$.

It is difficult to date the first attempts to extend the theory of linear programming to more general settings, but an early example of a linear-programming problem posed other than in R^n was due to Bellman (1957), who examined a particular linear optimal control problem arising from a model of a production system. The control problem was posed as a linear program in a space of functions of a continuous-time variable. Such problems have come to be called continuous-time linear programs, and have been the object of considerable study, along with many other linear programs posed in a variety of abstract vector spaces. Our object here is to examine the theory of linear programming in this more general sense, and particularly as it relates to the extension to infinite-dimensional problems of the simplex and related algorithms.

Accordingly, this book is set out as follows. We begin by looking, in the remainder of this chapter, at some problems which can be posed as infinite-dimensional linear programs. We then examine briefly the elements of the simplex algorithm for FLP, and formulate some questions which must be addressed in any attempt to extend to the infinite-dimensional case the concepts which underlie the simplex algorithm. Examples demonstrate some of the difficulties involved in making these extensions. The resolution of some of these difficulties is the aim of Chapters 2 and 3. The remaining chapters of the book are then devoted to working out the theory and deriving algorithms for a number of particular classes of infinite-dimensional problems.

1.2 SOME INFINITE LINEAR PROGRAMS

To motivate our study, we examine here some problems which are naturally modelled as infinite linear programs.

1.2.1 The bottleneck problem (Bellman (1957))

In an economy, n different goods, $G_1, G_2, \ldots, G_n$, are produced by m different types of plant or production facility, $P_1, P_2, \ldots, P_m$. At the beginning of a five-year plan, there is available a certain capacity in each of these types of plant, and more can be made by re-investing the goods produced. The aim of the plan is to maximize the productive capacity at the end of the period.

Let $x_i(t)$, $i = 1, 2, \ldots, m$, denote the rate of production of new capacity of type i at time t. Production of new plant requires the consumption of a certain quantity b_{ij} of good G_i for each additional unit of plant P_j. Thus the amounts of goods consumed in this way are given by $Bx(t)$, where $x(t)$ is the vector with components $x_1(t), x_2(t), \ldots, x_m(t)$ and B is the matrix whose i, jth element is b_{ij}. Let $z_i(t)$ denote the total productive capacity of type i available at time t. Denote by d_{ij} the rate of production of G_i for each unit of plant P_j. Then the total rates of production of goods at time t are given by $Dz(t)$, where D is the matrix whose i, jth element is d_{ij} and $z(t)$ is the vector $(z_1(t), z_2(t), \ldots, z_m(t))^{\mathrm{T}}$.

The constraint on investment in additional plant due to limitations in productive capacity is then given by

$$Bx(t) \leqslant Dz(t)$$

throughout the time period under consideration. If c_0 is the vector of initial productive capacities, we can write

$$z(t) = c_0 + \int_0^t x(\tau)\,\mathrm{d}\tau, \tag{1}$$

and hence

$$Bx(t) - \int_0^t Dx(\tau)\,d\tau \leqslant c, \tag{2}$$

where $c = Dc_0$. If we wish to maximize a weighted sum, $\Sigma a_i z_i(T)$, of the production capacities at the end of the time period, then we obtain the following linear program

$$\begin{aligned} \text{BP:}\quad \text{maximize} \quad & \int_0^T a^{\mathrm{T}} x(t)\,dt \\ \text{subject to} \quad & Bx(t) - \int_0^t Dx(\tau)\,d\tau \leqslant c, \\ & x(t) \geqslant 0, \qquad t \in [0, T]. \end{aligned}$$

The decision variables here are the functions x_i. If we wish to maximize a weighted sum of the total production of the various goods over the whole time interval, then the form of the problem is very similar. An integration by parts is necessary and the vector a in the objective function is replaced by a function of time of the form $(1-t)\,D^{\mathrm{T}}a$.

This type of problem is called a continuous or continuous-time linear program. Various extensions to the basic form of the problem posed here have been studied, particularly the case where a, c, B and D vary with time. The name *bottleneck problem* refers to the constraint on production capacity.

1.2.2 Continuous-time network flow

A classical problem in finite linear programming is that of maximizing the flow of some commodity between two specified nodes (the *source* and the *sink*) in a transportation network whose arcs are subject to capacity limitations. When the capacity limitations do not vary with time the problem is solved by choosing a single set of constant flows, and this can be done by means of a simple algorithm due to Ford and Fulkerson (1962). This algorithm relies on an elegant duality theorem which is available for this particular problem. The duality theorem tells us that the problem of maximizing the flow in the network is equivalent to that of partitioning the network into two sets of nodes, one containing the source and the other containing the sink, in such a way as to minimize the total capacity of those arcs connecting nodes associated with the source to nodes associated with the sink. Such a partition is called a *cut*, and the duality theorem is sometimes called the *maximum-flow minimum-cut theorem*.

Infinite-dimensional versions of this problem arise in a number of contexts. Perhaps the most obvious is the case where the capacities of the arcs vary with time, and there is the possibility of storage at the nodes of the network. Consider, for example, a system of n reservoirs $R_1, R_2, \ldots, R_n$ from which a single, time-varying demand has to be met during some time period $[0, T]$. Suppose that the capacity of R_i is c_i, $i = 1, 2, \ldots, n$. Water flows into R_i at time t at a rate $r_i(t)$, and the demand at time t is $d(t)$. The maximum rate at which water can be fed from R_i to meet demand is f_i, a constant. If more water flows into any reservoir than it can hold, the rest is spilled to waste, and water may not be fed back into reservoirs. Subject to these constraints, we wish to meet as much as possible of the demand during the time interval $[0, T]$.

Let $x_i(t)$ denote the rate at which water is fed from R_i at time t. The feeder constraints can then be expressed

$$0 \leqslant x_i(t) \leqslant f_i, \qquad t \in [0, T], \quad i = 1, 2, \ldots, n. \tag{3}$$

Let $w_i(t)$ denote the rate of spillage from R_i at time t. The storage constraints are

$$0 \leqslant \int_0^t [r_i(\tau) - x_i(\tau) - w_i(\tau)]\, d\tau + s_i \leqslant c_i, \qquad t \in [0, T], \quad i = 1, 2, \ldots, n, \tag{4}$$

where s_i is the amount of water stored in R_i initially. The reservoir control problem can then be stated as

$$\text{maximize} \quad \int_0^T \left[\sum_{i=1}^{n} x_i(t) \right] dt$$

subject to (3) and (4), and the further constraints that the total rate of feed is no greater than the demand, that is

$$\sum_{i=1}^{n} x_i(t) \leqslant d(t). \qquad t \in [0, T],$$

and that

$$w_i(t) \geqslant 0, \qquad t \in [0, T], \quad i = 1, 2, \ldots, n.$$

This problem is obviously a special case of the bottleneck problem of the previous section. To see how it can be posed as a continuous-time network-flow problem, consider the capacitated network $\Omega(t)$ shown in Figure 1.1. This has a storage node representing each reservoir. An arc of capacity f_i connects R_i to a single node D_2 of zero storage capacity, which is in turn connected to the sink B_t by an arc of capacity $d(t)$. Conservation of flow at this zero-storage node ensures that the total feed is no greater than $d(t)$. Inflows are represented by an infinite-capacity arc connecting the source A_t to the infinite-storage node D_1, which is connected by arcs of capacity $r_1(t), r_2(t), \ldots, r_n(t)$ to the reservoirs. Spillage is represented by flows in infinite capacity arcs connecting the reservoirs to D_1.

Continuous-time network flow problems in which there is no storage are an essentially straightforward extension of the static case. The solution to such a problem is obtained by maximizing the instantaneous flow at each time. Assuming that the problem data are well enough behaved to allow us to choose these instantaneous flows appropriately, then the maximum flow over a time interval is just the integral of the instantaneous maxima. Of course, the problem cannot be solved like this in practice, as the number of flow maximizations involved is infinite. In reality, one uses parametric programming techniques to

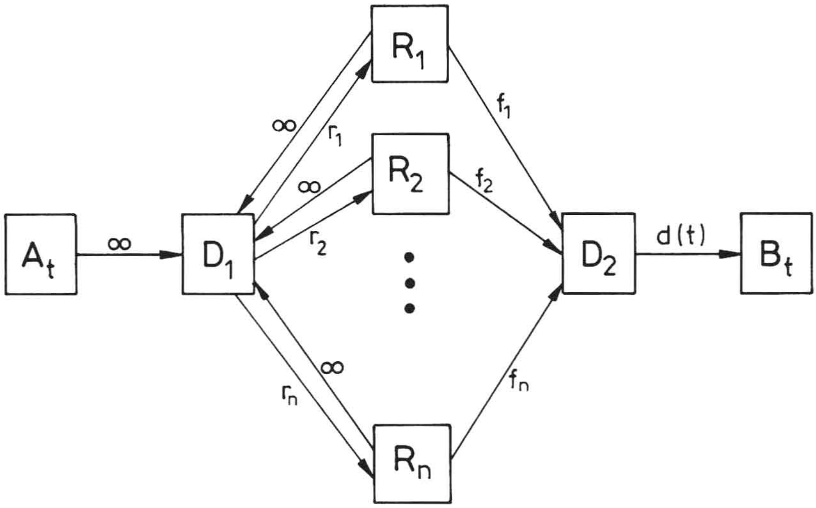

Figure 1.1 The network $\Omega(t)$ for the reservoir control problem

find the times when the maximum-flow regime changes, and relies on the good behaviour of the problem data to ensure that these are only finite in number.

The presence of non-zero storage capacities complicates the problem a good deal. The effect of storage is to link flows at different times. In a discrete-time formulation, this can be dealt with by considering a number of copies (one per time epoch) of the basic network, with directed arcs to represent storage capacity linking successive copies of each node at which storage is allowed. The result of this construction is an augmented network for which the maximum flow can be computed by the Ford–Fulkerson algorithm, and this provides the solution to the original problem.

One approach to the continuous-time problem is to approximate it using a discrete-time formulation. While this is clearly a workable method, the augmented network will be very large if a fine discretization is needed. Moreover, one feels that the discretization may hide essential structural features of the solution, and it is natural to ask whether a treatment in continuous time is possible. We shall see in Chapter 6 that the Ford–Fulkerson algorithm can be extended to the continuous-time problem.

1.2.3 Cutting and filling

A road built on undulating terrain does not usually follow exactly the contours of the land, but is supported on embankments and run through cuttings, so as to reduce the number and severity of the gradients. In building the road, large masses of earth have to be moved to create the cuttings and embankments. It is advantageous if the total amount of earth needed for embankments is roughly equal to the total amount removed from cuttings, to avoid large-scale importation or dumping of earth. Even when this is so, it is necessary to plan the movements of earth to minimize their total cost.

Suppose we have a stretch of road from A to B. Let s denote distance along the route from A, and let $\Psi(s)$ be the planned elevation of the road above the terrain. Let $c(s, s')$ denote the unit cost of moving earth between points at distances s and s' along the route from A. Let $\pi(s, s')\,\mathrm{d}s\,\mathrm{d}s'$ be the quantity of earth moved from within the section $(s, s+\mathrm{d}s)$ to within the section $(s', s'+\mathrm{d}s')$. Then the total cost of moving earth is $\int\int c(s, s')\pi(s, s')\,\mathrm{d}s\,\mathrm{d}s'$. This has to be minimized, subject to

$$\int \pi(s, s')\,\mathrm{d}s = \Psi_1(s') \qquad \text{for all } s',$$

$$\int \pi(s, s')\,\mathrm{d}s' = \Psi_2(s) \qquad \text{for all } s,$$

$$\pi(s, s') \geqslant 0 \qquad \text{for all } s, s',$$

where

$$\Psi_1(s) = \max\{\Psi(s), 0\},$$

$$\Psi_2(s) = \max\{-\Psi(s), 0\}.$$

Similar 'cutting and filling' problems arise in levelling a plot of land, or moving a mass of earth in three dimensions to form an earthwork. The study of such problems has a long history, actually initiated in the study of military operations involving earthworks. As is apparent, the problem is an infinite-dimensional version of the well-known transportation problem of linear programming.

We shall give a full description of this problem in Chapter 5. Here we just note that the problem in more general form is

$$\begin{aligned} &\text{minimize} && \int_{X\times Y} c(x, y)\,\mathrm{d}\pi \\ &\text{subject to} && P_1\pi = \rho_1, \\ & && P_2\pi = \rho_2, \\ & && \pi \geqslant 0, \end{aligned}$$

where ρ_1 and ρ_2 are prescribed measures on two sets X and Y; π is a measure on $X\times Y$, to be found; P_1 and P_2 are the operators which project measures on $X\times Y$ onto measures on X and Y respectively; and $c(x, y)$ is a given continuous function on $X\times Y$. The form of the problem stated originally occurs when π is an absolutely continuous measure, so that for any measurable set $S\subset X\times Y$

$$\pi(S) = \int_S \pi(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$