Systems and Control E_2

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Analysis and Design of Nonlinear Control Systems

（非线性控制系统的分析与设计）

Science Press
Beijing

Springer

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With 63 figures

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Chapter 1
Introduction

In this chapter we give an introduction to control theory and nonlinear control systems. In Section 1.1 we briefly review some basic concepts and results for linear control systems. Section 1.2 describes some basic characteristics of nonlinear dynamics. A few typical nonlinear control systems are presented in Section 1.3.

1.1 Linear Control Systems

A control system can be described as a black box in Fig. 1.1 (a), with input (or control) u and output y. In this sense, a control system is considered as a mapping from the input space to the output space. Before 1950's, primarily due to the nature of many electrical and electronic engineering problems then, control problems were largely treated as filtering problems and in the frequency domain, which is particularly suitable for single-input and single-output systems.

During 1950's Rudolf E. Kalman proposed a state space description for control systems. A set of state variables were introduced to describe the box. Intuitively, the black box is split into two parts: the first part is a set of differential (or difference) equations, which are used to describe the dynamics from control u to state variables x, and then a static equation is used to describe the mapping from state variables x to output y. See Fig. 1.1 (b).

There are many different state space descriptions that realize the same input-output mapping. These are called the state space realizations of the input-output mapping. A realization is minimum if there is no other realization that has less dimension of the state space.

Feedback is perhaps the most fundamental concept in automatic control and has a long history. Feedback means that the control strategy relies on the current status of the system. Depending on what information on the current status is available, it can be classified as state feedback control, see Fig. 1.1 (c), and output feedback control, see Fig. 1.1 (d).

A nonlinear control system considered throughout this book is described by

$$\begin{cases} \dot{x} = F(x,u), & x \in M,\ u \in U \\ y = h(x), & y \in N, \end{cases} \tag{1.1}$$

where M, U, and N are manifolds of dimensions n, m, p respectively. $F(x,u)$ and $h(x)$ are smooth mappings (C^∞ mappings unless elsewhere stated). In fact, we con-

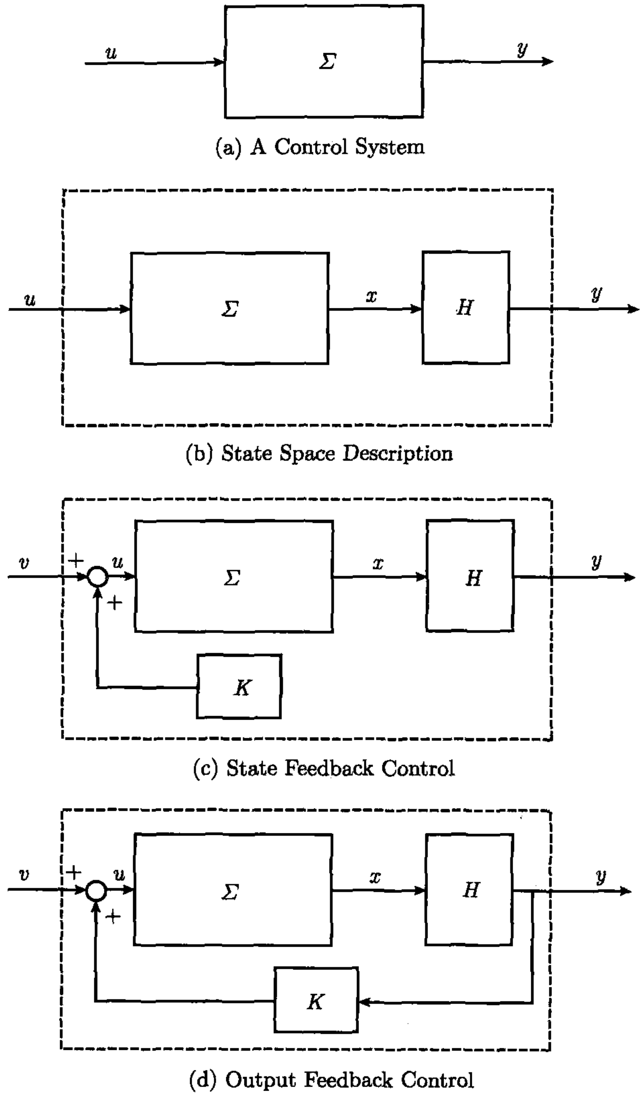

Fig. 1.1 A Control System

sider (1.1) as an expression of the system in a coordinate chart, and under the local coordinates x. In many applications we simply have $M = \mathbb{R}^n$, $U = \mathbb{R}^m$, and $N = \mathbb{R}^p$.

Particularly, when $F(x,u)$ is affine with respect to u, system (1.1) becomes an affine nonlinear system. It is commonly expressed as

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i := f(x) + g(x)u, \quad x \in M,\ u \in U \\ y = h(x), \quad y \in N. \end{cases} \tag{1.2}$$

System (1.2) becomes a linear control system if $f(x) = Ax$, $h(x) = Cx$ are linear and $g_i(x)$ are constant. That is

$$\begin{cases} \dot{x} = Ax + \sum_{i=1}^{m} b_i u_i := Ax + Bu, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y = Cx, & y \in \mathbb{R}^p. \end{cases} \tag{1.3}$$

It is clear that all the definitions and results for (1.1) are also applicable to (1.2) and (1.3).

Observing the history of modern control theory, one sees that the theory was firstly developed for linear systems, and then as the theory for linear control systems had become considerably mature, theory for nonlinear control systems started to take off. So in nonlinear control theory, there are many concepts and results that are originated, inherited, developed, or extended from their linear counterparts. For the sake of easy reference, we give a brief survey on linear control theory in the following. Most of the proofs are omitted in the survey. We refer to standard references such as [11, 6, 4] for details of linear control systems.

1.1.1 Controllability, Observability

Definition 1.1. System (1.1) is said to be controllable (on a subset $V \subset M$) if for any two points $x, z \in M$ (respectively, $x, z \in V$) there exists a control u such that under this control the trajectory goes from $x(0) = x$ to $x(T) = z$ for some $T > 0$ with $x(t) \in M$ (respectively, $x(t) \in V$), $t \in [0, T]$.

Theorem 1.1. *For linear system* (1.3) *the controllable subspace* $\mathscr{C}$ *is*

$$\mathscr{C} = \text{Span}\{B, AB, \cdots, A^{n-1}B\}. \tag{1.4}$$

Consequently, the system is controllable, if and only if $\mathscr{C} = \mathbb{R}^n$.

Proof. Define the following matrix

$$\Phi(t_0, t) = \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} B B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}(t-\tau)} \mathrm{d}\tau, \quad t > t_0. \tag{1.5}$$

We claim that $\mathscr{C}$ has full rank, if and only if $\Phi(t_0, t)$ is nonsingular. If $\text{rank}(\mathscr{C}) < n$, there exists $0 \neq X \in \mathbb{R}^n$ such that

$$X^{\mathrm{T}} A^k B = 0, \quad k = 0, 1, \cdots, n-1.$$

It follows from Taylor expansion and Cayley-Hamilton's theorem that

$$X^{\mathrm{T}} \mathrm{e}^{A(t-\tau)} B = 0.$$

So $\Phi(t_0, t)$ is singular. Conversely, if $\Phi(t_0, t)$ is singular, then there exists $0 \neq X \in \mathbb{R}^n$ such that

$$\int_{t_0}^{t} \|X^{\mathrm{T}} \mathrm{e}^{A(t-\tau)} B\|^2 \mathrm{d}\tau = 0.$$

Hence,

$$X^{\mathrm{T}} \mathrm{e}^{A(t-\tau)} B \equiv 0, \quad t_0 \leqslant \tau \leqslant t.$$

Differentiate it with respect to τ for k times, and set $\tau = t$, we have

$$X^{\mathrm{T}}A^kB = 0, \quad k = 0,1,\cdots,n-1.$$

The claim is proved.

For system (1.3), the trajectory can be expressed as

$$x(t) = \mathrm{e}^{A(t-t_0)}x_0 + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau. \tag{1.6}$$

Now if $\operatorname{rank}(\mathscr{C}) < n$, and assume the system is controllable. Let $0 \neq X \in \mathbb{R}^n$, such that $X^{\mathrm{T}}A^kB = 0$, $k = 0,1,\cdots,n-1$. Set $x_0 = X$ and $x(t) = 0$, then by controllability, (1.6) yields

$$x_0 = -\int_{t_0}^{t} \mathrm{e}^{A(t_0-\tau)}Bu(\tau)\mathrm{d}\tau.$$

Multiply both sides by X^{T} yields

$$\|X\|^2 = 0,$$

which is absurd.

Next, assume $\operatorname{rank}(\mathscr{C}) = n$. For any x_0 and x_t, we can choose u as

$$u(\tau) = B^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}(t-\tau)}v. \tag{1.7}$$

Then (1.6) yields

$$x_t = \mathrm{e}^{A(t-t_0)}x_0 + \Phi v.$$

Using the claim, Φ is invertible. So

$$v = \Phi^{-1}(x_t - \mathrm{e}^{A(t-t_0)}x_0). \tag{1.8}$$

That is, the control (1.7) with v as in (1.8) drives x_0 to x_1. □

We gave the proof here since the above theorem is fundamental and the proof is constructive.

Remark 1.1. 1. Throughout this book we assume the set of admissible controls is the set of measurable functions, unless elsewhere stated.

2. Let $x(0) = x_0$ and $u = u(t)$. Then the solution of (1.3) is

$$x(t) = \mathrm{e}^{At}x_0 + \int_{0}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau. \tag{1.9}$$

3. System (1.3) is controllable, if and only if

$$W(t_0,t_1) = \int_{t_0}^{t_1} \mathrm{e}^{-A\tau}BB^{\mathrm{T}}\mathrm{e}^{-A^{\mathrm{T}}\tau}\mathrm{d}\tau \tag{1.10}$$

is nonsingular for $t_1 > t_0$. Moreover, in this case the control u, which drives x from x_0 at t_0 to x_1 at t_1 is:

$$u(t) = B^{\mathrm{T}}\mathrm{e}^{-A^{\mathrm{T}}t}W^{-1}(t_0,t_1)\left[\mathrm{e}^{-At_1}x_1 - \mathrm{e}^{-At_0}x_0\right]. \tag{1.11}$$

Definition 1.2. System (1.1) is said to be observable (on a subset $V \subset M$) if for any two points $x,z \in M$ (respectively, $x,z \in V$) there exists a control u such that under this control the output $y(t,u,x) \neq y(t,u,z)$ for some $t > 0$.

Theorem 1.2. *For linear system* (1.3) *the observable subspace* $\mathscr{O}$ *is*

$$\mathscr{O} = \mathrm{Span}\{C^{\mathrm{T}}, A^{\mathrm{T}}C^{\mathrm{T}}, \cdots, (A^{\mathrm{T}})^{n-1}C^{\mathrm{T}}\}. \tag{1.12}$$

Consequently, the system is observable, if and only if $\mathscr{O} = \mathbb{R}^n$.

Remark 1.2. 1. From Theorem 1.2 one sees that for linear systems the observability is independent of the inputs. However, this is in general not true for nonlinear systems.

2. System (1.3) is observable, if and only if

$$Q(t_0,t_1) = \int_{t_0}^{t_1} \mathrm{e}^{A^{\mathrm{T}}\tau}C^{\mathrm{T}}C\mathrm{e}^{A\tau}\mathrm{d}\tau \tag{1.13}$$

is nonsingular for $t_1 > t_0$.

Spliting the state space into four parts as: $\mathscr{C}\cap\mathscr{O}$, $\mathscr{C}\cap\mathscr{O}^{\perp}$, $\mathscr{C}^{\perp}\cap\mathscr{O}$, $\mathscr{C}^{\perp}\cap\mathscr{O}^{\perp}$, and choosing coordinates $x = (x^1,x^2,x^3,x^4)$ in such a way that each block of state variables corresponds to the above four subspaces, then we have the following Kalman decomposition:

Theorem 1.3. (Kalman Decomposition) *The system* (1.3) *can be expressed as*

$$\begin{cases} \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \\ \dot{x}^4 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix}. \end{cases} \tag{1.14}$$

Moreover,

$$\begin{cases} \dot{z} = A_{11}z + B_1 u \\ y = C_1 z \end{cases} \tag{1.15}$$

is a minimum realization of the system (1.3).

From the above argument it is clear that a linear system is a minimum realization, if and only if it is controllable and observable.

A controllable linear system can be expressed into a canonical form, called the Brunovsky canonical form.

Theorem 1.4. (Brunovsky Canonical Form) *Assume system* (1.3) *is controllable. Then its state equations can be expressed as*

$$\begin{cases} \dot{x}_1^1 = x_2^1 \\ \vdots \\ \dot{x}_{n_1-1}^1 = x_{n_1}^1 \\ \dot{x}_{n_1}^1 = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \alpha_{ij}^1 x_j^i + \sum_{i=1}^{m} \beta_i^1 u_i \\ \vdots \\ \dot{x}_1^m = x_2^m \\ \vdots \\ \dot{x}_{n_m-1}^m = x_{n_m}^m \\ \dot{x}_{n_m}^m = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \alpha_{ij}^m x_j^i + \sum_{i=1}^{m} \beta_i^m u_i. \end{cases} \tag{1.16}$$

Remark 1.3. It is a common assumption that the columns of B are linearly independent, otherwise some input channel(s) will be redundant. In this case, if we use a state feedback

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \beta_1^1 & \cdots & \beta_m^1 \\ \vdots & & \vdots \\ \beta_1^m & \cdots & \beta_m^m \end{bmatrix}^{-1} \left(\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} - \begin{bmatrix} \sum_{i=1}^{m}\sum_{j=1}^{n_i} \alpha_{ij}^1 x_j^i \\ \vdots \\ \sum_{i=1}^{m}\sum_{j=1}^{n_i} \alpha_{ij}^m x_j^i \end{bmatrix} \right),$$

then the system (1.16) has a very neat form, called the feedback Brunovsky canonical form as

$$\begin{cases} \dot{x}_1^k = x_2^k \\ \vdots \\ \dot{x}_{n_k-1}^k = x_{n_k}^k \\ \dot{x}_{n_k}^k = v_k, \quad k = 1, 2, \cdots, m. \end{cases} \tag{1.17}$$

1.1.2 Invariant Subspaces

Definition 1.3. 1. For linear system (1.3), a subspace V is called an (A,B)-invariant subspace if there exists an $m \times n$ matrix F such that

$$(A + BF)V \subset V. \tag{1.18}$$

V is called an A-invariant subspace if

$$AV \subset V. \tag{1.19}$$

2. The matrix F satisfying (1.18) is said to be a friend to V. The set of matrices that are friends to V, is denoted by

$$\mathscr{F}(V) = \{F | (A+BF)V \subset V\}.$$

Proposition 1.1. (Quaker Lemma) *V is an (A,B)-invariant subspace, if and only if*

$$AV \subset V + \text{Im}(B). \tag{1.20}$$

If we denote by $\langle A|W \rangle$ the smallest subspace that contains W and is A-invariant, then the controllable subspace of system (1.3) can be expressed as $\mathscr{C} = \langle A \,|\, \text{Im}(B) \rangle$. Based on this observation, we define the reachability subspace as follows: A subspace V is called a reachability subspace if there exist matrices F and G with proper dimensions such that

$$V = \langle A+BF \,|\, \text{Im}(BG) \rangle .$$

It is obvious that a reachability subspace is an (A,B)-invariant subspace.

Proposition 1.2. *A subspace V is a reachability subspace, if and only if there exists F such that*

$$V = \langle A+BF \,|\, \text{Im}(B) \cap V \rangle . \tag{1.21}$$

Now let $Z \subset \mathbb{R}^n$ be a subspace. Then we have

1. There exists a maximal (A,B)-invariant subspace $S^* \subset Z$, denoted by $S^*(Z)$. That is, for any $S \subset Z$ being (A,B)-invariant, $S \subset S^*$.
2. There exists a maximal reachability subspace $R^* \subset Z$.
3. Let $F \in \mathscr{F}(S^*)$, then

$$R^* = \langle A+BF \,|\, \text{Im}(B) \cap S^* \rangle . \tag{1.22}$$

As an application, we consider the following example.

Example 1.1. Consider a system

$$\begin{cases} \dot{x} = Ax + Bu + Ew \\ y = Cx, \end{cases} \tag{1.23}$$

where w is disturbance. The disturbance decoupling problem is to find a control

$$u = Fx,$$

such that for the closed-loop system the disturbance w does not affect y. It is easy to prove that the disturbance decoupling problem is solvable, if and only if

$$\mathrm{Im}(E) \subset S^*(\mathrm{Ker}(C)). \tag{1.24}$$

1.1.3 Zeros, Poles, Observers

Consider a single-input single-output (SISO) system, namely system (1.3) with $m = p = 1$. Then

$$W(s) := c(sI - A)^{-1}b \tag{1.25}$$

is called the transfer function of system (1.3). The transfer function can be expressed as a proper rational fraction of polynomials as

$$W(s) = \frac{q(s)}{p(s)},$$

where $p(s)$ and $q(s)$ are polynomials with $\deg(p(s)) > \deg(q(s))$. The zeros of $q(s)$ and $p(s)$ are called the zeros and the poles of the system respectively. It is easy to see that the poles are the eigenvalues of A.

Proposition 1.3. *Consider the system* (1.3). *The followings are equivalent:*

- *It is a minimum realization;*
- *It is controllable and observable;*
- *Its transfer function has no zero/pole cancellation.*

If there is no zero/pole cancellation, it is the presence of zeros on the transfer function $W(s)$ that makes $S^*(\mathrm{Ker}(C))$ nontrivial. Hence for a SISO system it follows that

$$\dim(S^*(\mathrm{Ker}(C))) = \text{Number of}\{\text{zeros of } W(s)\}. \tag{1.26}$$

While the zeros can not be changed by the state feedback control, the poles may. In fact, pole (eigenvalue) assignment is very important for control design.

Theorem 1.5. *Given a set of n numbers $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$.*

1. *Assume* (1.3) *is controllable (briefly, (A,B) is a controllable pair), there exists an F, such that*
$$\sigma(A + BF) = \Lambda.$$
2. *Assume* (1.3) *is observable (briefly, (C,A) is an observable pair), there exists an L, such that*
$$\sigma(A + LC) = \Lambda.$$

When the system state is not completely measurable, an observer can be designed to estimate the state. A full-dimensional observer has the following form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{1.27}$$

Then the error $e := x - \hat{x}$ satisfies

$$\dot{e} = (A - LC)e. \tag{1.28}$$

As long as the system is observable, Theorem 1.5 assures that there exists L such that $A - LC$ is a Hurwitz matrix. Hence

$$\lim_{t\to\infty} e(t) = 0.$$

In fact, y is a part of the state that is already available. Thus, it may suffice to construct just a reduced-dimensional observer. Choosing R such that

$$T = \begin{bmatrix} C \\ R \end{bmatrix}$$

is a nonsingular matrix. Let $z = Tx$. Then (1.3) becomes

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = TAT^{-1}z + TBu := \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} z + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u \\ y = z_1. \end{cases} \tag{1.29}$$

Let

$$v := \bar{A}_{21}y + \bar{B}_2 u, \quad w := \dot{y} - \bar{A}_{11}y - \bar{B}_1 u,$$

we have

$$\begin{cases} \dot{z}_2 = \bar{A}_{22}z_2 + v \\ w = \bar{A}_{12}z_2. \end{cases} \tag{1.30}$$

As long as (C,A) is observable, it is easy to prove that $(\bar{A}_{12}, \bar{A}_{22})$ is observable [4]. So we can choose an L such that $\bar{A}_{22} - L\bar{A}_{12}$ is Hurwitz. Then we can construct an observer for z_2 from (1.30) as

$$\begin{aligned} \dot{\hat{z}}_2 &= (\bar{A}_{22} - L\bar{A}_{12})\hat{z}_2 + Lw + v \\ &= (\bar{A}_{22} - L\bar{A}_{12})\hat{z}_2 + L(\dot{y} - \bar{A}_{11}y - \bar{B}_1 u) + (\bar{A}_{21}y + \bar{B}_2 u). \end{aligned} \tag{1.31}$$

This equation contains the derivative of y. It can be eliminated by defining

$$\xi = \hat{z}_2 - Ly. \tag{1.32}$$

Then the observer becomes

$$\dot{\xi} = (\bar{A}_{22} - L\bar{A}_{12})\xi + \left[(\bar{A}_{22} - L\bar{A}_{12})L + (\bar{A}_{21} - L\bar{A}_{11})\right] y + (\bar{B}_2 - L\bar{B}_1)u. \tag{1.33}$$

Finally, the estimator is

$$\hat{x} = T^{-1} \begin{bmatrix} I_p & 0 \\ L & I_{n-p} \end{bmatrix} \begin{bmatrix} y \\ \xi \end{bmatrix}. \tag{1.34}$$

1.1.4 Normal Form and Zero Dynamics

Next, we consider the zeros and zero dynamics of a linear control system. For a SISO system, assume its transfer function is expressed as

$$W(s) = \frac{q(s)}{p(s)} = \frac{\alpha(s^m + q_1 s^{m-1} + \cdots + q_m)}{s^n + p_1 s^{n-1} + \cdots + p_n}. \tag{1.35}$$

The roots of $q(x)$–the zeros of the system, are also called the transmission zeros of the system. The relative degree ρ is defined as

$$\rho = \deg(p(s)) - \deg(q(s)) = n - m. \tag{1.36}$$

Assume $q(s)$ and $p(s)$ are coprime, then a minimum state space realization is

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha \end{bmatrix} u \\ y = \begin{bmatrix} q_m & \cdots & q_1 & 1 & 0 & \cdots & 0 \end{bmatrix} x. \end{cases} \tag{1.37}$$

The problem is: Find, if possible, a control u and a set Z^* of initial conditions x_0, such that $y(t) = 0$, $\forall t \geqslant 0$. If such Z^* exists, the restriction of (1.37) on Z^* is called the zero dynamics.

Calculating the derivatives of the outputs, $y^{(i)}$, $i = 0, 1, \cdots$, one sees that

$$cA^i b = 0,\ i = 0, 1, \cdots, \rho - 2; \quad cA^{\rho-1} b \neq 0. \tag{1.38}$$

That is

$$\begin{aligned} y^{(i-1)} &= cA^{i-1}x, \quad i = 1, \cdots, \rho; \\ y^{(\rho)} &= cA^{\rho}x + cA^{\rho-1}bu. \end{aligned} \tag{1.39}$$

It is easy to see that cA^{i-1}, $i = 1, \cdots, \rho$ are linearly independent. So we can choose a new coordinate frame by letting

$$\begin{aligned} \xi_i &:= cA^{i-1}x, \quad i = 1, \cdots, \rho; \\ z_i &:= x_i, \quad i = 1, \cdots, m. \end{aligned} \tag{1.40}$$

Then the system becomes

$$\begin{cases} \dot{z} = Nz + M\xi_1 \\ \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_{\rho-1} = \xi_\rho \\ \dot{\xi}_\rho = Rz + S\xi + \alpha u, \\ y = \xi_1. \end{cases} \tag{1.41}$$

where

$$N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -q_m & -q_{m-1} & -q_{m-2} & \cdots & -q_1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{1.42}$$

In order to keep $y(t) = 0$, we must have

$$\xi_1 = \xi_2 = \cdots = \xi_\rho = 0,$$

and

$$u = -\frac{1}{\alpha}(Rz + S\xi).$$

It follows that the zero dynamics is defined on the subspace

$$Z^* = \left\{ x | cA^i x = 0,\ i = 0, \cdots, \rho - 1 \right\},$$

and represented by

$$\dot{z} = Nz.$$

The eigenvalues of N are the zeros of $q(s)$.

Next, we consider multiple input-multiple output (MIMO) systems. Similar to the SISO case, we define the transfer function matrix as

$$C(sI - A)^{-1}B.$$

The relative degree vector, denoted by $\rho = (\rho_1, \cdots, \rho_p)$, is defined as

$$\begin{array}{c} c_i A^k B = 0,\ k = 0, \cdots, \rho_i - 2; \quad c_i A^{\rho_i - 1} B \neq 0, \\ i = 1, \cdots, p. \end{array} \tag{1.43}$$

Denote the decoupling matrix

$$D = \begin{bmatrix} c_1 A^{\rho_1 - 1} B \\ \vdots \\ c_p A^{\rho_p - 1} B \end{bmatrix}. \tag{1.44}$$

Define

$$\xi_j^i = c_i A^{j-1} x, \quad i = 1, \cdots, p, j = 1, \cdots, \rho_i.$$

It is easy to prove that if D has full row rank (which implies that $p \leqslant m$), then

$$L := \{ c_i A^{j-1} | i = 1, \cdots, p; j = 1, \cdots, \rho_i \}$$

are linearly independent. Choosing $\{z_k = p_k x | p_k \in L^\perp\}$ (there are $n - \sum_{i=1}^{p} \rho_i$ linearly independent p_k). Using (ξ_j^i, z_k) as coordinates, the system (1.3) can be converted to