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Daniel B. Dix

Large-Time Behavior of Solutions of Linear Dispersive Equations

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INTRODUCTION

This work concerns the large-time asymptotic behavior of solutions of the initial-value problem for the linear dispersive equation

$$\begin{aligned} u_t + iR(D)u &= 0, \\ u(x,0) &= u_0(x), \end{aligned}$$

where the dispersive symbol $R(k)$ is homogeneous of degree $r > 1$ and is generally of the form

$$R(k) = \begin{cases} \rho_+ k^r & k > 0, \\ \rho_- (-k)^r & k < 0, \end{cases}$$

where the constants ρ_+, ρ_- are real numbers. We define $[R(D)u]\hat{}(k) = R(k)\hat{u}(k)$ where $\hat{u}(k) = \int_{-\infty}^{\infty} e^{-ikx} u(x)\,dx$ is the Fourier transform of u. Different choices for r, ρ_+, ρ_- allow us to incorporate the following special cases:

Equation	$iR(D)u$	$R(k)$	r	ρ_+	ρ_-
Linear Schrödinger	$-i\rho u_{xx}$	ρk^2	2	ρ	ρ
Linearized Benjamin-Ono	$-\rho \mathcal{H} u_{xx}$	$-\rho k\|k\|$	2	$-\rho$	ρ
Linearized KdV	ρu_{xxx}	$-\rho k^3$	3	$-\rho$	ρ

KdV is an abbreviation for Korteweg-de Vries, $\rho \in \mathbb{R}$ and $\mathcal{H}$ denotes the Hilbert transform. The corresponding nonlinear dispersive wave equations

$$\begin{aligned} u_t \pm i|u|^2 u - i\rho u_{xx} &= 0, && \text{Nonlinear Schrödinger} \\ u_t + uu_x - \rho \mathcal{H} u_{xx} &= 0, && \text{Benjamin-Ono} \\ u_t + uu_x + \rho u_{xxx} &= 0, && \text{Korteweg-de Vries} \end{aligned}$$

have been the subjects of intensive investigation in recent years because of their importance to physical applications (wave motions), because of the many fascinating properties of their solutions (such as the existence of solitons), and because there are methods (inverse scattering transforms) by which exact solutions may be computed and the pure initial-value problem solved in the same manner that the Fourier transform allows us to solve the above linear equations; see Newell [N]. It has also been found that solutions of more general nonlinear dispersive equations possess many of the same properties as solutions of these three *integrable* equations; however, the proofs of these properties cannot be

based on any special structure of integrability. Frequently the proofs are based on an analysis of solutions of the linear dispersive equation, coupled with a perturbative argument allowing one to deal with the nonlinearity. In regard to large-time behavior of solutions, no comprehensive account exists in the literature dealing with the solutions of constant coefficient linear dispersive equations (although limited and scattered results for special cases have appeared). This work is an attempt to fill this gap. One of our main goals has been to provide a reference work which researchers and applied scientists in this area could consult for the detailed large-time asymptotic expansions (and their proofs) of solutions of these linear dispersive equations. It is hoped that these results might be a stimulus to further progress in the analysis of the large-time asymptotic behavior of solutions of nonlinear dispersive equations.

Before proceeding, we offer the following as motivation for the detailed study of the large-time asymptotic behavior of solutions of linear or nonlinear wave equations. In any physical situation where wave motion is encountered after sufficient time has elapsed for the transient and incidental features of the initial disturbance to have faded, one observes the "large-time" behavior of the wave motion. Actually, the elapsed time between the initial disturbance and the point at which the transient features have become relatively insignificant need not be that large. Thus large-time asymptotic expansions can be an effective way of approximating the wave motion over even intermediate time scales. Of course there is a definite limit to the accuracy of such an approximation to the solution at a fixed time. Greater accuracy can usually only be obtained by numerical calculations. In contrast to the results of numerical computations, the large-time asymptotic expansion can frequently be determined as a closed form expression. As a consequence of this closed form expression, one gains significant and deep insight into the solution; e.g. how it decays in various regions, how various functionals of the solution behave, where the solution oscillates and where it does not, which features of the initial disturbance are preserved and which ones are not, etc.. Such knowledge is often impractical to obtain by means of numerical calculations alone. Hence the large-time asymptotic behavior is an important and frequently accessible component of knowledge about the solution which is complementary to knowledge obtained by direct numerical computations.

We make the assumption that $u_0 = |D|^\gamma(v_0 + \mathcal{H}v_1)$, where v_0, v_1 are integrable complex-valued functions with compact support, and the real number γ satisfies $\gamma > -1$. We assume that $|\hat{v}_0(0)| + |\hat{v}_1(0)| \neq 0$. These assumptions will allow us, under different choices for γ, v_0, v_1 to understand several different types of solutions.

(1) **Derivatives of solutions**: $u_0 = \partial_x^j U_0$, where j is a nonnegative integer. These can be obtained by letting $\gamma = j$ and

$$(v_0, v_1) = \begin{cases} (U_0, 0) & j \equiv 0 \mod 4, \\ (0, -U_0) & j \equiv 1 \mod 4, \\ (-U_0, 0) & j \equiv 2 \mod 4, \\ (0, U_0) & j \equiv 3 \mod 4. \end{cases}$$

The understanding of the large-time behavior of derivatives is relevant to the problem of the asymptotic balance in solutions of nonlinear dispersive equations (see section 5.3).

(2) **Fractional Derivatives or Integrals of solutions**: $u_0 = |D|^\gamma U_0$, where $\gamma > -1$ is a real number. Understanding of the case $\gamma > 0$ has been found to be useful in the study of the large-time behavior of nonlinear dispersive equations. The case $-1 < \gamma < 0$ exposes some of the variety of large-time asymptotic behaviors that can result from initial data in L^p spaces (see section 5.2).

(3) **Solutions which decay slowly as** $|x| \to \infty$: see section 5.4.

The compact support condition is convenient for us, but not absolutely necessary; our results could be generalized somewhat. Also, it should be noted that many interesting initial data which decay as $|x| \to \infty$ like a negative power of $|x|$ can be put into the above form. It is not our purpose to prove results about the solution arising from an arbitrary initial datum in some commonly used function classes (such as $L^p(\mathbb{R})$); such an assumption on u_0 is compatible with a large variety of different asymptotic behaviors. It is our purpose to prove sharp results for initial conditions of the above type because such results map out the types of asymptotic behavior which can occur, with a minimum amount of extraneous complications. In fact, our results show why it is so hard to state or prove a sharp result valid for all initial data in spaces such as $L^p(\mathbb{R})$, $1 < p < \infty$. For more details, consult chapter five.

The solution of this initial-value problem can be expressed as the following Fourier integral

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-iR(k)t}|k|^\gamma[\hat{v}_0(k) - i\,\mathrm{sgn}(k)\hat{v}_1(k)]\,dk.$$

This integral for $u(x,t)$ breaks naturally into the sum of two integrals:

$$u(x,t) = \frac{1}{2\pi}\int_0^{\infty} e^{ikx}e^{-i\rho_+ k^r t}k^\gamma[\hat{v}_0(k) - i\hat{v}_1(k)]\,dk + \frac{1}{2\pi}\int_0^{\infty} e^{-ikx}e^{-i\rho_- k^r t}k^\gamma[\hat{v}_0(-k) + i\hat{v}_1(-k)]\,dk.$$

Thus it suffices to analyze the large-time behavior of the single integral

$$I(x,t) = I(x,t;r,\rho,\gamma,V) \stackrel{\text{def}}{=} \int_0^{\infty} e^{ikx - i\rho k^r t}k^\gamma V(k)\,dk,$$

where $V(k)$ is an entire function of $k \in \mathbb{C}$. In fact, since

$$I(x,t;r,\rho,\gamma,V) = \overline{I(-x,t;r,-\rho,\gamma,\tilde{V})}$$

where $\tilde{V}(k) = \overline{V(\bar{k})}$, we see that it suffices to consider this integral when $\rho > 0$. Rather than define this integral as a limit of a family of absolutely convergent

artificially regularized integrals, we will exploit the freedom we have (by Cauchy's theorem) to change the contour of integration, so that on the new contour the integral will converge absolutely, and to the same value. The exact nature of these new contours will be discussed later.

Although our ultimate goal is to analyze the solution $u(x,t)$, we set as an intermediate goal the complete asymptotic description of the integral $I(x,t;r,\rho,\gamma,V)$. This endeavor will occupy the first three chapters. In chapters four and five we will return to the analysis of $u(x,t)$. We approach the study of the integral $I(x,t)$ first from the standpoint of matched asymptotic expansions (chapters one and two) and then we obtain an expansion which is uniformly valid as $t \to \infty$ and $(-1)^\alpha x \geq 0$ is arbitrary (chapter three). Both of these approaches have significant redeeming features and we view them as complementary.

All this is necessary because the asymptotic expansion as $t \to \infty$, $b = xt^{-1} \in \mathbb{R}$ fixed, of the function $I(bt,t)$ is essentially different from the asymptotic expansion as $t \to \infty$, $\xi = xt^{-1/r} \in \mathbb{R}$ fixed, of $I(\xi t^{1/r}, t)$. The asymptotic expansions of $I(bt,t)$ are called the *outer expansions* (studied in chapter one); there are two of them, one for $b < 0$ and the other for $b > 0$. The terms of the outer expansions involve only elementary functions and the derivatives of $V(k)$. The derivation of the outer expansions and control of the associated error terms when b is fixed can be done using the classical method for Laplace contour integrals as discussed in Olver [O1], Bleistein and Handelsman [BH], Wong [W], and other places as well. We outline this method in section 1.1. In fact, in that same section we derive some general formulae for the coefficients which appear in Laplace expansions; these formulae are useful when one is studying (as we are) the dependence of the expansion on a parameter (such as b). The study of the outer expansions is rather lengthy because there is a different expansion for each connected component of the *steepest descent contour* which the initial path of integration is deformed into. Several authors have pointed out that the complete determination of the steepest descent contours is unnecessary in order to compute the Laplace expansion as well as to control the error in the classical case where b is constant. Nevertheless, in section 1.2 we have given a fairly complete study of these contours, and a proof that the original contour can be deformed into a union of these special contours. Our error estimates must be proved under significantly weaker assumptions on b than in the classical case (for reasons which will be explained below) and the proofs (sections 1.4 and 1.7) use our detailed knowledge of the steepest descent contours. In order that we might prove sharp error estimates, it is essential that we know exactly how the derivatives of $V(k)$ behave as $k \to \infty$ on rays and in sectors in the k plane. Thus in section 1.5 we study of the leading-order behavior of $V(k) = \hat{v}(k)$, under carefully formulated conditions on a function $v(x)$ with compact support. These assumptions on $v(x)$, which we call the *standard assumptions*, are invoked extensively in chapters four and five.

Unfortunately, the asymptotic character of the outer expansions break down and the error estimates obscure the "true" leading-order terms if b is allowed to tend to 0 too rapidly as $t \to \infty$. Indeed this must happen, since as we said

above, there is an *inner expansion* of $I(\xi t^{1/r}, t)$, completely distinct from the outer expansions, which holds as $t \to \infty$, $\xi \in \mathbb{R}$ fixed. The inner expansion (studied in chapter two) is in descending powers of t with coefficients which involve the derivatives of V at $k = 0$ and certain special functions of ξ. These special functions appear naturally, and cannot be avoided. The inner expansion is derived by expanding $V(k)$ in a Taylor series at $k = 0$, and then integrating term-by-term (see section 2.1). The proofs of the sharp error estimates (sections 2.2 and 2.3) again utilize our detailed knowledge of the steepest descent contours.

The asymptotic character of the inner expansion breaks down as $|\xi| \to \infty$; hence we must live with three distinct expansions, each valid in its own asymptotic region. A complete matched asymptotic description of $I(x,t)$ is obtained by proving the validity of the outer and inner expansions in *extended* asymptotic regions ([KC]) in the (x,t) plane—the outer expansions in the disjoint regions R_+ and R_-, and the inner expansion in the inner region R_0. These extended regions will be defined precisely in sections 1.1 and 2.1, but they have the following properties. $R_\pm$ eventually (i.e for sufficiently large t) contains every ray $x = bt$, where $\pm b > 0$ is a constant. The inner region, R_0 eventually contains every curve of the form $x = \xi t^{1/r}$, where ξ is a real constant. R_+ and R_0 overlap for all $t \geq 1$; also R_- and R_0 overlap for all $t \geq 1$. Furthermore, we must prove that in the overlap of two regions the two expansions which are valid there can be matched to one another. We perform this matching in section 2.4.

The necessity of concerning oneself with carefully defined asymptotic regions and matching different expansions in an overlap region is the disagreeable aspect of the matched asymptotic description. This could be avoided if one had a single expansion which retains its asymptotic character independently of the value of x. Such an expansion is said to be *uniformly valid*. Methods for obtaining uniformly valid expansions have been studied for some time, see Wong [W] for references. When r is an integer, one finds that the phase function $kb - \rho k^r$ has exactly $r-1$ complex stationary points, some of which contribute to the asymptotic expansion of $I(x,t)$, all of which are simple and distinct for $b \neq 0$, but which coalesce as $b \to 0$. This coalescence is the opportunity for the Laplace expansions to become nonuniformly valid. The usual approach to this problem of non-uniformity, as pioneered by Chester Friedman Ursell [CFU], and developed by many others, most notably Bleistein [B], is to write $V(k)$ as the sum of a polynomial in k which agrees with $V(k)$ at *all* the problem points (stationary points, endpoints, singularities, etc.) and a remainder. Thus $I(x,t)$ breaks into two integrals: the integral involving the polynomial approximation can be expressed in terms of special functions, and provides the first term(s) of an uniformly-valid expansion of $I(x,t)$; the integral involving the remainder decays more rapidly in t than the first term(s), as one discovers by integrating by parts, whereupon a factor of t^{-1} appears multiplied by an integral of the exact same appearance as $I(x,t)$ but with a different function substituted for $V(k)$. This process can then be repeated any number of times to generate as many terms as desired. See §3.1 for further discussion.

When r is not an integer, it is not exactly clear how this procedure is to be

carried out. One problem when r is irrational is that there are now *infinitely many* stationary points on the Riemann surface forming the domain of the phase function, and it is not clear how to construct an approximating function to $V(k)$. Even when r is rational, when there are still finitely many stationary points, it is not clear how to construct the approximating function so that the remainder has good enough behavior so that the process can be repeated indefinitely, thereby generating a complete asymptotic expansion. When r is not an integer, it is not clear that the integration by parts procedure can be generalized.

Thus in section 3.1 we use a different procedure to generate a complete uniformly-valid asymptotic expansion of $I(x,t)$, an approach which will work for all real $r \geq 2$. The basic idea is simple. We separate $I(x,t)$ as a sum of integrals over the connected steepest descent contours employed in the analysis of the outer region. Then we expand $V(k)$ in a Taylor series about the saddle point; each connected contour passes through at most one saddle point. When this Taylor expansion is integrated term-by-term, certain special functions of the variable ξ emerge because of the scaling properties of the phase function. This yields an uniformly-valid expansion of the integral over each steepest descent contour. These expansions are then added together to obtain an uniformly-valid expansion of $I(x,t)$. These expansions are written down in full detail, together with their sharp error estimates, in section 3.2. The method of integrating Taylor expansions (or other expansions closely related to Taylor expansions) for generating uniformly-valid asymptotic expansions was used in the original paper of Chester, Friedmann, and Ursell [CFU], and on pages 352–357 in the book of Olver [O1]. As far as we can tell, the idea of applying this method to each connected contour of steepest descent is new. Since the nature of the steepest descent contours depends on whether $b > 0$ or $b < 0$, the resulting uniformly-valid expansions look different in these two cases. But they retain their asymptotic character even as $x \to 0$. In the inner region they are shown to agree (in section 3.1) with the inner expansion, which is the same for both $x > 0$ and $x < 0$. We obtain full asymptotic expansions as $|\xi| \to \infty$ of the special functions involved, and show (in section 3.1) that the uniformly-valid expansions reduce to the outer expansions in the outer regions.

The only disadvantage of the uniform expansions is that they involve a significant number of special functions, which are themselves defined by integrals. When $r = 2$ (section 4.1) or $r = 3$ (section 4.5) these can be identified with or expressed in terms of well-known special functions, such as confluent hypergeometric functions, or Airy functions; but for other values of r these special functions have not been studied as much. We need the fundamental special function

$$F_n(r,\delta;y) \stackrel{\text{def}}{=} \int_0^\infty \exp(-\tfrac{1}{r}\sigma^r + y\sigma)\sigma^{\delta-1}(\sigma - y^{\frac{1}{r-1}})^n \, d\sigma,$$

where $n \geq 0$ is an integer, $\delta > 0$ is a real number, and y lies on the Riemann surface of the function $y^{\frac{1}{r-1}}$, i.e. $y \in S\{y^{\frac{1}{r-1}}\}$. When $r \geq 3$ is an odd integer we also need two incomplete forms of this function. All the other special functions

we need can be expressed in terms of these. When $r = 2$ or $r = 3$ we also derive recursion relations for these functions in the variable n, which allow us to reduce the number of special functions to a small number (i.e. r) of truly essential functions. These recursion relations also allow us to compare our uniformly-valid expansions to the uniformly-valid expansions which can be obtained (when $r = 2, 3$) by the integration-by-parts procedure mentioned above. Especially in the case $r = 3$ (in section 4.5) we discuss the relative merits of these two types of uniformly-valid expansions. As mentioned above, an expansion generated by an integration-by-parts procedure is not available when r is not an integer.

It turns out that even the leading-order terms of uniformly-valid asymptotic expansions are not uniquely determined. Since the higher-order terms depend very sensitively on the exact form of the leading-order term, they are also not uniquely determined. Despite this nonuniqueness it is worthwhile to compute all the terms of the asymptotic expansion. Knowledge of the higher terms in the expansion allows us to formulate sharp error estimates for finite term asymptotic approximations. Also, as we have already mentioned, asymptotic expansions can be used to compute the solution $u(x,t)$, and a variable number of terms of the expansion are needed in such calculations, depending on the particular point (x,t). We will not discuss the realistic computable *a priori* error bounds needed to decide if the desired accuracy has been obtained with a given number of terms.

In chapter four, in addition to the above topics, we also return to the original subject of the solution of the linear dispersive equation—but only in the special cases $r = 2$ and $r = 3$. We devote an entire section to each of the three linear equations listed as examples earlier, the Linear Schrödinger (section 4.3), the Linearized Benjamin-Ono (section 4.4), and the Linearized Korteweg-de Vries (section 4.6) equations. In each of these three cases we write down the complete uniformly-valid asymptotic expansions, together with the sharp error estimates. We also give the leading-order terms of the outer and inner expansions, and their error estimates. In the case $r = 2$ the outer expansion (and our uniformly-valid expansion) can lose its asymptotic character in the limit $|x| \to \infty$, $t \geq 1$ fixed. Thus we find the correct leading-order term in this limit, and prove an error estimate in section 4.2. This enables us to give the correct leading-order term of the asymptotics of the solutions of the Linear Schrödinger and Linearized Benjamin-Ono equations as $|x| \to \infty$, $t \geq 1$ fixed, when the initial data is only of moderate smoothness.

In chapter five we begin to discuss some applications of our results in the previous chapters to solutions of the linear dispersive equation with general $r \geq 2$. Although we restrict our discussion to equations possessing real-valued solutions, such as Linearized Benjamin-Ono and Linearized KdV, most of what we say can be reformulated to apply to Schrödinger-type equations, i.e. to $u_t + i\rho|D|^r u = 0$. Although we do not formulate or prove any results about solutions of nonlinear dispersive equations, everything in chapter five is directed toward the understanding of the nonlinear case. We have tried to make it possible to read chapter five without having to digest chapters one through four, although we have probably not succeeded completely. We make frequent references to results in the

literature on nonlinear dispersive equations, and attempt to describe succinctly the relevant features of the theorems appearing in those papers, but we are not attempting a survey of such results. We hope our discussion will be helpful to both the expert and the newcomer to the field. It will be apparent that there is a great deal which is still unknown about the nonlinear case, and one of our main goals is to formulate some reasonable conjectures about when we can expect solutions of a nonlinear dispersive equation to behave to the leading-order as $t \to \infty$ like solutions of the corresponding linear dispersive equation. We do not completely reach even this goal; but we do identify some of the main unresolved issues which we feel are of central importance.

We have tried to keep the prerequisites to a minimum for the reader. A reasonable knowledge of single variable complex analysis (for example on the level of the book of Ahlfors, [Ahl]) should be adequate preparation for chapters one through three. We refer to Riemann surfaces several times, but we only use the fundamental idea of them as natural domains of certain analytic functions, especially when we wish to avoid a detailed discussion of branches of multivalued functions. In chapters four and five we also assume a certain familiarity with Fourier analysis, and Fourier multiplier operators, such as $|D|^\gamma$ or $\mathcal{H} = -i\,\mathrm{sgn}(D)$. Whenever a line of argument is presented for the first time, we try to write it out in detail. Considerable effort has been expended to make our expansions explicit, useful in computations, and reliable. In particular, we have endeavored to relate our special functions as much as possible to those discussed in Olver [O1] and Abramowitz and Stegun [AS]. However, it is inevitable that some errors are present. The author would appreciate being made aware of any errors the reader discovers in the text or formulae.

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CONTENTS

CHAPTER 1

LAPLACE EXPANSIONS, OUTER REGIONS

1.1 Generalities, Notation

We define $b = x/t$ and $\varphi(k; b, r, \rho) = ikb - i\rho k^r$. We will usually suppress whatever parameters from our notation that are understood from the context. With our new notation we have

$$I(x, t; r, \rho, \gamma, V) = \int_0^\infty e^{\varphi(k;b)t} k^\gamma V(k)\, dk.$$

$V(k)$ is an entire function, the Fourier transform of an integrable function with compact support. Techniques for computing asymptotic expansions as $t \to \infty$ for this integral when b is fixed can be found in [O1], [BH], [W]. The outer regions $R_\pm$ are defined as follows:

$$R_\pm = \{(x, t) \in \mathbb{R} \times [1, \infty) \mid \pm x t^{-\beta} > \epsilon\},$$

where $\frac{1}{r} < \beta < 1$ and $0 < \epsilon < 1$ are constants. The principal contributions in this case arise from the singularity (or zero) and endpoint of integration at $k = 0$ and from the relevant saddle points $k_j \in \mathbb{C}$, where $\varphi'(k_j) = 0$. Since $\varphi'(k) = ib - ir\rho k^{r-1}$, we have that

$$k_j(b, r) = B^{\frac{1}{r-1}} e^{iS_j^\alpha}$$

where $\operatorname{sgn}(b) = (-1)^\alpha$, $\alpha \in \{0, 1\}$, $B = \left|\frac{b}{r\rho}\right|$, $S_j^\alpha = \frac{\pi}{r-1}(2j + \alpha)$, and $j \in \mathbb{Z}$. Not all of these saddle points are asymptotically relevant. Exactly which saddle points are relevant will be revealed when we deform the original contour of integration into a union of *steepest descent contours* (SDCs) starting at $k = 0$, (maybe) through some saddle points, and proceeding to infinity in the nearest *valley* to the ray $\arg k = 0$. A *valley* is a region in the complex k-plane where $\Re\varphi(k) \leq -M$, where $M \gg 1$. Likewise a *hill* is a region in the complex k-plane where $\Re\varphi(k) \geq M$, where $M \gg 1$. The asymptotic centers of the valleys (resp. hills) can be identified by rays along which $\Re\varphi(k)$ decreases (resp. increases) most rapidly as $k \to \infty$. If $k = Re^{i\theta}$ we have that

$$\Re\varphi(Re^{i\theta}) = -bR\sin(\theta) + \rho R^r \sin(r\theta).$$

Hence the asymptotic center of the valleys are along the rays $\theta = V_j = \frac{1}{r}(-\frac{\pi}{2} + 2\pi j)$, where $j \in \mathbb{Z}$. Likewise the asymptotic center of the hills are along the rays

$\theta = H_j = \frac{1}{r}(\frac{\pi}{2} + 2\pi j)$, where $j \in \mathbb{Z}$. We will refer to hills and valleys by the arguments of their asymptotic centers.

The initial contour of integration (i.e. $\theta = 0$) is between a hill (at $\theta = H_0 = \frac{\pi}{2r}$) and a valley (at $\theta = V_0 = -\frac{\pi}{2r}$). The ray of integration cannot be rotated in the direction of the hill, but it can be rotated toward the valley. But it cannot be rotated beyond that valley up onto the next hill, i.e. beyond $\theta = -\frac{\pi}{r}$. Thus the deformed path of integration, whatever else it does, must eventually go to infinity in the valley V_0. We call this valley the *final* valley.

A steepest descent contour (SDC) starting at a (simple) saddle point k_j leaves tangent to a certain line through the saddle point and proceeds downhill as fast as possible, i.e. always travelling in the opposite direction as the gradient vector of $\Re\varphi(k)$. In order to determine this line we must expand $\varphi(k)$ in the neighborhood of the saddle point. A short calculation will verify that

$$\varphi(k_j) = (-1)^\alpha i(r-1)\rho B^{\frac{r}{r-1}} e^{iS_j^\alpha},$$
$$-\varphi''(k_j) = ir(r-1)\rho B^{\frac{r-2}{r-1}} e^{i(r-2)S_j^\alpha}.$$

Suppose k is on the ray $k - k_j = Re^{i\omega}$. Then

$$\begin{aligned} t\varphi(k) &= t\varphi(k_j) + \frac{t}{2}\varphi''(k_j)R^2 e^{i2\omega} + \mathrm{O}(R^3) \\ &= t\varphi(k_j) - \frac{|t|}{2}|\varphi''(k_j)|R^2 \exp\left[i\arg(t) + i\omega_0 + i2\omega\right] \\ &\quad + \mathrm{O}(R^3), \end{aligned}$$

where $0 < R < 1$ and $\omega_0 = \arg(-\varphi''(k_j)) = \frac{\pi}{2} + (r-2)S_j^\alpha + 2\pi\mathbb{Z}$. Hence we must choose $\arg(t) + \omega_0 + 2\omega = 0$ if the ray is to be tangent to a SDC. This equation determines ω up to an integral multiple of π:

$$\omega = -\frac{\arg(t)}{2} - \frac{(2\alpha+1)}{4}\pi + \frac{1}{2}S_j^\alpha + \pi\mathbb{Z}.$$

Following Olver, we will first make a choice of ω according to the above. Since square roots of the quantity $-\varphi''(k_j)t$ appear in the terms of the expansion arising from this saddle point, it will be convenient to require that $\arg(t) + \omega_0$ be determined by the choice of ω by the relation $\arg(t) + \omega_0 + 2\omega = 0$. The value of the square root will then be determined by that choice of argument. So if ω changes by π, such as happens when one departs the saddle point in the opposite direction as before, the value of $[-\varphi''(k_j)t]^{1/2}$ changes by a factor of -1. Obviously, it is important that this choice of branch be used consistently.

Steepest descent contours turn out to be along level curves of $\Im\varphi(k)$ (see [BH]). We note that

$$\Im\varphi(Re^{i\theta}) = bR\cos(\theta) - \rho R^r\cos(r\theta).$$

Thus the SDC through the saddle point k_j is part of the polar locus

$$bR\cos(\theta) - \rho R^r\cos(r\theta) = (-1)^\alpha(r-1)\rho B^{\frac{r}{r-1}}\cos(S_j^\alpha).$$