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# PLOTTING GRAPHS

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*На английском языке*

ИЗДАТЕЛЬСТВО МИР

ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Г. Е. Шилов

# КАК СТРОИТЬ ГРАФИКИ

Издательство «Наука» Москва

*The graph of the sine, wave after wave,*
*Flows along the axis of abscissas...*
(FROM STUDENT LORE)

It would be hard to find a field of science or social life where graphs are not used. We have all often seen graphs, for example, of industrial growth or increase in labour productivity. Natural phenomena like daily or annual fluctuations of temperature or atmospheric pressure are also easier to describe by means of graphs. The plotting of graphs of that kind presents no difficulty, provided the appropriate tables have been compiled. But we shall be dealing here with graphs of a different kind, with graphs that must be plotted from given mathematical formulas. A need for such graphs often arises in various fields of knowledge. Thus, in analysing the theoretical course of some physical process, a scientist obtains a formula yielding some magnitude with which he is concerned, for example, the amount of product obtained relative to time. The graph plotted from this formula will provide a clear picture of the future process. Looking at it, the scientist may possibly introduce substantial changes into the scheme of his experiment in order to obtain better results.

In this booklet we shall consider some simple methods of plotting graphs from given formulas.

Let us draw two mutually perpendicular lines on a plane, one horizontal and the other vertical, denoting their intersection by $O$. We shall call the horizontal line the *axis of abscissas* and the vertical the *axis of ordinates*. Each axis will be divided by the point $O$ into two semi-axes, positive and negative, the right half of the axis of abscissas and the upper half of the axis of ordinates being taken as positive, while the left half of the axis of abscissas and the lower half of the axis of ordinates being taken as negative. Let us mark the positive semi-axes by arrows. The position of any point $M$ on the plane can now be determined by a pair of numbers. To find them we drop perpendiculars from $M$ to each of the axes; these perpendiculars intercept segments $OA$ and $OB$ (Fig. 1) on the axes. The length of segment $OA$, taken with a plus sign $+$ when $A$ is on the positive semi-axis and with a minus sign $-$ when it is on the negative semi-axis, we call the *abscissa* of point $M$ and denote by $x$. Similarly, the length of segment $OB$ (with the same sign rule) we call the *ordinate* of point $M$ and denote by $y$. The two numbers $x$ and $y$ are called the *coordinates* of point $M$. Every point on the plane has certain coordinates. The points on the axis of abscissas have zero ordinates and the points on the axis of ordinates have zero abscissas. The origin of the coordinates $O$ (the point of intersection of the

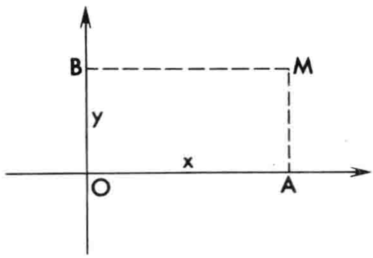


Fig. 1

axes) has both coordinates equal to zero. Conversely, when we are given any two numbers $x$ and $y$ (negative or positive), we can always plot a point $M$ with abscissa $x$ and ordinate $y$; to do that we measure off a segment $OA = x$ on the axis of abscissas and draw a perpendicular $AM = y$ at point $A$ (bearing in mind the sign); $M$ will be the point we want.

Suppose we are given a formula for which we are to plot the graph. The formula must indicate what operations must be carried out on the independent variable (denoted by $x$) in order to obtain the value we need (denoted by $y$). For example, the formula

$$y = \frac{1}{1 + x^2}$$

indicates that the value of $y$ can be obtained by squaring the independent variable $x$, adding it to unity and dividing unity by the result. If $x$ assumes some numerical value $x_0$, then in accordance with our formula $y$ will assume some numerical value $y_0$. The numbers $x_0$ and $y_0$ will determine a certain point $M_0$ in the plane of our picture. Instead of $x_0$ we can take some other number $x_1$ and use the formula to calculate a new value $y_1$; the pair of numbers $(x_1, y_1)$ will define a new point $M_1$ on the plane. The locus of *all* points whose ordinates are related to their abscissas by the given formula is called the *graph* of this formula.

The set of points on a graph, generally speaking, is infinite, and we cannot hope in fact to plot them all without exception according to the given rule. But we can manage without it. In most cases it is enough to know a small number of points in order to be able to judge the general form of the graph.

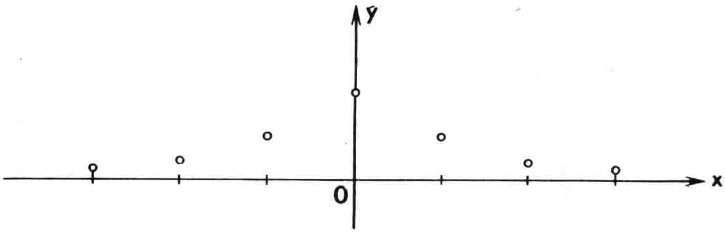


Fig. 2

The point-by-point plotting of a graph consists simply in plotting a certain number of points of the graph and then joining them (as far as possible) by a smooth curve.

As an example, let us consider the graph of the function

$$y = \frac{1}{1 + x^2} \tag{1}$$

Let us compile the following table:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | $-1$ | $-2$ | $-3$ |
| $y$ | 1 | $1/2$ | $1/5$ | $1/10$ | $1/2$ | $1/5$ | $1/10$ |

In the first line we have written down values for $x = 0$, 1, 2, 3, $-1$, $-2$, $-3$. We usually take integral numbers for $x$ because they are easier to operate with. In the second line we have

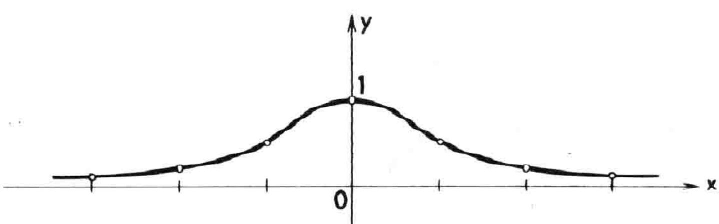


Fig. 3

written the corresponding values for $y$ obtained from formula (1). Let us plot the corresponding points on the plane (Fig. 2). Joining them by a smooth line, we obtain the graph (Fig. 3).

The point-by-point plotting, as we see, is extremely simple and requires no 'theory'. But, perhaps just because of that, we can make bad mistakes by blindly following it.

Let us plot the curve given by the equation

$$y = \frac{1}{(3x^2 - 1)^2} \tag{2}$$

using this rule.

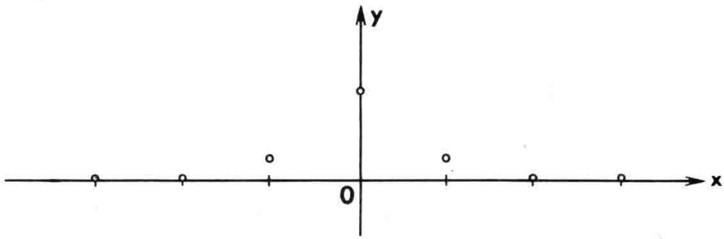


Fig. 4

The table of values for $x$ and $y$ corresponding to this equation is as follows:

| $x$ | 0 | 1 | 2 | 3 | $-1$ | $-2$ | $-3$ |
|---|---|---|---|---|---|---|---|
| $y$ | 1 | $^1/_4$ | $^1/_{121}$ | $^1/_{676}$ | $^1/_4$ | $^1/_{121}$ | $^1/_{676}$ |

The corresponding points on the plane are shown in Fig. 4. The outline looks very like the previous one; joining the points by a smooth line we obtain a graph (Fig. 5). Now it seems

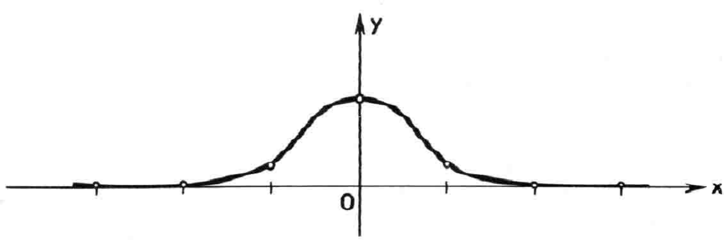


Fig. 5

we can put down our pens and take a deep breath; we have mastered the art of plotting graphs! But just in case, as a check, let us calculate $y$ for some intermediate value of $x$, say $x = 0.5$. We get a surprising result: for $x = 0.5$, $y = 16$. This sharply disagrees with our picture. And there is no guarantee that in calculating

$y$ for other intermediate values of $x$ – and there is an infinite number of such values – we would not obtain even more striking incongruities. Apparently the point-by-point plotting of a graph itself is not well-founded.

*

* *

Now we shall consider another method of plotting graphs, more reliable since it protects us from those unexpected things we have just come across. This method – let us call it 'plotting by operations' – consists in carrying out directly on the graph all the operations which are written down in the given formula: addition, subtraction, multiplication, division, etc.

Let us consider some of the simplest examples. We plot the graph corresponding to the equation

$$y = x \tag{3}$$

This equation shows that all the points of the sought line on

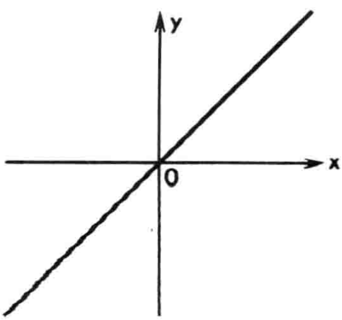


Fig. 6

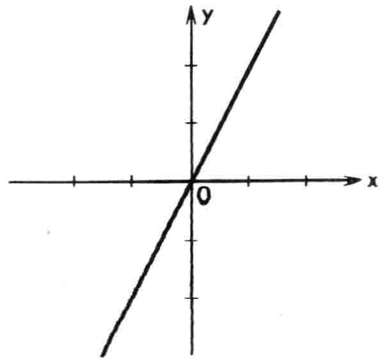


Fig. 7

the graph have equal abscissas and ordinates. The locus of points having an ordinate equal to an abscissa is the bisector of the angle formed by the positive semi-axes and the angle formed by the negative semi-axes (Fig. 6). The graph corresponding to the equation

$$y = kx$$

with some coefficient $k$ is obtained from the previous one by multiplying each ordinate by the same number $k$. Suppose, for example, $k = 2$; each ordinate of the previous graph must be doubled, and, as a result, we obtain a straight line more steeply rising upwards (Fig. 7). Each step to the right along the $x$-axis corresponds to two steps along the $y$-axis. Incidentally, it is very convenient to plot such graphs on squared or graph paper. In the general case we will also get a straight line in the equation $y = kx$. If $k > 0$, the line, with each step to the right, will move up $k$ steps along the $y$-axis. If $k < 0$, the line will slope down instead.

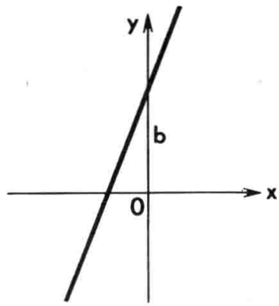


Fig. 8

Now let us consider a somewhat more complicated formula

$$y = kx + b \tag{4}$$

To plot a corresponding graph, we must add one and the same number $b$ to each ordinate of the already known line $y = kx$. As a result, the line $y = kx$ will move up, as a whole, by $b$ units if $b > 0$; if $b < 0$, the original line will, of course, move down instead. We will thus obtain a straight line parallel to the initial one, but no longer passing through the origin of coordinates and intercepting the segment $b$ on the axis of ordinates (Fig. 8).

Therefore, the graph of any polynomial of degree one in $x$ is a straight line which can be plotted according to the above rules.

Now let us consider the graph of polynomials of degree two. Let us consider the formula

$$y = x^2 \tag{5}$$

It can be represented in the form

$$y = y_1^2$$

where

$$y_1 = x$$

In other words, we will obtain the sought graph by squaring each ordinate of the known line $y = x$. Let us see what we get in that way.

Since $0^2 = 0$, $1^2 = 1$, $(-1)^2 = 1$, we have obtained three basic points $A$, $B$ and $C$ (Fig. 9). When $x > 1$, $x^2 > x$; therefore, to the right of point $B$, the graph will lie above the bisector of the quadrant angle (Fig. 10). When $0 < x < 1$, $0 < x^2 < x$; therefore, between points $A$ and $B$ the graph will lie below the bisector.

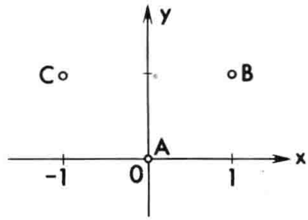


Fig. 9

Moreover, we claim that when we approach point $A$, the graph will be contained in any angle bounded from above by the straight line $y = kx$ (with an arbitrarily small $k$), and from below by the $x$-axis; indeed, the inequality

$$x^2 < kx$$

is satisfied only if $x < k$. This fact means that our curve is *tangent* to the axis of abscissas at point $O$ (Fig. 11). Now let

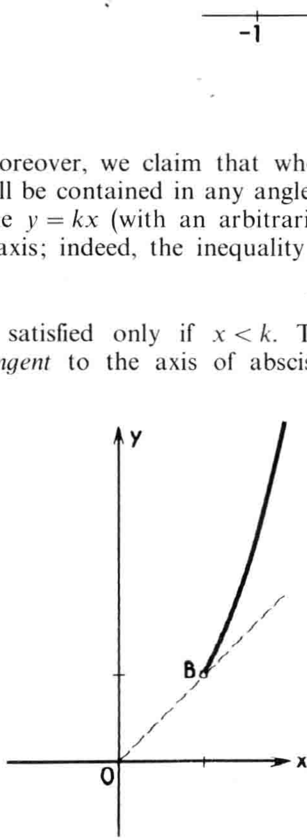


Fig. 10

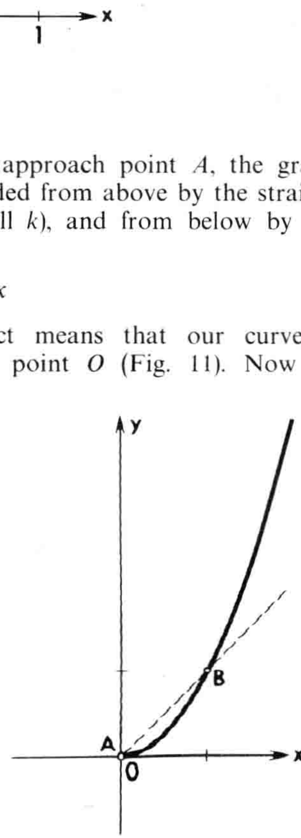


Fig. 11

us move along the $x$-axis to the left of point $O$. We know that the numbers $-a$ and $+a$ give the same square $(+a^2)$. Thus the ordinate of our curve for $x = -a$ will be the same as for $x = +a$. Geometrically this means that the graph of the curve in the left half-plane will be a mirror image of the already plotted graph in the right half-plane with respect to the $y$-axis. We have obtained a curve called a *parabola* (Fig. 12).

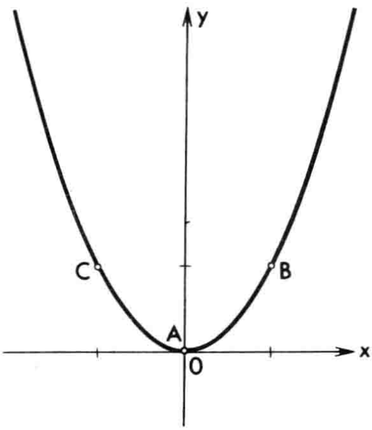


Fig. 12

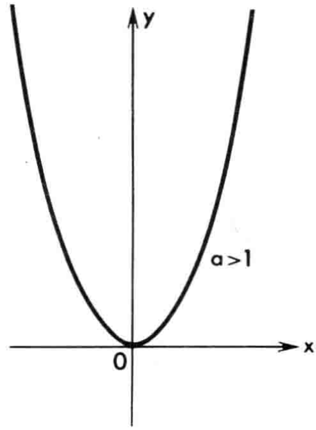


Fig. 13

Now by the same method we can plot a more complicated curve

$$y = ax^2 \tag{6}$$

and an even more complicated one

$$y = ax^2 + b \tag{7}$$

The first is obtained by multiplying all the ordinates of parabola (5), which we will call the *standard parabola*, by the number $a$.

For $a > 1$ we get a similar curve but more steeply rising upwards (Fig. 13). For $0 < a < 1$ the curve will be steeper (Fig. 14), and for $a < 0$ its branches will turn upside down (Fig. 15). Curve (7) is obtained from curve (6) by moving it up by a distance $b$ if $b > 0$ (Fig. 16). But if $b < 0$, we will have to move the curve down instead (Fig. 17). All these curves are also called parabolas.

Now let us consider a somewhat more complicated example of plotting a graph by multiplication. Suppose we are to plot the graph according to the equation

$$y = x(x-1)(x-2)(x-3) \tag{8}$$

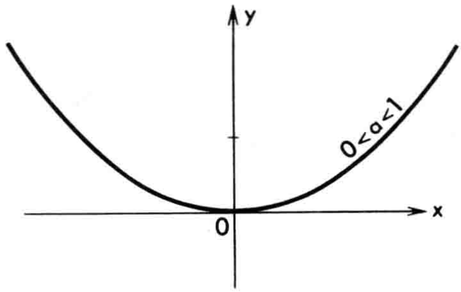


Fig. 14

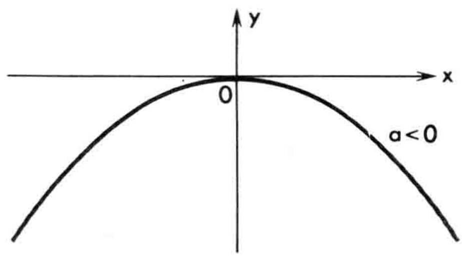


Fig. 15

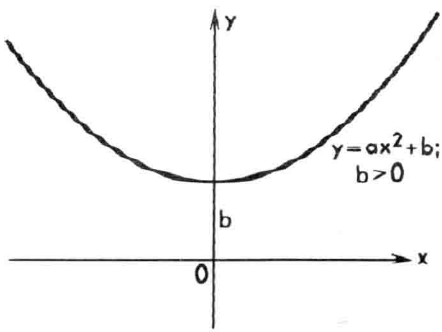


Fig. 16

Here we are given the product of four factors. Let us plot the graph of each of them separately: they are all straight lines, parallel to the bisector of the quadrant angle and intercepting the $y$-axis at points *0*, *−1*, *−2*, *−3*, respectively (Fig. 18). At points *0*, *1*, *2*, *3*, on the $x$-axis, our curve will have a zero ordinate, since a product is equal to zero if at least one of the factors is zero. At other points the product will be non-zero

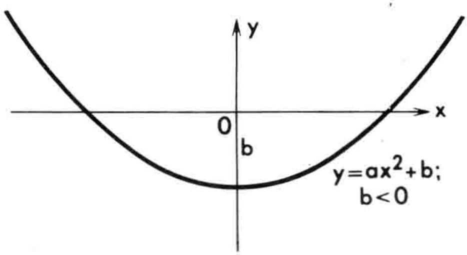


Fig. 17

and will have a sign which can easily be found from the signs of the factors. Thus, to the right of point *3* all the factors are positive, therefore, so is their product. Between points *2* and *3* one factor is negative, therefore, so is the product. Between points *1* and *2* there are two negative factors, so the product is positive, etc. We obtain the following distribution of signs of the product (Fig. 19). To the right of point *3* all the factors increase with


Fig. 18

$x$, therefore, the product will also increase and very rapidly. To the left of point $O$ all the factors increase in the negative direction, so the product (which is positive) will also rapidly increase.

Now it is easy to sketch the general form of the graph (Fig. 20).

So far we have used the operations of addition and multiplication. Now let us consider division. Let us plot the curve

$$y = \frac{1}{1 + x^2} \tag{9}$$

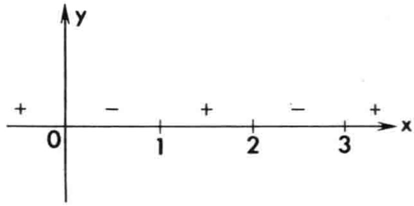


Fig. 19

To do that we separately plot the graphs of the numerator and the denominator. The graph of the numerator

$$y_1 = 1$$

is a straight line parallel to the $x$-axis and passing through unity. The graph of the denominator

$$y_2 = x^2 + 1$$

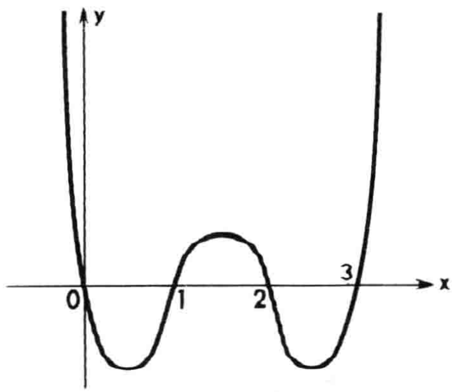


Fig. 20

is a standard parabola, moved upwards by unity. Both of these graphs are shown in Fig. 21.

Now we will carry out the division of each ordinate of the numerator by the corresponding (i. e. taken for the same $x$) ordinate of the denominator. When $x = 0$, we see that $y_1 = y_2 = 1$, which yields $y = 1$. When $x \neq 0$, the numerator is less than the denominator, so that the quotient is less than unity. Since the numerator and the denominator are positive everywhere, the quotient is positive, and, therefore, the graph is contained in the strip bounded by the $x$-axis and the line $y = 1$. When $x$ infinitely increases, the denominator also grows infinitely, whereas the numerator remains constant; therefore, the quotient tends to zero. All this gives the following graph of the quotient (Fig. 22). We have obtained the same picture as the one plotted previously by the point-by-point plotting of a graph (p. 7).

When carrying out division on the graph, particular attention should be paid to those values for $x$ at which the denominator becomes zero. If the numerator is non-zero at this point, the quotient becomes infinite. For example, let us plot the curve

$$y = \frac{1}{x} \tag{10}$$

Here, the graphs of the numerator and the denominator are already known (Fig. 23). For $x = 1$ we have $y_1 = y_2 = 1$, which yields $y = 1$. When $x > 1$, the numerator is less than the denominator and the quotient is less than unity as in the previous example. When $x$ increases infinitely, the quotient tends to zero and we obtain the part of the graph which corresponds to the values $x > 1$ (Fig. 24).

Let us consider now the values for $x$ between 0 and 1. When $x$ moves from 1 to 0, the denominator tends to zero, while the numerator remains equal to unity. Therefore, the quotient increases infinitely and we obtain the branch moving up to infinity (Fig. 25). When $x < 0$, the denominator as well as the whole fraction become negative. The general form of the graph is presented in Fig. 26. Now we can already start to plot the graph discussed on page 8:

$$y = \frac{1}{(3x^2 - 1)^2} \tag{11}$$

Let us first plot the graph of the denominator. The curve $y_1 = 3x^2$ is a 'triple' standard parabola (Fig. 27). Subtracting unity we move the graph down by unity (Fig. 28). The curve intersects the $x$-axis at two points which will be easily found by setting