



Systems and Control *E*₂

Daizhan Cheng
Xiaoming Hu
Tielong Shen

Analysis and Design of Nonlinear Control Systems

(非线性控制系统的分析与设计)



Science Press
Beijing



Springer

Daizhan Cheng
Xiaoming Hu
Tielong Shen

Analysis and Design of Nonlinear Control Systems

(非线性控制系统的分析与设计)

With 63 figures

 Science Press
Beijing

 Springer

Contents

1. Introduction	1
1.1 Linear Control Systems	1
1.1.1 Controllability, Observability	3
1.1.2 Invariant Subspaces	6
1.1.3 Zeros, Poles, Observers	8
1.1.4 Normal Form and Zero Dynamics	10
1.2 Nonlinearity vs. Linearity	14
1.2.1 Localization	14
1.2.2 Singularity	16
1.2.3 Complex Behaviors	18
1.3 Some Examples of Nonlinear Control Systems	20
References	27
2. Topological Space	29
2.1 Metric Space	29
2.2 Topological Spaces	34
2.3 Continuous Mapping	39
2.4 Quotient Spaces	44
References	46
3. Differentiable Manifold	47
3.1 Structure of Manifolds	47
3.2 Fiber Bundle	53
3.3 Vector Field	56
3.4 One Parameter Group	60
3.5 Lie Algebra of Vector Fields	62
3.6 Co-tangent Space	65
3.7 Lie Derivatives	66
3.8 Frobenius' Theory	70
3.9 Lie Series, Chow's Theorem	72
3.10 Tensor Field	75
3.11 Riemannian Geometry	79
3.12 Symplectic Geometry	85
References	89

4. Algebra, Lie Group and Lie Algebra	91
4.1 Group	91
4.2 Ring and Algebra	97
4.3 Homotopy	100
4.4 Fundamental Group	101
4.5 Covering Space	109
4.6 Lie Group	113
4.7 Lie Algebra of Lie Group	115
4.8 Structure of Lie Algebra	117
References	119
5. Controllability and Observability	121
5.1 Controllability of Nonlinear Systems	121
5.2 Observability of Nonlinear Systems	136
5.3 Kalman Decomposition	140
References	145
6. Global Controllability of Affine Control Systems	147
6.1 From Linear to Nonlinear Systems	147
6.2 A Sufficient Condition	150
6.3 Multi-hierarchy Case	163
6.4 $\text{Codim}(\mathcal{G}) = 1$	168
References	171
7. Stability and Stabilization	173
7.1 Stability of Dynamic Systems	173
7.2 Stability in the Linear Approximation	175
7.3 The Direct Method of Lyapunov	177
7.3.1 Positive Definite Functions	177
7.3.2 Critical Stability	179
7.3.3 Instability	180
7.3.4 Asymptotic Stability	180
7.3.5 Total Stability	182
7.3.6 Global Stability	182
7.4 LaSalle's Invariance Principle	183
7.5 Converse Theorems to Lyapunov's Stability Theorems	185
7.5.1 Converse Theorems to Local Asymptotic Stability	185
7.5.2 Converse Theorem to Global Asymptotic Stability	187
7.6 Stability of Invariant Set	188
7.7 Input-Output Stability	189
7.7.1 Stability of Input-Output Mapping	190
7.7.2 The Lur'e Problem	192
7.7.3 Control Lyapunov Function	193
7.8 Region of Attraction	194
References	205
8. Decoupling	207
8.1 (f, g) -invariant Distribution	207
8.2 Local Disturbance Decoupling	213
8.3 Controlled Invariant Distribution	218
8.4 Block Decomposition	223

8.5	Feedback Decomposition	232
	References	235
9.	Input-Output Structure	237
9.1	Decoupling Matrix	237
9.2	Morgan's Problem	240
9.3	Invertibility	243
9.4	Decoupling via Dynamic Feedback	247
9.5	Normal Form of Nonlinear Control Systems	253
9.6	Generalized Normal Form	256
9.7	Fliess Functional Expansion	264
9.8	Tracking via Fliess Functional Expansion	267
	References	277
10.	Linearization of Nonlinear Systems	279
10.1	Poincaré Linearization	279
10.2	Linear Equivalence of Nonlinear Systems	282
10.3	State Feedback Linearization	287
10.4	Linearization with Outputs	292
10.5	Global Linearization	295
10.6	Non-regular Feedback Linearization	306
	References	313
11	Design of Center Manifold	315
11.1	Center Manifold	315
11.2	Stabilization of Minimum Phase Systems	317
11.3	Lyapunov Function with Homogeneous Derivative	319
11.4	Stabilization of Systems with Zero Center	328
11.5	Stabilization of Systems with Oscillatory Center	335
11.6	Stabilization Using Generalized Normal Form	341
11.7	Advanced Design Techniques	349
	References	353
12	Output Regulation	355
12.1	Output Regulation of Linear Systems	355
12.2	Nonlinear Local Output Regulation	366
12.3	Robust Local Output Regulation	374
	References	377
13	Dissipative Systems	379
13.1	Dissipative Systems	379
13.2	Passivity Conditions	383
13.3	Passivity-based Control	388
13.4	Lagrange Systems	393
13.5	Hamiltonian Systems	397
	References	401
14	L_2-Gain Synthesis	403
14.1	H_∞ Norm and L_2 -Gain	403
14.2	H_∞ Feedback Control Problem	409
14.3	L_2 -Gain Feedback Synthesis	411
14.4	Constructive Design Method	417
14.5	Applications	423
	References	429

15 Switched Systems	431
15.1 Common Quadratic Lyapunov Function	431
15.2 Quadratic Stabilization of Planar Switched Systems	454
15.3 Controllability of Switched Linear Systems	467
15.4 Controllability of Switched Bilinear Systems	476
15.5 LaSalle's Invariance Principle for Switched Systems	483
15.6 Consensus of Multi-Agent Systems	492
15.6.1 Two Dimensional Agent Model with a Leader	493
15.6.2 n Dimensional Agent Model without Lead	495
References	508
16 Discontinuous Dynamical Systems	509
16.1 Introduction	509
16.2 Filippov Framework	510
16.2.1 Filippov Solution	510
16.2.2 Lyapunov Stability Criteria	513
16.3 Feedback Stabilization	517
16.3.1 Feedback Controller Design: Nominal Case	518
16.3.2 Robust Stabilization	521
16.4 Design Example of Mechanical Systems	523
16.4.1 PD Controlled Mechanical Systems	523
16.4.2 Stationary Set	524
16.4.3 Application Example	528
References	531
Appendix A Some Useful Theorems	533
A.1 Sard's Theorem	533
A.2 Rank Theorem	533
References	533
Appendix B Semi-Tensor Product of Matrices	535
B.1 A Generalized Matrix Product	535
B.2 Swap Matrix	537
B.3 Some Properties of Semi-Tensor Product	538
B.4 Matrix Form of Polynomials	539
References	540
Index	541

Chapter 1

Introduction

In this chapter we give an introduction to control theory and nonlinear control systems. In Section 1.1 we briefly review some basic concepts and results for linear control systems. Section 1.2 describes some basic characteristics of nonlinear dynamics. A few typical nonlinear control systems are presented in Section 1.3.

1.1 Linear Control Systems

A control system can be described as a black box in Fig. 1.1 (a), with input (or control) u and output y . In this sense, a control system is considered as a mapping from the input space to the output space. Before 1950's, primarily due to the nature of many electrical and electronic engineering problems then, control problems were largely treated as filtering problems and in the frequency domain, which is particularly suitable for single-input and single-output systems.

During 1950's Rudolf E. Kalman proposed a state space description for control systems. A set of state variables were introduced to describe the box. Intuitively, the black box is split into two parts: the first part is a set of differential (or difference) equations, which are used to describe the dynamics from control u to state variables x , and then a static equation is used to describe the mapping from state variables x to output y . See Fig. 1.1 (b).

There are many different state space descriptions that realize the same input-output mapping. These are called the state space realizations of the input-output mapping. A realization is minimum if there is no other realization that has less dimension of the state space.

Feedback is perhaps the most fundamental concept in automatic control and has a long history. Feedback means that the control strategy relies on the current status of the system. Depending on what information on the current status is available, it can be classified as state feedback control, see Fig. 1.1 (c), and output feedback control, see Fig. 1.1 (d).

A nonlinear control system considered throughout this book is described by

$$\begin{cases} \dot{x} = F(x, u), & x \in M, u \in U \\ y = h(x), & y \in N, \end{cases} \quad (1.1)$$

where M , U , and N are manifolds of dimensions n , m , p respectively. $F(x, u)$ and $h(x)$ are smooth mappings (C^∞ mappings unless elsewhere stated). In fact, we con-

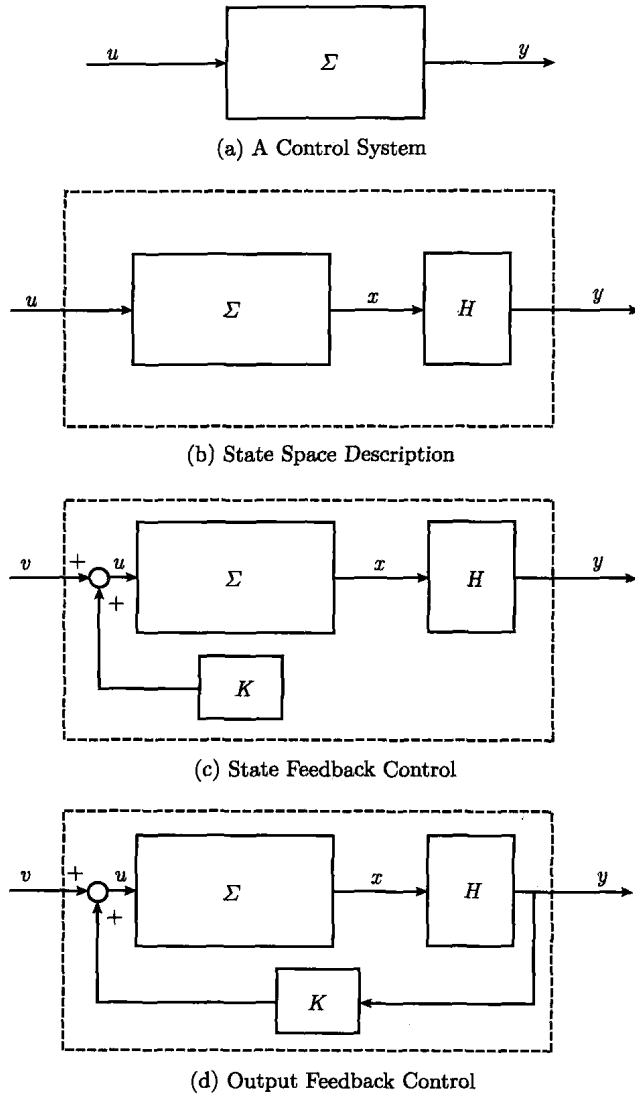


Fig. 1.1 A Control System

sider (1.1) as an expression of the system in a coordinate chart, and under the local coordinates x . In many applications we simply have $M = \mathbb{R}^n$, $U = \mathbb{R}^m$, and $N = \mathbb{R}^p$.

Particularly, when $F(x, u)$ is affine with respect to u , system (1.1) becomes an affine nonlinear system. It is commonly expressed as

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i := f(x) + g(x)u, & x \in M, u \in U \\ y = h(x), & y \in N. \end{cases} \quad (1.2)$$

System (1.2) becomes a linear control system if $f(x) = Ax$, $h(x) = Cx$ are linear and $g_i(x)$ are constant. That is

$$\begin{cases} \dot{x} = Ax + \sum_{i=1}^m b_i u_i := Ax + Bu, & x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y = Cx, & y \in \mathbb{R}^p. \end{cases} \quad (1.3)$$

It is clear that all the definitions and results for (1.1) are also applicable to (1.2) and (1.3).

Observing the history of modern control theory, one sees that the theory was firstly developed for linear systems, and then as the theory for linear control systems had become considerably mature, theory for nonlinear control systems started to take off. So in nonlinear control theory, there are many concepts and results that are originated, inherited, developed, or extended from their linear counterparts. For the sake of easy reference, we give a brief survey on linear control theory in the following. Most of the proofs are omitted in the survey. We refer to standard references such as [11, 6, 4] for details of linear control systems.

1.1.1 Controllability, Observability

Definition 1.1. System (1.1) is said to be controllable (on a subset $V \subset M$) if for any two points $x, z \in M$ (respectively, $x, z \in V$) there exists a control u such that under this control the trajectory goes from $x(0) = x$ to $x(T) = z$ for some $T > 0$ with $x(t) \in M$ (respectively, $x(t) \in V$), $t \in [0, T]$.

Theorem 1.1. For linear system (1.3) the controllable subspace $\mathcal{C}$ is

$$\mathcal{C} = \text{Span}\{B, AB, \dots, A^{n-1}B\}. \quad (1.4)$$

Consequently, the system is controllable, if and only if $\mathcal{C} = \mathbb{R}^n$.

Proof. Define the following matrix

$$\Phi(t_0, t) = \int_{t_0}^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau, \quad t > t_0. \quad (1.5)$$

We claim that $\mathcal{C}$ has full rank, if and only if $\Phi(t_0, t)$ is nonsingular. If $\text{rank}(\mathcal{C}) < n$, there exists $0 \neq X \in \mathbb{R}^n$ such that

$$X^T A^k B = 0, \quad k = 0, 1, \dots, n-1.$$

It follows from Taylor expansion and Cayley-Hamilton's theorem that

$$X^T e^{A(t-\tau)} B = 0.$$

So $\Phi(t_0, t)$ is singular. Conversely, if $\Phi(t_0, t)$ is singular, then there exists $0 \neq X \in \mathbb{R}^n$ such that

$$\int_{t_0}^t \|X^T e^{A(t-\tau)} B\|^2 d\tau = 0.$$

Hence,

$$X^T e^{A(t-\tau)} B \equiv 0, \quad t_0 \leq \tau \leq t.$$

Differentiate it with respect to τ for k times, and set $\tau = t$, we have

$$X^T A^k B = 0, \quad k = 0, 1, \dots, n-1.$$

The claim is proved.

For system (1.3), the trajectory can be expressed as

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau. \quad (1.6)$$

Now if $\text{rank}(\mathcal{C}) < n$, and assume the system is controllable. Let $0 \neq X \in \mathbb{R}^n$, such that $X^T A^k B = 0$, $k = 0, 1, \dots, n-1$. Set $x_0 = X$ and $x(t) = 0$, then by controllability, (1.6) yields

$$x_0 = - \int_{t_0}^t e^{A(t_0-\tau)}Bu(\tau)d\tau.$$

Multiply both sides by X^T yields

$$\|X\|^2 = 0,$$

which is absurd.

Next, assume $\text{rank}(\mathcal{C}) = n$. For any x_0 and x_t , we can choose u as

$$u(\tau) = B^T e^{A^T(t-\tau)}v. \quad (1.7)$$

Then (1.6) yields

$$x_t = e^{A(t-t_0)}x_0 + \Phi v.$$

Using the claim, Φ is invertible. So

$$v = \Phi^{-1}(x_t - e^{A(t-t_0)}x_0). \quad (1.8)$$

That is, the control (1.7) with v as in (1.8) drives x_0 to x_1 . $\square$

We gave the proof here since the above theorem is fundamental and the proof is constructive.

Remark 1.1. 1. Throughout this book we assume the set of admissible controls is the set of measurable functions, unless elsewhere stated.

2. Let $x(0) = x_0$ and $u = u(t)$. Then the solution of (1.3) is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \quad (1.9)$$

3. System (1.3) is controllable, if and only if

$$W(t_0, t_1) = \int_{t_0}^{t_1} e^{-A\tau}BB^T e^{-A^T\tau}d\tau \quad (1.10)$$

is nonsingular for $t_1 > t_0$. Moreover, in this case the control u , which drives x from x_0 at t_0 to x_1 at t_1 is:

$$u(t) = B^T e^{-A^T t} W^{-1}(t_0, t_1) [e^{-A t_1} x_1 - e^{-A t_0} x_0]. \quad (1.11)$$

Definition 1.2. System (1.1) is said to be observable (on a subset $V \subset M$) if for any two points $x, z \in M$ (respectively, $x, z \in V$) there exists a control u such that under this control the output $y(t, u, x) \neq y(t, u, z)$ for some $t > 0$.

Theorem 1.2. For linear system (1.3) the observable subspace $\mathcal{O}$ is

$$\mathcal{O} = \text{Span}\{C^T, A^T C^T, \dots, (A^T)^{n-1} C^T\}. \quad (1.12)$$

Consequently, the system is observable, if and only if $\mathcal{O} = \mathbb{R}^n$.

Remark 1.2. 1. From Theorem 1.2 one sees that for linear systems the observability is independent of the inputs. However, this is in general not true for nonlinear systems.

2. System (1.3) is observable, if and only if

$$Q(t_0, t_1) = \int_{t_0}^{t_1} e^{A^T \tau} C^T C e^{A \tau} d\tau \quad (1.13)$$

is nonsingular for $t_1 > t_0$.

Splitting the state space into four parts as: $\mathcal{C} \cap \mathcal{O}$, $\mathcal{C} \cap \mathcal{O}^\perp$, $\mathcal{C}^\perp \cap \mathcal{O}$, $\mathcal{C}^\perp \cap \mathcal{O}^\perp$, and choosing coordinates $x = (x^1, x^2, x^3, x^4)$ in such a way that each block of state variables corresponds to the above four subspaces, then we have the following Kalman decomposition:

Theorem 1.3. (Kalman Decomposition) The system (1.3) can be expressed as

$$\begin{cases} \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \\ \dot{x}^4 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u \\ y = [C_1 \ 0 \ C_3 \ 0]. \end{cases} \quad (1.14)$$

Moreover,

$$\begin{cases} \dot{z} = A_{11} z + B_1 u \\ y = C_1 z \end{cases} \quad (1.15)$$

is a minimum realization of the system (1.3).

From the above argument it is clear that a linear system is a minimum realization, if and only if it is controllable and observable.

A controllable linear system can be expressed into a canonical form, called the Brunovsky canonical form.

Theorem 1.4. (Brunovsky Canonical Form) Assume system (1.3) is controllable. Then its state equations can be expressed as

$$\begin{cases} \dot{x}_1^1 = x_2^1 \\ \vdots \\ \dot{x}_{n_1-1}^1 = x_{n_1}^1 \\ \dot{x}_{n_1}^1 = \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij}^1 x_j^i + \sum_{i=1}^m \beta_i^1 u_i \\ \vdots \\ \dot{x}_1^m = x_2^m \\ \vdots \\ \dot{x}_{n_m-1}^m = x_{n_m}^m \\ \dot{x}_{n_m}^m = \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij}^m x_j^i + \sum_{i=1}^m \beta_i^m u_i. \end{cases} \quad (1.16)$$

Remark 1.3. It is a common assumption that the columns of B are linearly independent, otherwise some input channel(s) will be redundant. In this case, if we use a state feedback

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \beta_1^1 & \cdots & \beta_m^1 \\ \vdots & & \vdots \\ \beta_1^m & \cdots & \beta_m^m \end{bmatrix}^{-1} \left(\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} - \begin{bmatrix} \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij}^1 x_j^i \\ \vdots \\ \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij}^m x_j^i \end{bmatrix} \right),$$

then the system (1.16) has a very neat form, called the feedback Brunovsky canonical form as

$$\begin{cases} \dot{x}_1^k = x_2^k \\ \vdots \\ \dot{x}_{n_k-1}^k = x_{n_k}^k \\ \dot{x}_{n_k}^k = v_k, \quad k = 1, 2, \dots, m. \end{cases} \quad (1.17)$$

1.1.2 Invariant Subspaces

Definition 1.3. 1. For linear system (1.3), a subspace V is called an (A, B) -invariant subspace if there exists an $m \times n$ matrix F such that

$$(A + BF)V \subset V. \quad (1.18)$$

V is called an A -invariant subspace if

$$AV \subset V. \quad (1.19)$$

2. The matrix F satisfying (1.18) is said to be a friend to V . The set of matrices that are friends to V , is denoted by

$$\mathcal{F}(V) = \{F | (A + BF)V \subset V\}.$$

Proposition 1.1. (Quaker Lemma) V is an (A, B) -invariant subspace, if and only if

$$AV \subset V + \text{Im}(B). \quad (1.20)$$

If we denote by $\langle A|W \rangle$ the smallest subspace that contains W and is A -invariant, then the controllable subspace of system (1.3) can be expressed as $\mathcal{C} = \langle A | \text{Im}(B) \rangle$. Based on this observation, we define the reachability subspace as follows: A subspace V is called a reachability subspace if there exist matrices F and G with proper dimensions such that

$$V = \langle A + BF | \text{Im}(BG) \rangle.$$

It is obvious that a reachability subspace is an (A, B) -invariant subspace.

Proposition 1.2. A subspace V is a reachability subspace, if and only if there exists F such that

$$V = \langle A + BF | \text{Im}(B) \cap V \rangle. \quad (1.21)$$

Now let $Z \subset \mathbb{R}^n$ be a subspace. Then we have

1. There exists a maximal (A, B) -invariant subspace $S^* \subset Z$, denoted by $S^*(Z)$. That is, for any $S \subset Z$ being (A, B) -invariant, $S \subset S^*$.
2. There exists a maximal reachability subspace $R^* \subset Z$.
3. Let $F \in \mathcal{F}(S^*)$, then

$$R^* = \langle A + BF | \text{Im}(B) \cap S^* \rangle. \quad (1.22)$$

As an application, we consider the following example.

Example 1.1. Consider a system

$$\begin{cases} \dot{x} = Ax + Bu + Ew \\ y = Cx, \end{cases} \quad (1.23)$$

where w is disturbance. The disturbance decoupling problem is to find a control

$$u = Fx,$$

such that for the closed-loop system the disturbance w does not affect y . It is easy to prove that the disturbance decoupling problem is solvable, if and only if

$$\text{Im}(E) \subset S^*(\text{Ker}(C)). \quad (1.24)$$

1.1.3 Zeros, Poles, Observers

Consider a single-input single-output (SISO) system, namely system (1.3) with $m = p = 1$. Then

$$W(s) := c(sI - A)^{-1}b \quad (1.25)$$

is called the transfer function of system (1.3). The transfer function can be expressed as a proper rational fraction of polynomials as

$$W(s) = \frac{q(s)}{p(s)},$$

where $p(s)$ and $q(s)$ are polynomials with $\deg(p(s)) > \deg(q(s))$. The zeros of $q(s)$ and $p(s)$ are called the zeros and the poles of the system respectively. It is easy to see that the poles are the eigenvalues of A .

Proposition 1.3. *Consider the system (1.3). The followings are equivalent:*

- *It is a minimum realization;*
- *It is controllable and observable;*
- *Its transfer function has no zero/pole cancellation.*

If there is no zero/pole cancellation, it is the presence of zeros on the transfer function $W(s)$ that makes $S^*(\text{Ker}(C))$ nontrivial. Hence for a SISO system it follows that

$$\dim(S^*(\text{Ker}(C))) = \text{Number of}\{\text{zeros of } W(s)\}. \quad (1.26)$$

While the zeros can not be changed by the state feedback control, the poles may. In fact, pole (eigenvalue) assignment is very important for control design.

Theorem 1.5. *Given a set of n numbers $\Lambda = \{\lambda_1, \dots, \lambda_n\}$.*

1. *Assume (1.3) is controllable (briefly, (A, B) is a controllable pair), there exists an F , such that*

$$\sigma(A + BF) = \Lambda.$$

2. *Assume (1.3) is observable (briefly, (C, A) is an observable pair), there exists an L , such that*

$$\sigma(A + LC) = \Lambda.$$

When the system state is not completely measurable, an observer can be designed to estimate the state. A full-dimensional observer has the following form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}). \quad (1.27)$$

Then the error $e := x - \hat{x}$ satisfies

$$\dot{e} = (A - LC)e. \quad (1.28)$$

As long as the system is observable, Theorem 1.5 assures that there exists L such that $A - LC$ is a Hurwitz matrix. Hence

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

In fact, y is a part of the state that is already available. Thus, it may suffice to construct just a reduced-dimensional observer. Choosing R such that

$$T = \begin{bmatrix} C \\ R \end{bmatrix}$$

is a nonsingular matrix. Let $z = Tx$. Then (1.3) becomes

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = TAT^{-1}z + TBu := \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} z + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u \\ y = z_1. \end{cases} \quad (1.29)$$

Let

$$v := \bar{A}_{21}y + \bar{B}_2u, \quad w := \dot{y} - \bar{A}_{11}y - \bar{B}_1u,$$

we have

$$\begin{cases} \dot{z}_2 = \bar{A}_{22}z_2 + v \\ w = \bar{A}_{12}z_2. \end{cases} \quad (1.30)$$

As long as (C, A) is observable, it is easy to prove that $(\bar{A}_{12}, \bar{A}_{22})$ is observable [4]. So we can choose an L such that $\bar{A}_{22} - L\bar{A}_{12}$ is Hurwitz. Then we can construct an observer for z_2 from (1.30) as

$$\begin{aligned} \dot{\hat{z}}_2 &= (\bar{A}_{22} - L\bar{A}_{12})\hat{z}_2 + Lw + v \\ &= (\bar{A}_{22} - L\bar{A}_{12})\hat{z}_2 + L(\dot{y} - \bar{A}_{11}y - \bar{B}_1u) + (\bar{A}_{21}y + \bar{B}_2u). \end{aligned} \quad (1.31)$$

This equation contains the derivative of y . It can be eliminated by defining

$$\xi = \hat{z}_2 - Ly. \quad (1.32)$$

Then the observer becomes

$$\dot{\xi} = (\bar{A}_{22} - L\bar{A}_{12})\xi + [(\bar{A}_{22} - L\bar{A}_{12})L + (\bar{A}_{21} - L\bar{A}_{11})]y + (\bar{B}_2 - L\bar{B}_1)u. \quad (1.33)$$

Finally, the estimator is

$$\hat{x} = T^{-1} \begin{bmatrix} I_p & 0 \\ L & I_{n-p} \end{bmatrix} \begin{bmatrix} y \\ \xi \end{bmatrix}. \quad (1.34)$$

1.1.4 Normal Form and Zero Dynamics

Next, we consider the zeros and zero dynamics of a linear control system. For a SISO system, assume its transfer function is expressed as

$$W(s) = \frac{q(s)}{p(s)} = \frac{\alpha(s^m + q_1 s^{m-1} + \dots + q_m)}{s^n + p_1 s^{n-1} + \dots + p_n}. \quad (1.35)$$

The roots of $q(s)$ —the zeros of the system, are also called the transmission zeros of the system. The relative degree ρ is defined as

$$\rho = \deg(p(s)) - \deg(q(s)) = n - m. \quad (1.36)$$

Assume $q(s)$ and $p(s)$ are coprime, then a minimum state space realization is

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -p_n & -p_{n-1} & -p_{n-2} & \dots & -p_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha \end{bmatrix} u \\ y = [q_m \dots q_1 \ 1 \ 0 \dots 0] x. \end{cases} \quad (1.37)$$

The problem is: Find, if possible, a control u and a set Z^* of initial conditions x_0 , such that $y(t) = 0, \forall t \geq 0$. If such Z^* exists, the restriction of (1.37) on Z^* is called the zero dynamics.

Calculating the derivatives of the outputs, $y^{(i)}, i = 0, 1, \dots$, one sees that

$$cA^i b = 0, \quad i = 0, 1, \dots, \rho - 2; \quad cA^{\rho-1} b \neq 0. \quad (1.38)$$

That is

$$\begin{aligned} y^{(i-1)} &= cA^{i-1}x, \quad i = 1, \dots, \rho; \\ y^{(\rho)} &= cA^\rho x + cA^{\rho-1}bu. \end{aligned} \quad (1.39)$$

It is easy to see that $cA^{i-1}, i = 1, \dots, \rho$ are linearly independent. So we can choose a new coordinate frame by letting

$$\begin{aligned} \xi_i &:= cA^{i-1}x, \quad i = 1, \dots, \rho; \\ z_i &:= x_i, \quad i = 1, \dots, m. \end{aligned} \quad (1.40)$$

Then the system becomes

$$\begin{cases} \dot{z} = Nz + M\xi_1 \\ \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_{\rho-1} = \xi_\rho \\ \dot{\xi}_\rho = Rz + S\xi + \alpha u, \\ y = \xi_1. \end{cases} \quad (1.41)$$

where

$$N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -q_m & -q_{m-1} & -q_{m-2} & \cdots & -q_1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \quad (1.42)$$

In order to keep $y(t) = 0$, we must have

$$\xi_1 = \xi_2 = \cdots = \xi_\rho = 0,$$

and

$$u = -\frac{1}{\alpha}(Rz + S\xi).$$

It follows that the zero dynamics is defined on the subspace

$$Z^* = \{x | cA^i x = 0, i = 0, \dots, \rho - 1\},$$

and represented by

$$\dot{z} = Nz.$$

The eigenvalues of N are the zeros of $q(s)$.

Next, we consider multiple input-multiple output (MIMO) systems. Similar to the SISO case, we define the transfer function matrix as

$$C(sI - A)^{-1}B.$$

The relative degree vector, denoted by $\rho = (\rho_1, \dots, \rho_p)$, is defined as

$$c_i A^k B = 0, k = 0, \dots, \rho_i - 2; \quad c_i A^{\rho_i - 1} B \neq 0, \quad i = 1, \dots, p. \quad (1.43)$$

Denote the decoupling matrix

$$D = \begin{bmatrix} c_1 A^{\rho_1 - 1} B \\ \vdots \\ c_p A^{\rho_p - 1} B \end{bmatrix}. \quad (1.44)$$

Define

$$\xi_j^i = c_i A^{j-1} x, \quad i = 1, \dots, p, j = 1, \dots, \rho_i.$$

It is easy to prove that if D has full row rank (which implies that $p \leq m$), then

$$L := \{c_i A^{j-1} | i = 1, \dots, p; j = 1, \dots, \rho_i\}$$

are linearly independent. Choosing $\{z_k = p_k x | p_k \in L^\perp\}$ (there are $n - \sum_{i=1}^p \rho_i$ linearly independent p_k). Using (ξ_j^i, z_k) as coordinates, the system (1.3) can be converted to