Lecture Notes in Mathematics 1541

D.A. Dawson B. Maisonneuve J. Spencer

Ecole d´ Eté de Probabilités de Saint-Flour XXI - 1991

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Editorial Policy

§ 1. Lecture Notes aim to report new developments - quickly, informally, and at a high level. The texts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes manuscripts from journal articles which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character. For similar reasons it is unusual for Ph. D. theses to be accepted for the Lecture Notes series.

§ 2. Manuscripts or plans for Lecture Notes volumes should be submitted (preferably in duplicate) either to one of the series editors or to Springer- Verlag, Heidelberg . These proposals are then refereed. A final decision concerning publication can only be made on the basis of the complete manuscript, but a preliminary decision can often be based on partial information: a fairly detailed outline describing the planned contents of each chapter, and an indication of the estimated length, a bibliography, and one or two sample chapters - or a first draft of the manuscript. The editors will try to make the preliminary decision as definite as they can on the basis of the available information.

§ 3. Final manuscripts should preferably be in English. They should contain at least 100 pages of scientific text and should include

- a table of contents;
- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

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INTRODUCTION

Ce volume contient les cours donnés à l'Ecole d'Eté de Calcul des Probabilités de Saint-Flour du 18 Août au 4 Septembre 1991.

Nous remercions les auteurs qui ont effectué un gros travail de rédaction définitive qui fait de leurs cours un texte de référence.

L'Ecole a rassemblé soixante cinq participants dont 33 ont présenté, dans un exposé, leur travail de recherche.

On trouvera ci-dessous la liste des participants et de ces exposés dont un résumé pourra être obtenu sur demande.

Afin de faciliter les recherches concernant les écoles antérieures, nous redonnons ici le numéro du volume des "Lecture Notes" qui leur est consacré :

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MEASURE-VALUED MARKOV PROCESSES

Donald A. DAWSON

MEASURE-VALUED MARKOV PROCESSES

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CONTENTS

1. INTRODUCTION

The central theme of these lectures is the construction and study of measure-valued Markov processes. This subject is in the midst of rapid development and has been stimulated from several different directions including branching processes, population genetics models, interacting particle systems and stochastic partial differential equations. The objective of these notes is to provide an introduction to these different aspects of the subject with some emphasis on their interrelations and also to outline some aspects currently under development. Chapters 1-9 provide an introduction to some of the main ideas and tools in the theory of measure-valued processes. Chapters 10-12 cover topics currently under active development and are primarily intended as an introduction to the growing literature devoted to these aspects of measure-valued processes. Throughout the emphasis is given to outlining the main lines of development rather than attempting a systematic detailed exposition. In Section 1.5 we describe in more detail the structure of these notes (including interrelations between the chapters) as well as mention the principal methods. In the remaining sections of this introduction we outline the roots of the theory of measure-valued Markov processes and the major topics to be discussed in these notes.

By a *measure-valued Markov process* we will always mean a Markov process whose state space is M(E), the space of Radon measures on $(E,\mathcal{E})$, where E is a Polish space and $\mathcal{E} = \mathcal{B}(E)$ is the σ-algebra of Borel subsets of E.

1.1. Particle Systems and their Empirical Measures

Consider a system of N E-valued processes $\{Z_i(t): i=1,\ldots,N,\ t\geq 0\}$. The associated (*normalized*) *empirical measure process* is defined by

$$(1.1.1)\qquad X(t) = N^{-1}\sum_{i=1}^{N} \delta_{Z_i(t)}.$$

We will show in the next chapter that empirical measure processes of type (1.1.1) in which the $\{Z_i\}$ form an exchangeable Markov system are in fact measure-valued Markov processes. Exchangeable particle systems arise naturally in many fields including statistical physics, population biology and genetic algorithms.

In Section 2.10 and Chapter 4 we will proceed to study a related class of measure-valued processes on $\mathbb{R}^d$ which arise from spatially distributed population models in which the number of particles, N(t), at time t, is no longer conserved but in which particles undergo birth and death. In this case we consider atomic measures of the form

$$(1.1.2)\qquad X(t) = m(N(0)) \sum_{i=1}^{N(t)} \delta_{Z_i(t)},$$

where $m(N(0))$ is the mass of each particle.

1.2. Limits of Particle Systems and Stochastic Partial Differential Equations

Non-atomic measure-valued processes arise naturally in the limit, $N \to \infty$, of systems of the form (1.1.1) or in the corresponding *high density* limit of (1.1.2). In many such cases a law of large numbers phenomenon (*propagation of chaos)* occurs and the limiting process is deterministic. For example this occurs in the usual *mean-field* or *McKean-Vlasov limit* (cf. Gärtner (1986), Léonard (1986), Sznitman (1989)). The high density limit $N(0) \longrightarrow \infty$, $m(N(0))\cdot N(0) \longrightarrow m$, of systems of the form (1.1.2) can also give rise to (deterministic) linear or nonlinear partial differential equations including reaction diffusion equations, hydrodynamic equations, etc. (cf. e.g. Oelschläger (1989), De Masi and Presutti (1991), Sznitman (1989)).

The main emphasis in these lectures will be to study such limits when the limiting process is itself random. However we do not consider the frequently studied case of fluctuations around a law of large numbers limit in which normalized and centered sequences are studied (cf. Holley and Stroock (1979)), but rather study the non-centered and therefore *non-negative* limits. For example in the context of symmetrically interacting diffusions *random* McKean-Vlasov limits can arise (cf. Sect. 5.8.1). In the next chapter we will consider sequences of finite exchangeable systems which arise in the study of population genetics and genetic algorithms which converge to a random measure-valued limit.

It should be emphasized that in many applications it is the finite particle systems themselves which are of primary interest. However qualitative properties of the limiting process can often provide insight into the collective behavior of the former. In addition these limiting processes possess rich mathematical structures which are of interest in their own right.

If we begin with the empirical measure of a system of particles in $\mathbb{R}^d$ one possibility is that the limiting measure-valued process has the form $X(t,dx) = \tilde{X}(t,x)dx$, where $\tilde{X}(t,x)$ denotes the *density process* and that $\tilde{X}(t,.)$ belongs to an appropriate linear space, V, of non-negative measurable functions. It would also be reasonable to expect that $\tilde{X}(t,x)$ is described as the solution of a *stochastic* *partial differential equation.* There are a number of different ways to formulate a stochastic partial differential equation (see e.g. Walsh, (1986)). For example the *integral form* of such an equation is given in terms of an $\mathscr{S}'(\mathbb{R}^d)$-valued Wiener process $\{W(t):t\geq 0\}$ with covariance operator Q_0 as follows: for each $\phi\in V^*$, a linear space of test functions in duality with V, with canonical bilinear form $\langle .,. \rangle$ on $V\otimes V^*$,

(1.2.1) $$\langle X(t),\phi\rangle-\langle X(0),\phi\rangle = \int_0^t \langle A(s,X(s)),\phi\rangle ds + \int_0^t \langle B(s,X(s))W(ds),\phi\rangle$$

where for each s $A(s,.):V \longrightarrow V$, and for each $v\in V$ $B(s,v)$ is a linear operator such that the Itô integral $\int_0^t \langle B(s,X(s))W(ds),\phi\rangle$ is well-defined yielding a martingale with increasing process $\int_0^t Q(s,X(s))(\phi,\phi)ds$ where $Q(s,v) := B(s,v)Q_0(s,v)B^*(s,v)$. Under certain natural conditions on $A(s,.)$, $B(s,.)$ and $Q(s,.)$ the solution to equation (1.2.1) is non-negative and hence measure-valued. An important class of stochastic partial differential equations of this form were first developed by Pardoux (1975) and generalized by Krylov and Rozovskii (1981). Equations of this type do occur in some cases including the study of stochastic flows in $\mathbb{R}^d$ (cf. Kunita (1986), Rozovskii (1990)), turbulent flows (cf. Chow (1978)) and random McKean-Vlasov limits. However it turns out that we must also consider more general measure-valued diffusions, that is, *singular* measure-valued as well as *density-valued* processes.

1.3 Some Basic Classes of Measure-valued Processes

To introduce these notions let us consider at a purely formal level the measure-valued analogue, $\{X_t:t\geq 0\}$, of a finite dimensional diffusion process associated with a second order elliptic operator. For functions, F, in an appropriate domain $D(G) \subset bC(M(E))$ (the bounded continuous functions on M(E)), a second order infinite dimensional differential operator would have the form

(1.3.1) $$G(t)F(\mu) = \int_E A(t,\mu,dx)(\delta F(\mu)/\delta\mu(x)) + 1/2\int_E\int_E (\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y))\ Q(t,\mu;dx,dy)$$

where $\frac{\delta F(\mu)}{\delta\mu(x)} := \lim_{\varepsilon\downarrow 0} (F(\mu+\varepsilon\delta_x)-F(\mu))/\varepsilon$, $A(t,.)$ generates a deterministic evolution on $M(E)$, and $Q(t,\mu;dx,dy)$ is a symmetric signed measure on $E\times E$ such that