Lecture Notes in Mathematics 1604

Annette Huber

Mixed Motives and their Realization in Derived Categories

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The reader will see from the mere number of citations how much this monograph owes to the ideas of Deligne, Beilinson and Jannsen. The deep results on Hodge structures and the theory of weights in the l-adic situation are due to Deligne. Without them my construction would break down. From his paper [12] I also take the techniques in working with simplicial varieties.

The idea how to give a good definition of absolute cohomology without a formalism of sheaves is Beilinson's in his construction of absolute Hodge cohomology [3].

On the other hand my definition of Chern classes is built from a sketch in [2]. I got the necessary background on K-theory from Thomason's papers. I am particularly thankful for his patient explanations during a visit in Münster. Proposition 18.1.5 is due to him.

The systematic treatment of mixed realizations was introduced in the book [39] by Jannsen. My work starts from the considerations in his §6. I thank him for his encouraging interest in my work and for a number of useful comments.

I thank Prof. Deligne for pointing out to me a number of misconceptions in an earlier version [18]. I very much appreciate the help of Prof. Scholl who corrected my english grammar. I thank him heartily.

Finally, I am very much indebted to Prof. Deninger. He acquainted me with the theory of motives and their realizations. His interest was the starting point for the project of my research. In particular the sections on 1-extensions would not have been written without his questions. I thank him for constant encouragement and for making the working atmosphere in Münster what it is. G. Kings and M. Schröter were always ready to discuss problems I had. K. Künnemann took on the tedious task of reading the whole manuscript.

Beside the ones mentioned above I use results of whole generations of topologists and geometers. Often the references I give are not the original ones but text books or survey articles. Important contributions of other authors might not be pointed out. I apologize for this.

Annette Huber

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Introduction

Imagine the best of all possible worlds, for us the motivic world. We consider the category $\mathcal{V}$ of varieties over some ground field k ($\mathbb{Q}$ say). For any variety V there is an abelian category $\mathcal{MM}(V)$ of motivic sheaves ("sheaves of motives"). Further there is a triangulated category $D_{\mathcal{MM}(V)}$ equipped with a t-structure with heart $\mathcal{MM}(V)$. It is easiest to think of the derived category of $\mathcal{MM}(V)$ but even the best of all worlds might not be that nice. There is a formalism of six Grothendieck functors i.e. a tensor product on $D_{\mathcal{MM}(V)}$, a notion of duals and for a morphism $f : V \to V'$ of varieties functors

$$Rf^* \quad Rf^! \quad Rf_* \quad Rf_!$$

between the triangulated categories. Motivic sheaves should have further good properties: Any object M of $D_{\mathcal{MM}(V)}$ has a canonical filtration W_* by subobjects, characterized by the property that the graded parts $\mathrm{Gr}_n^W(M)$ are pure of weight n (where "pure sheaf of weight n" has some intrinsic meaning). The functors as well as the weight filtration fulfill a set of properties that I do not want to give in detail. Motivic sheaves would be the universal theory fulfilling this formalism. The category

$$\mathcal{MM} := \mathcal{MM}(\mathrm{Spec}(k))$$

in particular is called the category of mixed motives. Its subcategory of objects whose weight filtration splits should agree with Grothendieck's category of pure motives.

The best of all worlds comes with two special cohomology theories, the geometric and the absolute motivic cohomology. By geometric cohomology I mean the system of objects

$$\underline{\mathrm{H}}^i_{\mathcal{MM}}(V) := R^i_{\mathcal{MM}} s_*(\mathbb{1}_V) \quad \text{in } \mathrm{Ob}(\mathcal{MM}),$$

where s is the structural morphism of the variety V, and $\mathbb{1}_V = s^*\mathbb{1}$ the constant sheaf for the unit object in the Tannaka category of pure motives. We might call $\mathbb{1}_V$ the local system $\mathbb{Q}$ on V. Usually the higher direct image is called the i-motive of the variety V. However, it is important to understand motives as an analogue of sheaves, not of groups. The other cohomology theory, whose definition one might have thought of first, is given by

$$\mathrm{H}^i_{\mathcal{MM}}(V) := R^i\Gamma(V, \mathbb{1}_V) \quad \text{(an abelian group)}.$$

I want to call it the absolute motivic cohomology. By a conjecture of Beilinson et. al. [4], absolute motivic cohomology should agree with the Adams eigen-spaces of algebraic K-theory. The global section functor is defined as usual by

$$\Gamma(V, \,\cdot\,) = \mathrm{Hom}_{\mathcal{MM}(V)}(\mathbb{1}_V, \,\cdot\,) \ .$$

Having a nice formalism of functors, we are going to use it. We have

$$\begin{aligned}
&(1) & R^i\Gamma(V, \mathbb{1}_V) &\overset{\text{Def.}}{=} \mathrm{Ext}^i(\mathbb{1}_V, \mathbb{1}_V)\\
&(2) & &\overset{\text{Yon.}}{=} \mathrm{Hom}_{D_{\mathcal{MM}(V)}}(\mathbb{1}_V, \mathbb{1}_V[i])\\
&(3) & &= \mathrm{Hom}_{D_{\mathcal{MM}(V)}}(s^*\mathbb{1}, \mathbb{1}_V[i])\\
&(4) & &\overset{\text{Adj.}}{=} \mathrm{Hom}_{D_{\mathcal{MM}}}(\mathbb{1}, Rs_*\mathbb{1}_V[i])\\
&(5) & &= \mathrm{Ext}^i(\mathbb{1}, Rs_*\mathbb{1}_V)\\
&(6) & &= R^i\Gamma(\mathbb{1}, Rs_*\mathbb{1}_V) \ .
\end{aligned}$$

The classical expression for this computation in derived categories is the Leray spectral sequence

$$(7) \qquad E_2^{pq} = R^p\Gamma(\mathrm{Spec}(k), R^q s_*\mathbb{1}_V) \ \Rightarrow \ R^{p+q}\Gamma(V, \mathbb{1}_V) \ .$$

Let us go back to the real world. Unfortunately, there is no definition of motivic sheaves. What we do have are their realizations.

The theory of l-adic sheaves has all properties of our motivic world (the t-structure being the perverse one), save universality, of course. What is usually called l-adic cohomology i.e.

$$\mathrm{H}^i_{l-\mathrm{ad.}}(V \times \mathrm{Spec}(\bar{k}), \mathbb{Q}_l)$$

has an additional operation of the absolute Galois group and should rather be interpreted as the sheaf

$$R^i s_*(\mathbb{Q}_l)$$

on the variety $\mathrm{Spec}(k)$. The corresponding absolute cohomology, defined as derived functor of the global section functor, is precisely the continuous étale cohomology of Jannsen [38].

The second example are Saito's Hodge modules [55], [56]. The category of "Hodge sheaves" on $\mathrm{Spec}(k)$ is given by the category of mixed Hodge structures. The geometric cohomology of a variety is its De Rham cohomology, equipped with a Hodge structure following Deligne [11], [12]. The absolute cohomology is Deligne cohomology, also called absolute Hodge cohomology (cf. Beilinson [3]).

These two examples are not independent of each other, but there are strong comparison theorems. In fact, their agreement gives evidence for the existence of the motivic world. And vice versa the belief in the motivic objects leads to definitions and theorems for which surprising parallels hold.

At this stage there seems to be no satisfactory definition of $\mathcal{MM}$. A definition of the subcategory of pure objects is due to Grothendieck, but even in this case important conjectures are open. Progress has been made by Jannsen [41], though. There are also heuristic ideas as to which constructions of algebraic geometry should give non-trivial extensions of pure motives. There are recent important attempts to construct the category of mixed motives by Levine, Lichtenbaum, Voedvodsky, Suslin and others. However, the properties of these categories depend on the open conjectures on algebraic cycles. The smaller category of mixed Tate motives is approached by Bloch and Goncharov. This subcategory seems to be more manageable.

Jannsen and Deligne proposed to make do with an approximation of the category $\mathcal{MM}$. As mixed motives should be a universal (geometric) cohomology theory, we can construct a category $\mathcal{MR}$ of mixed realizations from the known cohomological data by means of linear algebra. The category has the advantage of being well defined and allows a calculus as expected from the category of true motives. The somewhat vague question what an extension in the unknown category of motives might be, becomes the concrete one, whether a given extension of mixed realizations is "motivic". The great hope is of course to find conditions such that the subcategory given by them is isomorphic to the category of true motives.

In [39] Jannsen not only gives a precise definition of the category $\mathcal{MR}$, he also constructs the realizations $\underline{\mathrm{H}}^i_{\mathcal{MR}}(V, \mathbb{1})$ of a variety. In the above vocabulary these are the geometric cohomology objects. In fact, it is rather the other way round: the definition of the category is made to suit the intended definition of the functor.

In this book, I am going to construct the absolute cohomology for these mixed realizations. The global section functor is defined as in the other theories by

$$\Gamma_{\mathcal{MR}} = \mathrm{Hom}_{\mathcal{MR}}(\mathbb{1}, \ \cdot\ )$$

where $\mathbb{1}$ is the unit object in the Tannakian category of mixed realizations. Again geometric and absolute cohomology are to be linked by a Leray spectral sequence. In contrast to the étale situation, I do not have a theory of sheaves at hand, everything only behaves as if there was one. My approach is modelled after the paper [3] and uses the Leray spectral sequence as definition.

This means: First we construct a triangulated category $D_{\mathcal{MR}}$, which is going to take the role of a derived category of $\mathcal{MR}$. Then we define a functor

$$R_{\mathcal{MR}} : \{\text{Varieties}\} \to D_{\mathcal{MR}} \ .$$

$R_{\mathcal{MR}}(V)$ represents the complex $Rs_*(\mathbb{1}_V)$ i.e. the complex which computes the higher direct images of the local system $\mathbb{1}_V$ under the structure map. Note: we are not considering $\mathcal{MR}$-sheaves, only their shadows in the form of the complexes $R_{\mathcal{MR}}$.

As a first test of whether the construction is reasonable, we check that the i-th cohomology object of $R_{\mathcal{MR}}(V)$ (an object of $\mathcal{MR}$) agrees with what was known already as the i-th mixed realization of V. To agree with equation (4) the absolute cohomology has to be defined by

$$\mathrm{H}^i_{\mathcal{MR}}(V,0) := \mathrm{Hom}_{D_{\mathcal{MR}}}(\mathbb{1}, R_{\mathcal{MR}}(V)[i]) \ .$$

The theory is universal in the same sense that Jannsen's mixed motives are, i.e. universal for the (absolute) cohomology theories used in the construction.

The Bloch-Ogus-axioms hold for the absolute cohomology of mixed realizations - in particular cohomology with compact support, homology, cup- and cap-product are defined. Further Chern classes from higher K-theory exist. All these are "universal" in the above sense. Where constructions are possible in one of the realizations, one expects them to be possible in the others and moreover compatible under the comparison morphisms. The absolute cohomology of mixed realizations gives a machine for checking these properties easily. The tedious technical problems are already contained in the definition of the cohomology.

Beilinson's conjectures ([2]) relate the regulators from Deligne cohomology with special values of L-functions. Scholl has reformulated these conjectures in terms of extensions of mixed motives ([59]). Hence we wish to find elements in K-theory whose images under the Chern class maps can be calculated. The Abel-Jacobi map treated in [39] attaches motivic extensions to cycles which are homologically equivalent to zero. We generalize this to higher algebraic K-theory:

Theorem: *If k is an element of $K_p(X_.)$ (where $X_.$ is a simplicial variety) such that the induced map under the Chern class c_j^p*

$$\mathbb{1}(-j) \longrightarrow \underline{\mathrm{H}}^{2j-p}_{\mathcal{MR}}(X_.)$$

vanishes, then a motivic element of the Yoneda group

$$\mathrm{Ext}^1_{\mathcal{MR}}(\mathbb{1}(-j), \underline{\mathrm{H}}^{2j-p-1}_{\mathcal{MR}}(X_.))$$

is induced.

Scholl has constructed similar extensions starting from Bloch's higher Chow groups instead of K-theory ([60]).

The heart of the book is the construction of the category used as a replacement of the derived category of mixed realizations. As it is very technical, I want to sketch the main ideas. We use singular, étale and de Rham cohomology. In each of these theories there are not only isolated cohomology objects but whole complexes to compute them. For example in the case of singular cohomology we might use the complex of singular cochains, and in the l-adic one the complex of l-adic sheaves $Rs_{V*}\mathbb{Q}_l$. There are comparison isomorphisms between the complexes in the derived category. An object of the triangulated category $D_{\mathcal{MR}}$ consists of a tuple of complexes (all coming from different categories) rigidified by comparison morphisms. It is constructed such that it is in a natural way the range of a functor on the category of varieties.

As we want to have natural weight filtrations on our objects, we work consistently with filtered complexes. The weight filtration on the realization of a smooth variety comes from the Leray spectral sequence of a smooth compactification. To deal with the singular case, we use the method of Deligne in [12]. A singular variety behaves for cohomological questions as a certain smooth but simplicial variety does. We make sure that all our constructions are functorial on the level of filtered complexes. This allows to take the total complex of the cosimplicial complex attached to a simplicial variety. So we get a functor on the category of smooth simplicial varieties. This theory already includes the theory for all simplicial varieties. Working with simplicial varieties has advantages in the sequel. Many question usually treated with sheaf theoretic methods can also dealt with using simplicial techniques. This is the case when checking the Bloch-Ogus axioms but also when treating algebraic K-theory.

Overview

The book assumes knowledge of simplicial techniques and of derived categories.

Part I deals with questions from homological algebra. Most of it should only be used as a reference. The short §1 fixes the terminology for the whole book. §3 is the technical heart of the construction of the triangulated category $D_{\mathcal{MR}}$.

In part II we go through the well known constructions of different cohomology theories. We make sure that everything is functorial on the level of complexes (and not only in the derived category) and construct the filtrations. Those who only want to get the general idea should restrict to the paragraphs 6 and 11.

Part III is treating the Bloch-Ogus axioms. Everything works out as one might expect it to do. We have to develop at the same time the properties of the category $D_{\mathcal{MR}}$ (homological algebra) and the realization functor $R_{\mathcal{MR}}$.

In part IV the Chern classes on higher K-theory are constructed. For this we first study the mixed realization of the classifying scheme $B_{.}Gl$. In §18 we give a somewhat unusual presentation of algebraic K-theory, in which techniques using sheaves are replaced by techniques using simplicial schemes.

The final part V does not rely on parts III and IV. The relation to the category of mixed motives is studied. We show that the functor $R_{\mathcal{MR}}$ can be extended to a functor on the category of Chow motives. Finally we give a better approximation of the true category $D_{\mathcal{MM}}$ by taking polarizability into account.

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