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PREFACE

Gauge theory is generally recognized to provide us with the adequate picture of the fundamental interactions. The gauge approach to the gravitation interaction establishes two main features of gravity as a physical field.

Gravitation phenomena are described by two geometric fields. These are an Einstein (tetrad or metric) gravitational field and a Lorentz connection. A Lorentz connection plays the role of a gauge gravitational potential induced by a gauge potential of fermion fields.

An Einstein gravitational field is a Higgs field which accompanies the spontaneous breaking of world symmetries. This spontaneous breaking takes place because of the coexistence of the Dirac fermion matter with exact Lorentz symmetries and the world geometric arena. Moreover, in contrast with Higgs fields of the Grand Unification models, Einstein gravitational field is a macroscopic Higgs field due to the peculiarity of gauge world transformations.

This book concerns only the gauge theory of the classical gravity. In the algebraic quantum theory, Higgs fields characterize nonequivalent Gaussian states on the algebras of quantum fields. They are "fictitious" fields describing collective phenomena. These fields fail to be quantized in the framework of the conventional quantum field theory. The Higgs nature of gravity therefore may open the door to many unexpected quantum effects.

Our formulation of gauge theory uses the machinery of modern differential geometry. Preliminary and Chapter 1 of this book are intended as an introduction to the jet bundle formalism and to the geometric theory of classical fields. In this book, we consider those aspects of gauge theory which explain local phenomena, although the theory itself is formulated in global terms.

INTRODUCTION

The geometric nature of classical gravity as a metric field has been established by Einstein's General Relativity. Its physical feature as a Higgs-Goldstone field corresponding to spontaneous breakdown of world symmetries is clarified owing to the gauge reformulation of gravitation theory in fibre bundle terms. Thus, gravity joins the unified gauge picture of the fundamental interactions.

The main problem of the gauge gravitation theory consists in that an Einstein gravitational field is a metric (or tetrad) field, whereas gauge potentials are connections. To settle this dilemma, many authors attempted to use the seeming identity of the tensor ranks of tetrad functions h^a_μ and gauge potentials σ^ν_μ of the translation subgroup of the Poincaré group (Section 4.1). They lost sight of Higgs-Goldstone fields appearing in gauge models due to spontaneous symmetry breaking. Moreover, the standard Yang-Mills scheme of gauge theory based on replacing global symmetries by the local ones appeared to be unsatisfactory for gauge theory of world symmetries. For instance, the holonomic transformations fail to be reproduced in this way. Besides, there are different types of gauge transformations (atlas transformations, principal morphisms, gauge freedom transformations etc.) which the conventional gauge principle fails to discern.

We therefore are based directly on the fibre bundle reformulation of classical field theory (Section 1.1). The necessary mathematical machinery can be exhausted by references [KOB, SUL, SAU, MAN 1991].

In bundle terms, classical fields are described by sections ϕ of some differentiable bundle E over a world manifold X. To construct differential operators and field Lagrangians one may use the jet bundle formalism. In its framework, a Lagrangian density $\mathcal{L}$ of fields ϕ is defined on the 1-jet manifold J^1E of the bundle E. Elements of J^1E are equivalence classes $j^1_x\phi$ of sections ϕ possessing the same values $\phi(x)$ and the same values of their first derivatives $\partial_\mu\phi(x)$ at $x \in X$. Given the world coordinates (x^μ) on X and the bundle coordinates (x^μ, y^i) on E, the jet manifold J^1E of the bundle E is endowed with the so-called adapted coordinates

$$(x^\lambda, y^i, y^i_\lambda) \circ j^1\phi = (x^\lambda, \phi^i(x), \partial_\lambda\phi^i(x)).$$

The jet manifold plays the role of a finite-dimensional configuration space of classical fields ϕ.

The corresponding finite-dimensional momentum space of fields ϕ is represented by the Legendre manifold Π provided with the so-called standard coordinates $(x^\lambda, y^i, p^\lambda_i)$ (Section 1.3). Given a Lagrangian density $\mathcal{L}$, we have the Legendre morphism

$$(x^\lambda, y^i, y^i_\lambda) \to (x^\lambda, y^i, p^\lambda_i = \partial^\lambda_i \mathcal{L}).$$

and the multimomentum Hamiltonian

$$\mathcal{H}(x^\lambda, y^i, p^\lambda_i) = p^\lambda_i y^i_\lambda - \mathcal{L}.$$

Let us remark that, in the multimomentum Hamiltonian formalism, time and spatial coordinates are considered on the same footing and so, this machinery does not require the preliminary (3+1) decomposition of a world manifold.

The jet bundle formalism enables us to manipulate general connections defined as sections Γ of the bundle $J^1E \to E$. Principal connections keep their physical importance because, to construct a gauge invariant Lagrangian, one must reduce the bundle E to the one associated with some principal bundle. We use general connections in the models of spontaneous symmetry breaking and in the multimomentum Hamiltonian formalism. For instance, there is the canonical splitting of a multimomentum Hamiltonian

$$\mathcal{H} = p^\lambda_i \Gamma^i_\lambda(y) + \widetilde{\mathcal{H}}$$

where Γ is some general connection on the bundle E.

In fibre bundle form, the gauge principle is reduced to the natural requirement of Lagrangians (or multimomentum Hamiltonians) be invariant under transformations of the adapted coordinates on the configuration space J^1E (or the standard coordinates on the Legendre bundle Π). Such coordinates are induced by atlases of the bundle E and the tangent bundle TX over a world manifold X. In field theory, these atlases define internal and world reference frames. The gauge principle thus makes the sense of a relativity principle.

To construct a gauge invariant Lagrangian, one needs a metric a^E in fibres of the bundle E and a world metric g in fibres of the cotangent bundle T^*X. By gauge transformations, a fibre metric a^E can be always brought into a canonical form invariant under a structure group G of the bundle E. In contrast with a^E, a world metric g takes a canonical form η only with respect to nonholonomic atlases of T^*X in general and η is invariant only under some subgroup of world symmetries. It means that, in gauge theory of world symmetries there is a dynamic metric field besides a gauge potential Γ.

The relativity principle however does not require g be a pseudo-Riemannian metric and Γ be a gauge potential of the Lorentz group. One therefore needs a supplementary principle besides the gauge one in order to reduce gauge theory of world symmetries to the gauge gravitation theory. This is the equivalence principle.

In Einstein's General Relativity, the equivalence principle is called to guarantee the transition to the Special Relativity with respect to some reference frames. There exist various formulations of this principle. Most of them are corollaries of geometrization of a gravitational field by components of a pseudo-Riemannian metric. The equivalence principle that we need must result in the existence of a pseudo-Riemannian metric itself. In geometric terms, we have formulated this principle as follows [IVA 1983].

In the Minkowski space, a time coordinate parameterizes the set of events ordered by the genetic relations. Lorentz transformations describe the transformations of these relations under changing a reference frame. In the spirit of Klein's Erlanger program, the Minkowski space geometry can be characterized as the geometry of Lorentz invariants. The geometric equivalence principle then postulates that, with respect to some reference frames, Lorentz invariants can be defined everywhere on a world manifold X^4 and are preserved by parallel displacement. This principle has the adequate fibre bundle formulation. It requires that the principal linear frame bundle LX with the structure group

$$GL_4 = GL(4, \mathbb{R})$$

be reduced to some subbundle L^hX with the structure Lorentz group

$$L = SO(3,1)$$

and so, that a gravitational field h exist on a world manifold X^4 (Section 2.2). There is 1:1 correspondence between the reduced L-subbundles L^hX and the tetrad gravitational fields h represented by global sections of the LX-associated Higgs bundle Σ with the standard fibre GL_4/L. This bundle is isomorphic to the 2-fold covering of the bundle Λ of pseudo-Riemannian forms in cotangent spaces T^*_xX to X^4. A global section of Λ is a pseudo-Riemannian metric g on X^4. The geometric equivalence principle thereby provides X^4 with the so-called L-structure [SUL]. This means the following.

A principal connection Γ^h on the linear frame bundle LX is assumed to be an extension of some connection A on the reduced subbundle L^hX. A world manifold X^4 is a pseudo-Riemannian space with the metric g corresponding to the reduced subbundle L^hX. Atlases of L^hX are extended to the atlases Ψ^h of LX possessing Lorentz transition functions. With respect to Ψ^h, metric functions of g are reduced to the Minkowski metric η and the local connection form Γ^h_κ takes its values in the Lie algebra of the Lorentz group, that is, its coefficients represent components of a Lorentz gauge potential. We call Γ^h a Lorentz connection. It plays the role of a gauge gravitational potential. There is the canonical splitting of Γ^h in the sum

$$\Gamma^h = \{\ \} + S.$$

of the Christoffel symbols $\{\ \}$ of the metric g and the contortion form S. The gauge gravitation theory thereby is the theory of gravity with torsion in general [HEL, IVA 1983, OBU].

The geometric equivalence principle defines some space-time structure on a world manifold X^4 (Section 2.3). For every reduced subbundle L^hX, there exist reduced subbundles $L^{\mathbf{F}}X$ of LX with the structure group $SO(3) \subset L$. There is 1:1 correspondence between these subbundles and the smooth distributions $\mathbf{F}$ of

3-dimensional spatial subspaces of tangent spaces T_xX. Such a distribution yields the (3+1) decomposition of the tangent bundle TX over X^4 into the direct sum of the 3-dimensional spatial subbundle $\mathbf{F}$ and its time-like orthocomplement T^0X. This decomposition turns a world manifold into a space-time. In particular, some types of gravitation singularities can be described as singularities of space-time distributions.

The geometric equivalence principle singles out the Lorentz group as the exact symmetry subgroup of world symmetries broken spontaneously. The corresponding Higgs-Goldstone field is a classical metric (or tetrad) gravitational field.

Spontaneous symmetry breaking is the quantum phenomenon. It takes place if, given a symmetry group G and its subgroup H, a Gaussian state F on an algebra of matter fields is H-stable and nonequivalent to any G-stable state [SAR']. There are two types of spontaneous symmetry breaking:

(i) States Fg, $g \in G$, are equivalent to F.

(ii) States Fg, $g \in G$, are nonequivalent to F, e.g., if matter fields possess only the exact symmetry group H.

Spontaneous breaking of world symmetries belongs to the type (ii). The corresponding matter fields are Dirac fermion fields on which the Clifford algebra of Dirac's γ-matrices and the Dirac operator act. There are various spinor models of the fermion matter. For instance, infinite-dimensional representations of the group $SL(4,\mathbb{R})$ are examined [NEE 1985] and, in this case, the above-mentioned spontaneous breakdown of world symmetries takes no place. All observable fermion particles however are Dirac fermions.

Let E be a spinor bundle whose sections describe classical Dirac fermion fields ϕ on a world manifold X^4. There is an associated bundle E_M of Minkowski spaces with the structure Lorentz group so that the bundle morphism

$$\gamma_E\colon\ E_M \otimes E \to E$$

exists and defines representation of elements of E_M by Dirac's γ-matrices on elements of E. To define the Dirac operator on sections of E, one must require E_M be isomorphic to the cotangent bundle T^*X over a world manifold X^4. Since the structure group of T^*X is GL_4, it takes place only if there is some reduced L-subbundle L^hX of the linear frame bundle LX and E_M is associated with L^hX, that is, if the geometric equivalence principle holds. The cotangent bundle T^*X provided only with atlases Ψ^h possesses the structure of the Minkowski space bundle M^hX associated with the reduced subbundle L^hX. For different tetrad fields h and h', bundles M^hX and $M^{h'}X$ are not isomorphic to each other. Their fibres M_x and M'_x are cotangent spaces T^*_xX, but provided with different Minkowski space structures.

The peculiarity of gravitational field thus is clarified. In contrast to the other fields, a tetrad gravitational field itself defines reference frames and these reference frames corresponding to different gravitational field are nonequivalent in a sense.

Let the Minkowski space bundle E_M associated with a spinor bundle E be isomorphic to the bundle

$$T^*X \cong M^h X.$$

Then, one can define the representation

$$\gamma_h\colon M^h X \otimes E \to E,$$
$$\gamma_h(dx^\mu) = h^\mu_a(x)\gamma^a$$

of cotangent vectors to X^4 (that is, 1-forms) by Dirac's γ-matrices on sections of a spinor bundle E. We denote such a spinor bundle (endowed with the representation morphism γ_h) by E^h (Section 2.2). Sections of E^h describe Dirac fermion fields ϕ_h in the presence of the tetrad gravitational field h.

Moreover, each principal connection A_s on the spinor bundle E^h induces a certain principal connection A on the reduced subbundle $L^h X$ of LX and A is uniquely extended to a Lorentz connection Γ^h on the linear frame bundle LX. In other words, gauge potentials A_s of fermion fields generate gauge gravitational potentials.

The Higgs character of gravity issues from the fact that different gravitational fields h and h' define the nonisomorphic representations γ_h and $\gamma_{h'}$. It follows that Dirac fermion fields must be considered only in a pair with a certain gravitational field. These pairs fail to be represented by sections of the bundle product $\Sigma \times E$ of the Higgs bundle Σ and some spinor bundle E, but form *sui generis* a fermion-gravitation complex (Section 3.1). To describe this complex, we use the fact that the total space P of the principal bundle LX represents the total space of the L-principal bundle P^L over the Higgs manifold

$$\Sigma = P/L.$$

The Higgs manifold Σ is parameterized both by coordinates x^μ of a world manifold X^4 and by values σ^μ_a which tetrad gravitational fields take in the quotient space GL_4/L. The manifold Σ is the finite-dimensional analogue of the Wheeler-DeWitt superspace in a sense.

Let $E^L \to \Sigma$ be a spinor bundle associated with P^L. Fermion-gravitation pairs can be represented by sections of the composite bundle

$$\widetilde{E} = E^L \to \Sigma \to X$$

over X. This bundle however is not associated with a principal bundle and so, does not admit a principal connection. To define a connection on $\widetilde{E}$, one uses principal connections on the bundles Σ and P^L and the canonical jet bundle morphism

$$J^1\Sigma \underset{\Sigma}{\times} J^1 E^L \to J^1\widetilde{E}.$$

As a result, covariant derivatives of fermion fields include the additional term due to parallel displacement along the coordinates σ^μ_a of the Higgs bundle Σ.

Since, for different gravitational fields h and h', the representations γ_h and $\gamma_{h'}$ are not isomorphic, tetrad gravitational fields, unlike matter fields and gauge potentials, fail to form a linear space or an affine space modelled on a linear space of deviations from some background field. They thereby do not satisfy the superposition principle and can not be quantized by usual methods because, in accordance with the conventional quantum field theory, fields must form a linear space in order to be quantized.

This is the common feature of Higgs fields. In algebraic quantum field theory, different Higgs fields correspond to nonequivalent Gaussian states on an algebra of matter fields. Quantized deviations of a Higgs field can not change a Gaussian state of this algebra, and so fail to result in some new Higgs field. A Higgs field thereby is a classical field. If one considers its small classical deviations being superposable in some approximation, their quantums turn out to be quasi-particles, not true particles.

A classical tetrad gravitational field as a Higgs field also is "fictitious" in a sense. It describes a field of invariance relations which is preserved by parallel displacement. For instance, a momentum part of the multimomentum Hamiltonian form for the classical gravity is reduced only to the connection term. Thus, quantization of tetrad (metric) gravitational fields goes beyond the framework of the standard quantum field theory.

At the same time, one can examine superposable deviations σ of a tetrad gravitational field h such that

$$h + \sigma$$

is not a tetrad gravitational field (Section 3.2). For instance, they do not change atlases Ψ^h and the world metric g. These deviations are generated by non-Lorentz transformations of fibres of T^*X and thereby violate the isomorphism between E_M and T^*X. Such transformations look like deformations of a world manifold in the gauge theory of space-time translations (Section 4.2). A Lagrangian of superposable deviations σ differs from the familiar Lagrangians of gravitation theory. For instance, contains the mass-like terms.

In other words, the superposable deviations σ of a tetrad gravitational field can destroy the correlation between the Dirac fermion matter and the space-time geometric arena. At the same time, if world symmetries are not broken (e.g., there are no fermion fields), transmutations of $h+\sigma$ into a new gravitational field h' may take place and may be accompanied by violation of the usual energy conservation law.

In the Grand Unification models, appearance of a Higgs field is usually related to a phase transition. A gravitational field also might arise owing to some primary phase transition which had separated prematter and pregeometry.

PRELIMINARY

We assume that all morphisms are smooth (that is, of class $\mathbb{C}^\infty$) and manifolds are real, Hausdorff, finite-dimensional, and second-countable (as a consequence, paracompact). Unless otherwise stated, structure groups of bundles are real finite-dimensional Lie groups.

By $\wedge$, we denote exterior product (i.e., skew-symmetric tensor product) of cotangent vectors.

Interior product (pairing) of tangent vectors with cotangent vectors is denoted by $\rfloor$.

0.1 Bundles

By a bundle, we mean a locally trivial fibre bundle

$$\pi\colon E \to B$$

whose total space E and base B are manifolds. For the sake of simplicity, we denote a bundle by its total space symbol E.

We use y and x in order to denote points of E and B respectively.

Given a bundle E and another bundle

$$\pi'\colon E' \to B',$$

a bundle morphism of E to E' is defined to be a pair of manifold morphisms

$$\Phi\colon E \to E', \qquad \Phi_B\colon B \to B'$$

such that

$$\pi' \circ \Phi = \Phi_B \circ \pi.$$

One says that Φ is a bundle morphism over Φ_B. If

$$\Phi_B = \mathrm{id}\, B,$$

then Φ is called a bundle morphism over B.

Let E and E' be bundles over the same base B. We denote their bundle product over B by

$$E \underset{B}{\times} E'.$$

There are two bundle projections

$$\mathrm{pr}_1\colon E \underset{B}{\times} E' \to E, \qquad \mathrm{pr}_2\colon E \underset{B}{\times} E' \to E'.$$

Given a bundle E and a manifold morphism

$$\Phi_B\colon B' \to B,$$

the pull-back of E by Φ_B is defined to be the bundle

$$\Phi_B^* E = \{(y, x') \in E \times B';\ \pi(y) = \Phi_B(x')\}$$

with the base B' and projection

$$\Phi_B^*\pi\colon (y, x') \to x'.$$

In particular, each section e of E yields the pull-back section

$$\Phi_B^* e(x') = (e(\Phi_B(x')), x')$$

of $\Phi_B^* E$. There is the bundle morphism

$$\overset{\circ}{\Phi}_B\colon \Phi_B^* E \ni (y, x') \to y \in E. \tag{0.1}$$

We provide a bundle E with local bundle coordinates

$$(x^\lambda, y^i), \quad 1 \leq \lambda \leq n = \dim B,$$
$$1 \leq i \leq l = \dim E - \dim B,$$

which are compatible with the bundle fibration of E, that is,

$$\mathrm{pr}_1 \circ (x^\lambda, y^i) = x^\lambda \circ \pi.$$

In particular, if

$$\Psi = \left\{U_\kappa, \psi_\kappa\colon \pi^{-1}(U_\kappa) \to U_\kappa \times F\right\}$$

is a bundle atlas of E, coordinates y^i on E can be induced by coordinates v^i on a standard fibre F of the bundle E:

$$y^i = v^i \circ \psi_\kappa. \tag{0.2}$$

We call coordinates (0.2) the bundle coordinates associated with an atlas Ψ.

In field theory, one is usually concerned with bundles possessing additional algebraic structure.

A group bundle is defined to be a bundle E together with canonical bundle morphisms m and k over B and a global section e_E of E:

$$\begin{aligned} &m\colon\ E \underset{B}{\times} E \to E, \\ &k\colon\ E \to E, \\ &e_E\colon\ B \to E. \end{aligned} \tag{0.3}$$

They make each fibre

$$E_x = \pi^{-1}(x)$$

of the bundle E into a Lie group:

$$\begin{aligned} &m(e_E(x), y) = m(y, e_E(x)) = y, \\ &m(k(y), y) = m(y, k(y)) = e_E(x), \quad y \in E_x. \end{aligned}$$

For instance, a vector bundle E possesses the structure of an additive group bundle. In this case, e_E is the canonical zero section of E.

A general affine bundle is defined to be a triple (E, E', r) of a bundle E, a group bundle E' over B, and a bundle morphism

$$r\colon\ E \underset{B}{\times} E' \to E$$

which makes each fibre E_x of E into a general affine space with the associated group E'_x acting freely and transitively on E_x.

In particular, if a group bundle is a vector bundle $\overline{E}$, a general affine bundle is called an affine bundle modelled on a vector bundle $\overline{E}$:

$$\begin{aligned} &r_E\colon\ E \underset{B}{\times} \overline{E} \to E, \\ &r_E\colon\ (y, \overline{y}) \to y + \overline{y}. \end{aligned}$$

For instance, every vector bundle E can be provided with the canonical structure of an affine bundle (of translations in E) modelled on E by means of the morphism

$$r_E\colon\ (y, y') \to y + y'.$$

A principal bundle $P \to B$ with a structure group G is defined to be a general affine bundle with respect to the trivial group bundle $B \times G$ where the group G acts on P on the right:

$$r_g\colon\ P \to Pg, \qquad g \in G. \tag{0.4}$$

A standard fibre of a principal bundle P is its structure group G which acts on itself on the left. Fibres of P are diffeomorphic to the group space of G, but fail to be groups.

A principal bundle P is a general affine bundle also with respect to the principal group bundle $\widetilde{P}$. This is the P-associated bundle with the standard fibre G on which the structure group acts by the adjoint representation

$$\operatorname{ad} g\colon\ G \to gGg^{-1}, \qquad g \in G.$$

Fibres of $\widetilde{P}$ are groups isomorphic (but not canonically isomorphic) to the structure group G. Moreover, for any P-associated bundle E, the canonical bundle morphism

$$\widehat{P}_E\colon\ E \underset{B}{\times} \widetilde{P} \to E \tag{0.5}$$

is defined.

Remark. Given a principal bundle

$$\pi_P\colon\ P \to B$$

with a structure group G, a total space of a P-associated bundle E with a standard fibre F is defined to be the quotient $(P \times F)/G$ of the product $P \times F$ by identification of elements (p, v) and $(pg, g^{-1}v)$ for all $g \in G$. A global section e of E then is determined by a F-valued equivariant function f_e on P such that

$$e(\pi_P(p)) = [p]_F f_e(p), \qquad p \in P,$$
$$f_e(pg) = g^{-1} f_e(p), \qquad g \in G,$$

where $[p]_F$ denotes the restriction of the canonical map

$$P \times F \to E$$

to the subset $p \times F$.

Let (E_1, E_1', r_1) and (E_2, E_2', r_2) be general affine bundles. An affine bundle morphism $E_1 \to E_2$ is a pair of bundle morphisms

$$\Phi\colon\ E_1 \to E_2, \qquad \Phi'\colon\ E_1' \to E_2'$$

such that

$$r_2 \circ (\Phi, \Phi') = \Phi \circ r_1.$$

For instance, let $P \to B$ and $P' \to B'$ be principal bundles with a structure group G. Then, an affine bundle morphism of P to P' is defined to be a G-equivariant bundle morphism

$$(\Phi, \Phi' = (\Phi_B, \mathrm{id}\, G))$$

over a manifold morphism

$$\Phi_B\colon\ B \to B'$$

such that, whenever $g \in G$,

$$r'_g \circ \Phi = \Phi \circ r_g.$$

Every principal isomorphism of a principal bundle P (over the identity morphism of its base B) is expressed as

$$\Phi_P(p) = pf_s(p), \qquad p \in P, \tag{0.6}$$
$$f_s(pg) = g^{-1}f_s(p)g, \quad g \in G,$$

where f_s is a G-valued equivariant function on P corresponding to some global section s of the principal group bundle $\widetilde{P}$. Principal isomorphisms thus form the gauge group $G(B)$ which is canonically isomorphic to the group of global sections of the bundle $\widetilde{P}$.

Remark. There is no canonical embedding of G into $G(B)$ even if P is a trivial bundle. Elements of $G(B)$ take their values in fibres of the group bundle $\widetilde{P}$, but not in its standard fibre G.

Given a P-associated bundle E with a standard fibre F, every principal isomorphism (0.6) yields the associated principal morphism

$$\Phi_E\colon\ (P \times F)/G \to (\Phi_P(P) \times F)/G \tag{0.7}$$

of the bundle E so that

$$\Phi_E = \widehat{P}_E|_{E\times_B s(B)}.$$

Given affine bundles E and E' modelled on vector bundles $\overline{E}$ and $\overline{E}'$ respectively, a bundle morphism

$$\Phi\colon\ E \to E'$$

is affine if there exists a linear bundle morphism

$$\overline{\Phi}\colon\ \overline{E} \to \overline{E}'$$

satisfying the following condition

$$r_{E'} \circ (\Phi, \overline{\Phi}) = \Phi \circ r_E.$$

This linear bundle morphism $\overline{\Phi}$ is called the fibred derivative of Φ:

$$\Phi(y^i) + \overline{\Phi}(\overline{y}^i) = \Phi(y^i + \overline{y}^i). \tag{0.8}$$

Let E be a vector bundle. Bundle coordinates (x^λ, y^i) on E are called linear if functions y^i are linear on each fibre.

Let E be an affine bundle modelled on a vector bundle $\overline{E}$. Bundle coordinates (x^λ, y^i) on E are called affine if functions y^i are affine on each fibre. By taking their fibred derivatives, one obtains the linear bundle coordinates $(x^\lambda, \overline{y}^i)$ on $\overline{E}$:

$$\overline{y}^i(\overline{y}) = y^i(y + \overline{y}) - y^i(y).$$

If E is a vector bundle provided with the canonical structure of an affine bundle, we have

$$y^i(y) = \overline{y}^i(y).$$

Henceforth, when we deal with a vector bundle and an affine bundle modelled on a vector bundle, we shall refer to the linear bundle coordinates and to the affine bundle coordinates respectively.

Let us note that additional algebraic structure puts constraints on a bundle E. For instance, a bundle E with a standard fibre F can be regarded as a bundle with the structure group Diff F of all diffeomorphisms of F. If E is assumed to be associated with a G-principal bundle P, it means that the structure group Diff F of E is reducible to G and that only atlases of E associated with atlases of the principal bundle P are allowed. One must discern affine bundles and affine bundles with an affine structure group. Jet bundles described in Section 0.2 exemplify affine bundles which are not associated with an affine principal bundle.

Remark. Given a principal bundle P and a P-associated bundle E, we say that a bundle atlas

$$\Psi^P = \left\{U_\kappa, \psi^P_\kappa\right\}$$

of P and a bundle atlas

$$\Psi = \{U_\kappa, \psi_\kappa\}$$

of E are associated atlases if they are determined by the same family $\{z_\kappa(x), x \in U_\kappa\}$ of local sections of P, that is,

$$\begin{aligned}
&\psi^P_\kappa(z_\kappa(x)) = 1_G, \\
&\psi_\kappa(x) = [z_\kappa(x)]_F^{-1}, \quad \pi_P(p) = x \in U_\kappa, \\
&z_\kappa(p) = z_\nu(p)\rho_{\nu\kappa}(\pi_P(p)), \quad \pi_P(p) = x \in U_\kappa \cup U_\nu, \\
&\rho_{\kappa\nu}(x) = \psi_\kappa(x)\psi_\nu{}^{-1}(x).
\end{aligned}$$

Here, $\rho_{\kappa\nu}$ are G-valued transition functions of atlases Ψ^P and Ψ and 1_G is the unit element of the group G. By $\psi_\kappa(x)$, we denote the morphism $\mathrm{pr}_2 \circ \psi_\kappa$ restricted to a fibre $\pi^{-1}(x)$:

$$\psi_\kappa(x)\colon\ \pi^{-1}(x) \to F.$$

The tangent bundle over a bundle E possesses additional structure which is the vertical subbundle.

Remark. Given the tangent bundle

$$\pi_M\colon\ TM \to M$$

and the cotangent bundle T^*M over a manifold M, we denote the familiar induced coordinates on TM and T^*M by $(x^\lambda, \dot{x}^\lambda)$ and $(x^\lambda, \dot{x}_\lambda)$ respectively. Here, $\dot{x}^\lambda$ and

$\dot{x}_\lambda$ are the coordinates on fibres T_xM and T_x^*M with respect to their holonomic bases $\{\partial_\lambda\}$ and $\{dx^\lambda\}$. Let

$$\Phi\colon M \to N$$

be a manifold morphism. It gives rise to the following linear bundle morphism over Φ:

$$\Phi_*\colon TM \to TN,$$

$$\Phi_*\colon \tau^\mu \partial_\mu \to \tau^\mu \frac{\partial \Phi^\nu}{\partial x^\mu} \partial'_\nu,$$

which is called the tangent morphism to Φ.

Given a bundle E, we have the tangent bundle

$$\pi_E\colon TE \to E$$

and the bundle

$$\pi_*\colon TE \to TB.$$

Given the bundle coordinates (x^λ, y^i) on E, the induced coordinates on TE are

$$(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i).$$

The vertical bundle over a bundle E is defined to be the subbundle

$$VE = \ker \pi_* \subset TE.$$

The induced bundle coordinates on VE are

$$(x^\lambda, y^i, \dot{y}^i).$$

We have the following exact sequence of tangent bundles over E:

$$0 \to VE \to TE \to E \underset{B}{\times} TB \to 0 \tag{0.9}$$

where

$$E \underset{B}{\times} TB = \pi^*(TB)$$

is the pull-back of the tangent bundle TB by π. For instance, a bundle morphism Φ of a bundle E to E' gives rise to the vertical tangent morphism

$$V\Phi = \Phi_*|_{VE}\colon VE \to VE'$$

of the vertical bundle VE to VE'.