Lecture Notes in Mathematics 1561

Ilya S. Molchanov

Limit Theorems for Unions of Random Closed Sets

Springer-Verlag

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Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Author

Ilya S. Molchanov
Department of Mathematics
Kiev Technological Institute of the Food Industry
Vladimirskaya 68
252017 Kiev, Ukraine
and
FB Mathematik
TU Bergakademie Freiberg
Bernhard-v.-Cotta-Str. 2
D-09596 Freiberg, Germany

Mathematics Subject Classification (1991): 60-02, 60D05, 60G55, 60G70, 26E25, 28B20, 52A22

ISBN 3-540-57393-3 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-57393-3 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

2146/3140-543210 - Printed on acid-free paper

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- a subject index: as a rule this is genuinely helpful for the reader.

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PREFACE

The theory of geometrical probability is, certainly, one of the oldest branches of probability theory. It deals with probability distributions on spaces of geometrical objects (points, lines, planes, triangles, sets etc.) and the corresponding random elements, see Ambartzumian (1990). The notion of a random closed set was introduced by Kendall (1974) and Matheron (1975). Since their studies the concept of probability was defined in a satisfactory manner from the point of view of probability measure on a space of closed sets.

Although a random closed set is a special case of general random elements, random sets have special properties due to the topological structure of the space of closed sets and specific features of set-theoretic operations. Therefore, well-known theorems of classical probability theory gain new meanings and features within the framework of the theory of random sets.

The role and place of limit theorems in probability theory can scarcely be exaggerated. Many important distributions appear as limiting ones with respect to various operations. It is of great interest to derive limit theorems for random sets with respect to set-theoretic operations such as union, intersection or Minkowski (element-wise) addition. It should be noted that limit theorems for random vectors will naturally follow from limit theorems for random sets, since a random vector can be considered to be a single-point random set. On the other hand, limit theorems for random sets gain new features as long as we deal with shapes of limiting random sets and summands.

The limit theorems for random sets have been investigated mostly for the Minkowski addition. The properties of this operation imply that the limiting distribution corresponds to a convex random closed set. Since any convex set can be associated with its support function, limit theorems for Minkowski sums follow from the central limit theorem for sums of random support functions as Banach-space-valued random elements.

In these notes we consider limit theorems for unions of random sets. It should be noted that the union scheme for random sets generalizes the max-scheme for random vectors in a partially-ordered space, whereas Minkowski addition of random sets generalizes the additive scheme for random vectors in a linear space. Limiting random sets for normalized unions of independent identically distributed random sets are naturally said to be union-stable.

It is well-known that the distribution of a random closed set is determined by the corresponding capacity (or hitting) functional on the class of all compacts. This functional is a so-called alternating Choquet capacity of infinite order. Although there are many examples of capacities, sometimes they are not alternating or the corresponding random sets are difficult to construct and simulate. The main stumbling block in the theory of random sets and, especially, in statistics of random sets, is the shortage of convenient models of random sets. In fact, until now only the grain-germ (or Boolean) model provides suitable examples of random sets. In this connection, it should be noted that limit theorems for unions and convex hulls supply us with new models of random sets, which appear as limits.

Unlikely distribution functions of random variables, a principal problem in the theory of random sets is to reduce the number of compacts needed to determine the distribution of a random set by means of its capacity functional on the chosen class. Similar problems are of no interest in classical probability theory, since a distribution function or density are defined naturally on the whole space. The chosen class of compacts then appears in a strong law of large numbers for unions and in definitions of probability metrics for random sets.

Similarly to the max-scheme for random variables or coordinate-wise-maximum-scheme for random vectors, the analysis of unions of random closed sets uses the technique of regularly varying functions. On the other hand, the theory of random sets sparks the theory of regularly varying functions with new concepts such as regularly varying capacities or multivalued regularly varying functions.

The probability metrics method elaborated by Zolotarev (1986) has proved its efficiency in the study of limit theorems for random variables. We define some probability metrics for random closed sets and apply them to limit theorems for unions. The essence of this method lies in proving limit theorems with respect to the most "convenient" metric for the given operation. Then the speed of convergence is estimated with respect to other metrics by the instrumentality of the appropriate inequalities between probability metrics.

Many of the ideas of these notes originate in the pioneering work done by Matheron (1975), who introduced the first notion of union-stability and infinite-divisibility of random sets. Very general notions of infinite divisibility and stability of random sets with respect to various set-theoretic operations were introduced by Trader (1981). Some of the results presented in these notes are closely connected with recent works on general extremal processes, max-stable random vectors and lattice-valued random elements, see Norberg (1986b, 1987), Vervaat (1988), Pancheva (1988), Gerritse (1986, 1990).

The book begins with the introduction of the basic tools and known results on random sets distributions and their weak convergence. Although the book is devoted to the study of limit theorems for unions, in Chapter 2 we present several results on Minkowski sums of random compact sets in the Euclidean space. In Chapter 3 we bring the notions of union-stable and convex-stable random closed sets. Their distributions are characterized in terms of the corresponding capacity or inclusion functionals. In Chapter 4 we prove limit theorems for scaled unions and convex hulls of random sets. Limit theorems for unions of special random sets (random triangles, balls) are considered too. Almost sure stability of unions is investigated in Chapter 5. In Chapter 6 the limit theorems for unions are reformulated in terms of regularly varying multivalued functions, whose definition is introduced too. Chapter 7 is devoted to the development of the probability metrics method in the framework of random sets theory. In the last chapter we discuss several applications. The content of Chapter 8 ranges from the estimates of the volume of random samples and the corresponding statistical tests to the limit theorems for pointwise maxima of random functions and polygonal approximations of convex compact sets.

In each chapter we use notations introduced in it without any comments. While referring to theorems, propositions, examples, formulae etc. from the same chapter we use two-digit notations, e.g., (3.2) designates the second formula from the third section of the same chapter. Otherwise three-digit notations are used, e.g., Theorem 3.1.1 designates Theorem 1.1 from Chapter 3.

I am grateful to Professor V.M.Zolotarev for suggesting the idea of writing these notes and for his further encouragement. These notes appeared as a result of an attempt to generalize the probability metric method for random closed sets. The idea originated in the annual workshop on stability problems for stochastic models organized by V.M.Zolotarev and V.V.Kalashnikov. I thank the organizers and participants of this workshop for helpful comments.

This book was benefited from a lot of discussions with Professor D.Stoyan. His suggestions led to a substantial improvement of the text. The final stage of the work was carried out at the time of my stay at the Technical University Mining Academy of Freiberg. This stay would have been impossible without the financial assistance of the Alexander von Humboldt-Stiftung (Bonn, Germany) and the hospitality of the Mining Academy.

I am indebted to all my colleagues for invitations, comments and discussions of this work at different stages and sending me reprints and preprints, especially, to A.J.Baddeley, N.Cressie, W.F.Eddy, F.Hiai, N.V.Kartashov, V.S.Korolyuk, E.Omey, E.Pancheva, T.Norberg, R.Rebolledo, A.D.Roitgartz, V.Schmidt, F.Streit, W.Vervaat, R.Vitale, W.Weil, M.Zähle and many others.

Special thanks go out to my mother for her invaluable help and constant attention to my research work.

Freiberg, July 7th, 1993 Ilya Molchanov

Contents

Chapter 1

Distributions of Random Closed Sets

1.1 The Space of Closed Sets.

Roughly speaking, a random closed set is a random element in the space of all closed subsets of the basic setting space E. The setting space E in the classical theory of random sets (see Matheron (1975), Stoyan, Kendall and Mecke (1987), Cressie and Laslett (1987) as principal references) is supposed to be locally compact, Hausdorff and separable. It should be noted that Norberg and Vervaat (1989) recently showed that non-Hausdorff E is the natural setting too.

Everywhere below we consider random closed sets in $\mathbb{R}^d$ only, i.e. we suppose E to be equal to $\mathbb{R}^d$. Nevertheless, many results can be easily reformulated for random closed sets in a general finite-dimensional linear space E. The dimension d of the Euclidean space is supposed to be fixed. The Euclidean norm and metric in $\mathbb{R}^d$ are denoted by $\|.\|$ and $\rho(.,.)$ respectively. The ball of radius r centered at x is denoted by $B_r(x)$. We shortly write B_r instead of $B_r(0)$ and B instead of $B_1(0)$.

Define $\mathcal{F}$ to be the family of all closed subsets of $\mathbb{R}^d$ (including the empty set $\emptyset$). Introduce sub-classes of $\mathcal{F}$ by

$$\mathcal{F}^X = \{F \in \mathcal{F} : F \cap X = \emptyset\}, \mathcal{F}_X = \{F \in \mathcal{F} : F \cap X \neq \emptyset\}, \tag{1.1}$$

where $X \subset \mathbb{R}^d$. The class $\mathcal{F}$ is endowed with the topology $\mathbb{T}_f$ (sometimes called *hit-or-miss topology*) generated by

$$\mathcal{F}^K_{G_1,\dots,G_n} = \mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \dots \cap \mathcal{F}_{G_n}, \tag{1.2}$$

where $n \geq 0$, K runs through the class $\mathcal{K}$ of compacts in $\mathbb{R}^d$, $G_1, \dots, G_n$ belong to the family $\mathcal{G}$ of all open sets. It was proven that the space $\mathcal{F}$ furnished with the hit-or-miss topology is compact, separable and Hausdorff, see Matheron (1975).

A sequence of closed sets $F_n, n \geq 1$, converges in $\mathbb{T}_f$ to a certain closed set F if and only if the following conditions are valid

(F1) if $K \cap F = \emptyset$ for a certain compact K, then $K \cap F_n = \emptyset$ for all sufficiently large n;

(F2) if $G \cap F \neq \emptyset$ for a certain open set G, then $G \cap F_n \neq \emptyset$ for all sufficiently large n.

We then write $F = \mathcal{F}-\lim F_n$ or $F_n \xrightarrow{\mathcal{F}} F$.

Let $\mathbb{T}_k$ be the topology on $\mathcal{K}$ induced by $\mathbb{T}_f$. To ensure the convergence of a sequence K_n, $n \geq 1$, of compact sets in $\mathbb{T}_k$ an additional condition is required:

(F3) there exists a compact K' such that $K_n \subseteq K'$ for all $n \geq 1$.

We denote $K = \mathcal{K}-\lim K_n$ in case K_n converges to K in $\mathbb{T}_k$.

The convergence of compact sets in $\mathbb{T}_k$ can be metrized by means of the *Hausdorff metric* ρ_H on $\mathcal{K}$. The Hausdorff distance between two compacts K and K_1 is defined as

$$\rho_H(K, K_1) = \inf\{\varepsilon > 0 : K \subseteq K_1^\varepsilon, K_1 \subseteq K^\varepsilon\}, \tag{1.3}$$

where

$$K^\varepsilon = \cup\{B_\varepsilon(x) : x \in K\} = K \oplus B_\varepsilon(0)$$

is the ε-envelope of K, $\oplus$ is the Minkowski addition (see Section 1.5). The Hausdorff distance between two closed sets is defined similarly. However, it can be infinite.

The upper limit $\mathcal{F}-\limsup F_n$ is the largest closed set F which satisfies the condition **(F1)**. Similarly, $\mathcal{K}-\limsup$ is defined by combining **(F1)** and **(F3)**.

Lemma 1.1 *Let K_n, $n \geq 1$, be a sequence of compact sets. Then $K \subseteq \mathcal{K}-\limsup K_n$ if and only if*

$$\varepsilon_n = \inf\{\varepsilon > 0 \colon K \subseteq K_n^\varepsilon\} \to 0 \quad as \ n \to \infty.$$

PROOF. Let $\varepsilon_n \to 0$ as $n \to \infty$. For any x from K there exists a sequence of points $x_n \in K_n$, $n \geq 1$, such that $\|x - x_n\| \leq \varepsilon_n$. Thus, $x_n \to x$ as $n \to \infty$, so that $x \in \mathcal{K}-\limsup K_n$.

Let $K \subseteq \mathcal{K}-\limsup K_n$. Suppose that $\varepsilon_n \geq \delta > 0$, $n \geq n_0$. Then there exist points $x_n \in K$, $n \geq n_0$, such that $B_\delta(x_n) \cap K_n = \emptyset$. Without loss of generality suppose that $x_n \to x_0 \in K$ as $n \to \infty$. Then $B_{\delta/2}(x_0) \cap K_n = \emptyset$, $n \geq n_0$, i.e. $x_0 \notin \mathcal{K}-\limsup K_n$. Hence $\varepsilon_n \to 0$ as $n \to \infty$. □

For later use we denote by $\bar{M}$, Int$M$, ∂M, M^c, conv(M) respectively the closure, interior, boundary, complement in $\mathbb{R}^d$ and the convex hull of any set $M \subset \mathbb{R}^d$.

A set M is said to be *canonically closed* if M coincides with the closure of its interior, i.e. $M = \overline{\text{Int}M}$.

1.2 Random Closed Sets and Capacity Functionals.

According to what has been said, a *random closed set* is an $\mathcal{F}$-valued random element. To complete this definition the class $\mathcal{F}$ is endowed with the Borel σ-algebra σ_f generated by $\mathbb{T}_f$. Then a random element in $(\mathcal{F}, \sigma_f)$ is said to be a random closed set (RACS). Here are several examples of random closed sets: random points and point processes, random spheres and balls, random half-spaces and hyperplanes etc.

The distribution of a random closed set A is described by the corresponding probability measure $\mathbf{P}$ on σ_f. In this connection

$$\mathbf{P}\left\{\mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \ldots \cap \mathcal{F}_{G_n}\right\} = \mathbf{P}\left\{A \cap K = \emptyset, A \cap G_1 \neq \emptyset, \ldots, A \cap G_n \neq \emptyset\right\}.$$

Clearly, these probabilities determine the measure $\mathbf{P}$ on σ_f. Fortunately, $\mathbf{P}$ is determined also by its values on $\mathcal{F}_K$ for K running through $\mathcal{K}$ only. Let $T(K)$ be equal to $\mathbf{P}(\mathcal{F}_K)$, i.e.

$$T(K) = \mathbf{P}\{A \cap K \neq \emptyset\}, K \in \mathcal{K}. \tag{2.1}$$

The functional T is said to be the *capacity* (or hitting) *functional* of A. Sometimes we write $T_A(K)$ instead of $T(K)$. Considered as a function on $\mathcal{K}$ the capacity functional T is an *alternating Choquet capacity* of infinite order (briefly Choquet capacity). Namely, T has the following properties:

(T1) T is upper semi-continuous on $\mathcal{K}$, i.e. $T(K_n) \downarrow T(K)$ in case $K_n \downarrow K$ as $n \to \infty$.

(T2) The following functionals recurrently defined by

$$\begin{aligned} S_1(K_0; K) &= T(K_0 \cup K) - T(K_0) \\ \dots & \quad \dots \\ S_n(K_0; K_1, ..., K_n) &= S_{n-1}(K_0; K_1, ..., K_{n-1}) - S_{n-1}(K_0 \cup K_n; K_1, ..., K_{n-1}) \end{aligned}$$

are non-negative for all $n \geq 0$ and $K_0, K_1, ..., K_n$ from $\mathcal{K}$.

The value of $S_n(K_0; K_1, ..., K_n)$ is equal to the probability that A misses K_0 but hits $K_1, ..., K_n$. In particular, T is increasing, since S_1 is non-negative.

The properties of T resemble those of the distribution function. Property **(T1)** is the same as the right-continuity and **(T2)** is the extension of the notion of monotonicity. However, in contrast to measures, the functional T is not additive, but only subadditive.

EXAMPLE **2.1** Let $A = (-\infty, \xi]$ be a random set in $\mathbb{R}^1$, where ξ is a random variable. Then $T(K) = \mathbf{P}\{\xi > \inf K\}$ for all $K \in \mathcal{K}$.

EXAMPLE **2.2** Let $A = \{\xi\}$ be a single-point random set in $\mathbb{R}^d$. Then $T(K)$ is equal to $\mathbf{P}\{\xi \in K\}$ and coincides with the corresponding probability distribution of ξ. It can be proven that the capacity functional T is additive iff A is a single-point random set.

The powerful result derived by Matheron (1975) and Kendall (1974) establishes one-to-one correspondence between Choquet capacities and distributions of random closed sets.

Theorem 2.3 (Choquet) *Let T be a functional on $\mathcal{K}$. Then there is a (necessary unique) distribution $\mathbf{P}$ on $\mathcal{F}$ with*

$$\mathbf{P}\{\mathcal{F}_K\} = T(K), \quad K \in \mathcal{K},$$

if and only if T is an alternating Choquet capacity of infinite order such that $0 \leq T(K) \leq 1$ and $T(\emptyset) = 0$.

Capacity functionals play in the theory of random sets the same role as distribution functions in classical probability theory. However, the class $\mathcal{K}$ of all compacts is too large to define efficiently the capacity functional on it. In this connection an important problem arises to reduce the class of test sets needed. That is to say, is the distribution of a random closed set determined by the values $T(K)$, $K \in \mathcal{M}$, for a certain class $\mathcal{M} \subset \mathcal{K}$?

It was proven in Molchanov (1983) that if realizations of a random set belong to a certain sub-class $\mathfrak{S} \subset \mathcal{F}$ then this extra knowledge reduces the class $\mathcal{M}$ of test sets needed.

Theorem 2.4 *Let $\mathfrak{S} \subset \mathcal{F}$, and let $\mathcal{M} \subset \mathcal{K}$. Suppose that the following conditions are valid.*

1. *$\mathcal{M}$ is closed with respect to finite unions.*
2. *There exists a countable sub-class $\mathfrak{B} \subset \mathcal{G}$ such that any compact K from $\mathcal{M}$ is the limit of a decreasing sequence of sets from $\mathfrak{B}$, and also any G from $\mathfrak{B}$ is the limit of an increasing sequence from $\mathcal{M}$.*
3. *For any $G \in \mathfrak{B} \cup \{\emptyset\}$, $K_1, ..., K_n \in \mathcal{M}$, $n \geq 0$, the class*

   $$\mathcal{F}^G_{K_1,...,K_n} \cap \mathfrak{S}$$

   is non-empty, provided $K_i \setminus G$ is non-empty for all $1 \leq i \leq n$.
4. *The σ-algebra σ_m generated by*

   $$\{\mathcal{F}^K_{G_1,..,G_n} \cap \mathfrak{S} \colon K \in \mathcal{M} \cup \{\emptyset\}, G_i \in \mathfrak{B}, 1 \leq i \leq n\}$$

   coincides with the σ-algebra $\sigma_f \cap \mathfrak{S} = \{\mathcal{A} \cap \mathfrak{S} \colon \mathcal{A} \in \sigma_f\}$ induced by σ_f on the class $\mathfrak{S}$.

Let $\bar{\mathfrak{S}}$ be the closure of $\mathfrak{S}$ in $\mathbb{T}_f$. Then the functional T on $\mathcal{M}$ is a Choquet capacity of infinite order on $\mathcal{M}$ (i.e. the conditions **(T1)-(T2)** *are valid on $\mathcal{M} \cup \{\emptyset\}$) such that $0 \leq T \leq 1$ and $T(\emptyset) = 0$ if and only if there is a (necessary unique) probability* **P** *on σ_m such that*

$$\mathbf{P}\{\mathcal{F}_K \cap \bar{\mathfrak{S}}\} = T(K), K \in \mathcal{M}.$$

In general, the distribution of any random closed set is determined by the values of its capacity functional on the class $\mathcal{K}_{ub}$ of all finite unions of balls of positive radii, or on the class $\mathcal{K}_{up}$ of all finite unions of parallelepipeds, see Salinetti and Wets (1986), Lyashenko (1983). Norberg (1989) established deep relations between topological properties of continuous partially ordered sets and distributions of random closed sets.

The capacity functional T is said to be *maxitive* if

$$T(K_1 \cup K_2) = \max(T(K_1), T(K_2))$$

for all compacts K_1, K_2. Such capacities arise naturally in the theory of extremal processes, see Norberg (1986b, 1987).

EXAMPLE **2.5** Define a maxitive capacity T by

$$T(K) = \sup\{f(x) \colon x \in K\},$$

where $f: \mathbb{R}^d \to [0,1]$ is an upper semi-continuous function. Then T describes the distribution of the random set A defined as

$$A = \left\{x \in \mathbb{R}^d : f(x) \geq \eta\right\},$$

where η is a random variable uniformly distributed on $[0,1]$.

A random closed set A is said to be *stationary* if A and $A + x$ coincide in distribution, whatever x in $\mathbb{R}^d$ may be. Similarly, A is *isotropic* if A has the same distribution as its any non-random rotation. Of course, the capacity functional of a stationary (isotropic) random set is shift-invariant (rotation-invariant).

A random set is said to be *compact* if its realizations are almost surely compact.

1.3 Convex Random Sets.

Define $\mathcal{C}$ to be the class of convex closed sets in $\mathbb{R}^d$, and let $\mathcal{C}_0 = \mathcal{C} \cap \mathcal{K}$ be the class of all convex compact sets. A random closed set is said to be *convex* if its realizations are almost surely convex, i.e. A belongs to $\mathcal{C}$ almost surely. Of course, the distribution of any convex random closed set A is determined by the corresponding capacity functional (2.1). Fortunately, the additional properties of the realizations of A (see Theorem 2.4) yield the reduction of the class of test compacts needed. The following result is due to Vitale (1983). It was proven independently by Molchanov (1983), see also Trader (1981).

Theorem 3.1 *The distribution of any convex compact random set A is determined uniquely by the values of the functional*

$$\mathfrak{t}(K) = \mathbf{P}\{A \subset K\}$$

for K running through the class $\mathcal{C}_0$ of convex compact sets.

PROOF. Check the conditions of Theorem 2.4. Having considered a single-point compactification $E' = \mathbb{R}^d \cup \{\omega\}$, we can regard A to be a convex RACS in the compact space E'. Since A is supposed to be compact, it misses $\{\omega\}$ almost surely. Let $\mathcal{M}$ be the class of complements to all open bounded convex sets in $\mathbb{R}^d$, and let $\mathfrak{B}$ be the class of complements to convex polyhedrons with rational vertices. It is easy to show that the first and the second conditions of Theorem 2.4 are valid. The third one is valid too, since for all $G \in \mathfrak{B} \cup \{\emptyset\}$, $K_1, ..., K_n \in \mathcal{M}$ and x_i belonging to $K_i \setminus G$, $1 \leq i \leq n$, the convex hull of $\{x_1, ..., x_n\}$ misses G, so that

$$\mathcal{F}^G_{K_1,...,K_n} \cap \mathcal{C}_0 \neq \emptyset.$$

Verify the fourth condition. Let K be a compact set, and let $F \in \mathcal{F}^K \cap \mathcal{C}_0$. Then $F \in \mathcal{F}^{K_1} \cap \mathcal{C}_0$ for a certain K_1 from $\mathcal{M}$. E.g., K can be chosen to be the complement to a certain bounded neighborhood $U(F)$ of F such that $U(F) \cap K = \emptyset$.

Let $F \in \mathcal{F}_G \cap \mathcal{C}_0$ for a certain open G, and let

$$x_0 = (x_{01}, ..., x_{0d}) \in F \cap G.$$

Pick $\delta > 0$ such that

$$G_0 = \left\{x = (x_1, \dots, x_d) \colon \max_{1 \le i \le d} |x_i - x_{0i}| < \delta \right\} \subseteq G.$$

For each collection of numbers $l_i = \pm 1$, $1 \le i \le d$, define

$$\begin{aligned} H_j^\varepsilon &= H_{l_1,\dots,l_d}^\varepsilon \\ &= \left\{x = (x_1, \dots, x_d) \colon \sum_{i=1}^{d} (x_i - x_{0i}) l_i > 1 - \varepsilon \right\}, \ \varepsilon > 0, \ 1 \le j \le 2^d. \end{aligned}$$

If ε is sufficiently small, then every convex set, which misses G_0^c and hits H_j^ε, $1 \le j \le 2^d$, also contains x_0. Observe that G_0^c belongs to $\mathcal{M}$. Thus

$$F \in \mathcal{F}^{G_0^c}_{H_1^\varepsilon,\dots,H_{2^d}^\varepsilon} \cap \mathcal{C}_0 \subseteq \mathcal{F}_G \cap \mathcal{C}_0,$$

whence $\sigma_m = \sigma_f \cap \mathcal{C}_0$.

By Theorem 2.4, there exists the unique probability measure $\mathbf{P}$ on σ_f such that $\mathbf{P}\left\{\mathcal{F}_K \cap \bar{\mathcal{C}}_0\right\} = T(K)$, $K \in \mathcal{M}$. The closure $\bar{\mathcal{C}}_0$ consists of also convex sets containing the point $\{\omega\}$ (i.e. $\bar{\mathcal{C}}_0 = \mathcal{C}$). However, since the random set A is compact, the corresponding probability $\mathbf{P}$ is concentrated within $\mathcal{C}_0$. Thus, $\mathbf{P}\left\{\mathcal{F}_K \cap \mathcal{C}_0\right\} = T(K)$ for each compact K. Then the distribution of A is determined by the values $\mathbf{P}\left\{A \subseteq K^c\right\}$, whence the statement of Theorem easy follows. □

The functional $\mathfrak{t}(K)$, $K \in \mathcal{C}_0$, is naturally extended onto the class $\mathcal{C}$ by

$$\mathfrak{t}(F) = \mathbf{P}\left\{A \subset F\right\}, F \in \mathcal{C}. \tag{3.1}$$

This functional $\mathfrak{t}$ is said to be the *inclusion functional* of A. It is a so-called monotone capacity of infinite order (see Choquet, 1953/54). In other words, it satisfies the following conditions.

(I1) $\mathfrak{t}$ is upper semicontinuous, i.e. $\mathfrak{t}(F_n) \to \mathfrak{t}(F)$ if $F_n \downarrow F$ as $n \to \infty$ for F_n, F belonging to $\mathcal{C}$, $n \ge 1$.

(I2) The recurrently defined functionals

$$\begin{aligned} S_1^{\mathfrak{t}}(F; F_1) &= \mathfrak{t}(F) - \mathfrak{t}(F \cap F_1) \\ \dots & \quad \dots \\ S_n^{\mathfrak{t}}(F; F_1, \dots, F_n) &= S_{n-1}^{\mathfrak{t}}(F; F_1, \dots, F_{n-1}) - S_{n-1}^{\mathfrak{t}}(F \cap F_n; F_1, \dots, F_{n-1}) \end{aligned}$$

are non-negative, whatever $n \ge 1$ and $F, F_1, \dots, F_n$ from $\mathcal{C}$ may be.

In fact, $S_n^{\mathfrak{t}}(F; F_1, \dots, F_n)$ is the probability that $A \subseteq F$ and $A \not\subset F_i$, $1 \le i \le n$. Note that $\mathfrak{t}$ is expressed in terms of the capacity functional T by means of

$$\mathfrak{t}(F) = \mathbf{P}\left\{A \subseteq F\right\} = T(F^c), F \in \mathcal{C}. \tag{3.2}$$

The following example shows that the distribution of a non-compact convex RACS, in general, cannot be determined via the functional $\mathfrak{t}$ on $\mathcal{C}$.

EXAMPLE 3.2 Let A be the half-space which touches the unit ball B_1 at a random point uniformly distributed on its boundary. Then $\mathfrak{t}(F) = 0$ whenever $F \in \mathcal{C}$, $F \neq \mathbb{R}^d$. Thus, the inclusion functional of A coincides with the inclusion functional of the set $A = \mathbb{R}^d$.

Nevertheless, we sometimes consider the functional $\mathfrak{t}(F)$, $F \in \mathcal{C}$, even for unbounded A. If A is non-convex, then this functional does not determine the distribution of A, but $\mathrm{conv}(A)$.

For any convex F define the *support function*

$$s_F(u) = \sup\{u \cdot v\colon\ v \in F\}, \tag{3.3}$$

where $u \cdot v$ is the scalar multiplication, u runs through the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. The function s_F is allowed to take infinite values if F is unbounded. Of course, s_F is finite everywhere iff F is compact.

If A is a convex compact random set, then $s_A(u)$ is the random element in the space $\mathbf{C}(\mathbb{S}^{d-1})$ of continuous functions on $\mathbb{S}^{d-1}$.

Let $\mathcal{H}$ be the class of all finite intersections of half-spaces in $\mathbb{R}^d$.

Proposition 3.3 *The distribution of a compact convex random set is determined by the values of its inclusion functional on $\mathcal{H}$.*

PROOF. The statement follows from the fact that the values $\mathfrak{t}(F)$ for F running through $\mathcal{H}$ determine the finite-dimensional distributions of the random process $s_A(u)$, $u \in \mathbb{S}^{d-1}$. □

1.4 Weak Convergence of Random Closed Sets.

Weak convergence of random sets is a particular case of weak convergence of probability measures, since a random closed set is associated with a certain probability measure on σ_f. A sequence of random closed sets A_n, $n \geq 1$, is said to *converge weakly* if the corresponding probability measures $\mathbf{P}_n$, $n \geq 1$, converge weakly in the usual sense, see Billingsley (1968). Namely,

$$\mathbf{P}_n(\mathfrak{A}) \to \mathbf{P}(\mathfrak{A}) \ \text{ as } \ n \to \infty \tag{4.1}$$

for each $\mathfrak{A} \in \sigma_f$ such that $\mathbf{P}(\partial\mathfrak{A}) = 0$ for the boundary of $\mathfrak{A}$ with respect to $\mathbb{T}_f$ (i.e. $\mathfrak{A}$ is a continuity set for the limiting measure).

However, it is rather difficult to check (4.1) for all $\mathfrak{A}$ from σ_f. The first natural reduction is in letting $\mathfrak{A}$ to be equal to $\mathcal{F}_K$ for K running through $\mathcal{K}$. It was proven in Lyashenko (1983) and Salinetti and Wets (1986) that the class $\mathcal{F}_K$ is a continuity set for $\mathbf{P}$ if

$$\mathbf{P}\{\mathcal{F}_K\} = \mathbf{P}\{\mathcal{F}_{\mathrm{Int}K}\}.$$

In other words,

$$\mathbf{P}\{A \cap K \neq \emptyset, A \cap \mathrm{Int}K = \emptyset\} = 0 \tag{4.2}$$