J. Aguadé M. Castellet F. R. Cohen (Eds.)

Algebraic Topology

Homotopy and Group Cohomology

Proceedings, Barcelona 1990

J. Aguadé   M. Castellet   F. R. Cohen (Eds.)

Algebraic Topology
Homotopy and Group Cohomology

Proceedings of the 1990 Barcelona Conference on
Algebraic Topology, held in S. Feliu de Guíxols,
Spain, June 6–12, 1990

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Editors

Jaume Aguadé
Manuel Castellet
Departament de Matemàtiques
Unversitat Autònoma de Barcelona
E-08193 Bellaterra, Spain

Frederick Ronald Cohen
Department of Mathematics
University of Rochester
Rochester, New York 14620, USA

Editorial Policy

for the publication of proceedings of conferences
and other multi-author volumes

Lecture Notes aim to report new developments - quickly, informally and at a high level. The following describes criteria and procedures for multi-author volumes. For convenience we refer throughout to "proceedings" irrespective of whether the papers were presented at a meeting.

The editors of a volume are strongly advised to inform contributors about these points at an early stage.

§ 1. One (or more) expert participant(s) should act as the scientific editor(s) of the volume. They select the papers which are suitable (cf. §§ 2 - 5) for inclusion in the proceedings, and have them individually refereed (as for a journal). It should not be assumed that the published proceedings must reflect conference events in their entirety. The series editors will normally not interfere with the editing of a particular proceedings volume - except in fairly obvious cases, or on technical matters, such as described in §§ 2 - 5. The names of the scientific editors appear on the cover and title-page of the volume .

§ 2. The proceedings should be reasonably homogeneous i.e. concerned with a limited and welldefined area. Papers that are essentially unrelated to this central topic should be excluded. One or two longer survey articles on recent developments in the field are often very useful additions. A detailed introduction on the subject of the congress is desirable.

§ 3. The final set of manuscripts should have at least 100 pages and preferably not exceed a total of 400 pages . Keeping the size below this bound should be achieved by stricter selection of articles and NOT by imposing an upper limit on the length of the individual papers .

§ 4. The contributions should be of a high mathematical standard and of current interest. Research articles should present new material and not duplicate other papers already published or due to be published. They should contain sufficient background and motivation and they should present proofs, or at least outlines of such, in sufficient detail to enable an expert to complete them. Thus summaries and mere announcements of papers appearing elsewhere cannot be included, although more detailed versions of, for instance, a highly technical contribution may well be published elsewhere later.

Contributions in numerical mathematics may be acceptable without formal theorems/proofs provided they present new algorithms solving problems (previously unsolved or less well solved) or develop innovative qualitative methods, not yet amenable to a more formal treatment.

Surveys, if included, should cover a sufficiently broad topic, and should normally not just review the author's own recent research. In the case of surveys, exceptionally, proofs of results may not be necessary.

§ 5. "Mathematical Reviews" and "Zentralblatt für Mathematik" recommend that papers in proceedings volumes carry an explicit statement that they are in final form and that no similar paper has been or is being submitted elsewhere, if these papers are to be considered for a review. Normally, papers that satisfy the criteria of the Lecture Notes in Mathematics series also satisfy this requirement, but we strongly recommend that each such paper carries the statement explicitly.

§ 6. Proceedings should appear soon after the related meeting. The publisher should therefore receive the complete manuscript (preferably in duplicate) including the Introduction and Table of Contents within nine months of the date of the meeting at the latest.

§ 7. Proposals for proceedings volumes should be sent to one of the editors of the series or to Springer-Verlag Heidelberg. They should give sufficient information on the conference, and on the proposed proceedings. In particular, they should include a list of the expected contributions with their prospective length. Abstracts or early versions (drafts) of the contributions are helpful.

Further remarks and relevant addresses at the back of this book.

Algebraic Topology

Homotopy and Group Cohomology

Proceedings of the 1990 Barcelona Conference on algebraic Topology, held in S. Feliu de Guixols, Spain, June 6–12, 1990
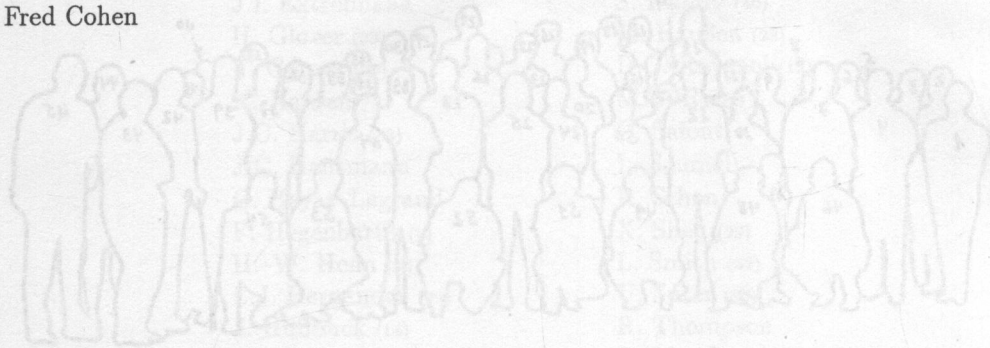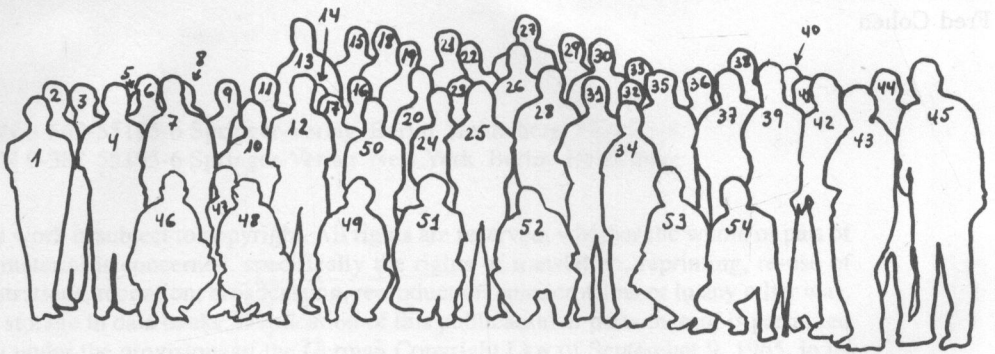
Springer-Verlag

INTRODUCTION

This volume represents the Proceedings of the Third Barcelona Conference on Algebraic Topology. The conference was held June 6-12, 1990, in Sant Feliu de Guíxols and was organized by the Centre de Recerca Matemàtica, a research institute sponsored by the Institut d'Estudis Catalans and located at the Universitat Autònoma de Barcelona. Financial support from the DGICYT and the CIRIT of the Generalitat de Catalunya made this Conference possible by bringing together mathematicians from around the world.

The scientific program consisted of 21 plenary lectures which are listed below. The Proceedings contains papers which were, for the most part, submitted by conference participants. All papers were refereed. We would like to take this opportunity to thank the participants for their contributions. The editors would also like to thank the referees for conscientious and thorough efforts.

We thank Carles Broto, Carles Casacuberta, and Laia Saumell for their generous help with the organization of the conference. We warmly thank the secretaries Consol Roca and Sylvia Hoemke. Most of the papers included in these Proceedings are prints obtained from TeX files submitted by the authors, suitably modified to uniformize headings and margins. We thank Maria Julià for her fine job in typing some of the manuscripts and for her help in solving TeX problems. Finally, we would like to thank the mayor of Sant Feliu de Guíxols for his help in stimulating an excellent mathematical environment.

Jaume Aguadé
Manuel Castellet
Fred Cohen

List of participants

The numbers in brackets refer to the group picture

A. Adem (9)

J. Aguadé (48)

N. Álamo

D. Benson (25)

J. Berrick (27)

M. Boileau

I. Bokor (16)

C. Broto (10)

J. Cabeza

C. Casacuberta (7)

M. Castellet (45)

A. Cavicchioli (3)

B. Cenkl (11)

F.R. Cohen (42)

W. Dicks

E. Dror Farjoun (43)

C. Elvira

J.I. Extremiana

H. Glover (33)

F. Gómez (22)

K. Hansen

J.C. Harris (53)

J.C. Hausmann

C. Hayat–Legrand

F. Hegenbarth (2)

H.-W. Henn (54)

L.J. Hernández (49)

J. Hubbuck (14)

K. Irie (40)

K. Ishiguro (35)

M. Izydorek (18)

S. Jackowski

A. Jeanneret (21)

B. Jessup (24)

R. Kane (13)

A. Kono (30)

J. Lannes (50)

I. Llerena (1)

L. Lomonaco (51)

H. Marcum (39)

J.R. Martino (37)

C.A. McGibbon (19)

G. Mislin

A. Murillo (20)

J.L. Navarro (52)

D. Notbohm (38)

B. Oliver

E. Ossa (17)

G. Peschke (6)

Ch. Peterson (34)

M. Pfenniger (41)

S. Priddy (46)

M. Raußen (28)

D.C. Ravenel (12)

M.T. Rivas

C. Safont

L. Saumell

R. Schon

X. Shen (29)

L. Smith (32)

U. Suter (36)

R. Thompson

K. Varadarajan (8)

E. Ventura (4)

A. Vidal (15)

A. Vučic (31)

Z. Wojtkowiak (44)

X. Zarati (26)

Titles of Talks
(in chronological order)

K. Ishiguro	Classifying spaces of compact simple Lie groups and p-tori.
D. Notbohm	Homotopy uniqueness of $BU(n)$.
J. Lannes	Representing modulo p homology.
A. Kono	Homology of the Kac-Moody groups.
A. Adem	The geometry and cohomology of some sporadic simple groups.
J.R. Martino	The complete stable splitting for classifying spaces of finite groups.
S. Priddy	On the dimension theory of summands of BG.
G. Mislin	Cohomologically central elements in groups.
C. Broto	On Sub-$\mathcal{A}_p^*$-algebras of H^*V.
H.-W. Henn	Refining Quillen's description of $H^*(BG; F_p)$.
J. Hubbuck	On Miller's stable splitting of $SU(n)$.
C.A. McGibbon	On spaces with the same n-type for all n.
L. Smith	Fake Lie groups and maximal tori.
D. Benson	Resolutions and Poincaré duality for finite groups.
R. Thompson	The fiber of secondary suspension.
J. Harris	Lannes' T functor on summands of $H^*(BZ/p)^s$.
D. Ravenel	Morava K-theory of BG and generalized group characters.
E. Dror Farjoun	Good spaces, bad spaces.
B. Oliver	Approximations of classifying spaces.
S. Jackowski	Vanishing theorems for higher inverse limits.

1990 Barcelona Conference
on Algebraic Topology.

Table of Contents

ON THE GEOMETRY AND COHOMOLOGY
OF FINITE SIMPLE GROUPS

ALEJANDRO ADEM*

§0. INTRODUCTION

Let G be a finite group and k a field of characteristic $p \mid |G|$. The cohomology of G with k-coefficients, $H^*(G, k)$, can be defined from two different points of view:

(1) $H^*(G, k) = H^*(BG; k)$, the singular cohomology of the classifying space BG

(2) $H^*(G, k) = Ext^*_{kG}(k, k)$.

From (1) it is clear that group cohomology contains important information about G-bundles and the rôle of G as a transformation group. From (2) we can deduce that $H^*(G, k)$ contains characteristic classes for representations over k, and indeed is the key tool to understanding modular representations via the method of cohomological varieties introduced by Quillen.

The importance of classifying spaces in topology is perhaps most evident in the method of "finite models". Roughly speaking, the classifying spaces of certain finite groups can be assembled to yield geometric objects of fundamental importance. For example, let Σ_m denote the symmetric group on m letters; then the natural pairing $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ induces an associative monoid structure on $C(S^\circ) = \coprod_{n \geq 0} B\Sigma_n$. We have the result due to Dyer and Lashof [D-L]:

THEOREM

$Q(S^\circ) = \Omega^\infty S^\infty$ is the homotopy group completion of $C(S^\circ)$ (using the loop sum), and hence

$$H_*(Q(S^\circ), \mathbf{F}_2) \cong \varinjlim H_*(\Sigma_m, \mathbf{F}_2) \otimes \mathbf{F}_2[Z].$$ ∎

Given that $\pi_i(Q(S^\circ)) = \pi_i^S(S^\circ)$ (the stable homotopy groups of spheres), $Q(S^\circ)$ is the "ground ring" for stable homotopy; and the result above indicates that it can be constructed from classifying spaces of finite groups.

A similar approach (due to Quillen [Q1]) uses the $GL_n(\mathbf{F}_q)$ to construct the space Im J, which essentially splits off $Q(S^\circ)$ in the natural way. From this space we obtain the algebraic K-theory of the field $\mathbf{F}_q$.

Now from the point of view of cohomology of groups, very few noteworthy examples have been calculated. Most strikingly, the important computations have been for groups involved in the geometric scheme outlined above. The understanding of the cohomology

---

of the symmetric and alternating groups (only additively) is linked to the geometry of $Q(S^\circ)$; indeed this space can be identified with $B\Sigma_\infty^+$ (the plus construction), and the homology of the finite symmetric groups can be expressed in a precise manner using Dyer-Lashof operations. Analogously, Quillen constructed $\mathrm{BGL}(\mathbf{F}_q)^+ = F\psi^q$ by comparing its cohomology to what he calculated for $\mathrm{BGL}_n(\mathbf{F}_q)$ away from q, for all n.

From a purely calculational point of view, the cases above are very special: the cohomology is detected on abelian subgroups. Hence these results cannot truly reflect the general difficulties of the subject and may not be good examples for relating $H^*(G, k)$ to geometric and algebraic properties of G.

At this point we introduce an ingredient usually ignored by topologists — the classification of finite simple groups. From the topological point of view, only the alternating and groups associated to those of Lie type have been considered (as outlined above). The classification theorem indicates that there are 26 additional groups which are as essential as the two infinite families — the so-called sporadic simple groups. A natural question arises: what rôle do the sporadic simple groups play in this framework?

In this paper I will discuss some recent work on this problem which I have done jointly with R. J. Milgram. By now it is clear that the first step in any such project should be to understand the cohomology first as a calculation and then as a source of algebraic and geometric data. In §1 I will describe the combination of techniques required to calculate the cohomology of groups of this complexity. Then, in §2 I will present some calculations. The particular example which will be dealt with in detail is the Mathieu group M_{12}, a group of order 95,040. The cohomology ring $H^*(M_{12}, \mathbf{F}_2)$ will be fully described. In the following section, a geometric analysis of the results in §2 will be provided. This example will show how well cohomology carries the local structure of the group, as well as plentiful topological information. Finally in §4 I will discuss current work on M_{22}, M_{23}, M^cL and $O'N$, which are even larger groups.

The aim of this paper is to bring the reader up to date on recent progress, hence he will be spared the technical details of the calculations, which are either available ([AMM], [AM]) or will appear soon. All coefficients are taken to be $\mathbf{F}_2$, so they will be suppressed.

A note of caution: it is not the author's viewpoint to encourage mindless calculation, which is seen so often. Rather, the main point is that with some work we can extend our knowledge of group cohomology by understanding the key models on which the whole framework of finite group theory is now based. The sporadic simple groups seen to be the natural objects we should now turn our attention to, if there is any possibility of understanding behaviour in algebraic topology which is modelled on finite group theory.

The results in this paper were presented at a talk given by the author during the 1990 Barcelona Conference on Algebraic Topology. The author is very grateful to the organizers for being invited and in particular to M. Castellet for his kind hospitality.

§1. CALCULATIONAL TECHNIQUES

From the landmark work of Quillen [Q2] it is known that $H^*(G)$ is detected up to nilpotence on its elementary abelian subgroups. Hence a clear understanding of these subgroups is necessary before embarking on a calculation. The relevant data can all be assembled into a finite complex (first introduced by K. Brown) as follows [Q3]. Let

$A_p(G)$ denote the partially ordered set of p-elementary abelian subgroups of G under inclusion. In the usual way $A_p(G)$ can be given the structure of a simplicial complex; the n-simplices correspond to flags of the form $O \neq (\mathbf{Z}/2)^{j_1} \subsetneq \cdots \subsetneq (\mathbf{Z}/2)^{j_n}$. Then G acts on this simplicial set by conjugation, and its realization $|A_p(G)|$ has the structure of a finite G-CW complex, in fact of dimension $\mathrm{rank}_p(G) - 1$. If $P \leq G$ is a p-subgroup, then this complex satisfies one key property: $|A_p(G)|^P$ is contractible. We can then apply the following theorem, which is a reformulation of a result due to P. Webb [We]:

THEOREM 1.1

If X is a finite G-CW complex such that $X^P \simeq *$ for each p-subgroup $P \leq G$, then the mod p Leray spectral sequence associated to the map $X \times_G EG \to X/G$ satisfies

$$E_2^{p,q} \cong \begin{cases} 0 & \text{for } p > 0 \\ H^q(G, \mathbf{F}_p) & \text{for } p = 0. \end{cases} \quad \blacksquare$$

Using the usual combinatorial description of the E_1-term of this spectral sequence yields:

COROLLARY 1.2

$$\left(\bigoplus_{\substack{\sigma_i \in (X/G)^{(i)} \\ i \text{ odd}}} H^*(G_{\sigma_i}) \right) \oplus H^*(G) \cong \bigoplus_{\substack{\sigma_i \in (X/G)^{(i)} \\ i \text{ even}}} H^*(G_{\sigma_i})$$

where X may be taken to be $|A_p(G)|$, and $\mathbf{F}_p$ coefficients are used. $\quad \blacksquare$

The isotropy subgroups for the poset space are intersections of normalizers of p-tori. This formula in general can be of no use either because of its size or the fact that G may normalize a p-torus. However, in case G is simple, this reduces the calculation of $H^*(G)$ to calculations for proper, local subgroups.

With this reduction in mind, it seems likely that a cohomology calculation will involve analyzing a local subgroup which contains the 2-Sylow subgroup (recall that we take only $p = 2$ in this paper). To understand how to use the cohomology of such a subgroup, the classical double coset formula of Cartan-Eilenberg is often extremely useful [C-E]. We recall how that goes.

Assume $K \subseteq G$ is a subgroup of odd index. Take a double coset decomposition $G = \coprod_{i=1}^{n} K g_i K$. Using a simple transfer-restriction argument, it is easy to see that $H^*(G) \overset{\mathrm{res}_H^G}{\hookrightarrow} H^*(K)$ is injective. Let $c_x^* : H^*(k) \to H^*(xKx^{-1})$ denote the usual isomorphism induced by conjugation. Recall that $\alpha \in H^*(k)$ is said to be *stable* if $\mathrm{res}_{H \cap Hx^{-1}}^{H}(\alpha) = \mathrm{res}_{H \cap Hx^{-1}}^{xHx^{-1}} \circ c_x^*(\alpha)$ for $x = g_1, \ldots, g_n$. Then we have, for [G:K] odd.

THEOREM 1.3 (Cartan-Eilenberg)

$\alpha \in H^*(K)$ is in im res_H^G if and only if α is stable. $\quad \blacksquare$

Applying the techniques described above it is in principle possible to reduce the calculation of $H^*(G)$ to understanding the cohomology of its local subgroups, together with the stability conditions. At this point one must dispense with formal reductions and calculate the cohomology of the key local subgroups using the spectral sequences associated to the different extensions they may appear in. Aside from the usual difficulties of understanding the differentials, often some very delicate modular invariant theory must be used to understand the E_2-terms or images of different restriction maps.

A combination of these methods will be used to calculate $H^*(M_{12})$, the main example in §2.

§2. COHOMOLOGY CALCULATIONS FOR M_{12}

M_{12} is the Mathieu group of order 95,040. We will give a complete analysis of its cohomology using the methods outlined in §1.

First, we have that $|A_2(M_{12})|/M_{12}$ is the following 2-dimensional cell-complex:
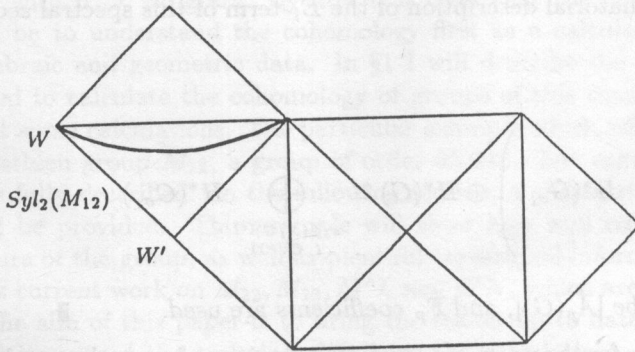
We have only labelled the isotropy on one edge, because 1.2 leads to

Proposition 2.1

Let $H = Syl_2(M_{12})$; then there exist two non-isomorphic subgroups $W, W' \subseteq M_{12}$ of order 192, such that

$$H^*(M_{12}) \oplus H^*(H) \cong H^*(W) \oplus H^*(W').$$ ∎

Next, we use 1.3 to prove

Proposition 2.2

The image of $res_H^{M_{12}}$ in $H^*(H)$ is the intersection of the two subalgebras,

$$H^*(M_{12}) = H^*(W) \cap H^*(W').$$ ∎

In other words, the configuration

$$W \cap W' = \begin{array}{ccc} H & \hookrightarrow & W' \\ \downarrow & & \downarrow \\ W & \hookrightarrow & M_{12} \end{array}$$

completely controls the cohomology of M_{12}.

We now give explicit descriptions of W, W'.

$$W: \qquad 1 \to Q_8 \to W \to \Sigma_4 \to 1$$

a split semidirect product (the holomorph of Q_8).

$$W': \qquad 1 \to (\mathbf{Z}/4 \times \mathbf{Z}/4) \underset{T}{\times} \mathbf{Z}/2 \to W' \to \Sigma_3 \to 1$$

also a split semidirect product.

Although $|H| = 64$, its cohomology is very complicated. It is obtained by using two index two subgroups with accessible cohomology. We have

THEOREM 2.3

$$H^*(Syl_2(M_{12})) \cong \mathbf{F}_2[e_1, s_1, t_1, s_2, t_2, L_3, k_4]/R$$

where R is the set of relations

$$s_1 e_1 = s_2 e_1 = t_1 e_1 = 0, \quad s_1^3 = s_1^2 t_1 + s_1 t_2,$$

$$s_2^2 = s_1 s_2 t_1 + s_2 t_1^2, \quad s_1^2 s_2 = t_1 L_3 + s_2 t_1^2 + s_2 t_2,$$

$$s_1 L_3 = t_1 L_3 + s_1 s_2 t_1 + s_2 t_1^2 + s_2 t_2,$$

$$s_2 L_3 = s_1 s_2 t_1^2 + s_2 t_1 t_2 + s_2 t_1^3,$$

$$L_3^2 = t_1 t_2 L_3 + s_1 s_2 t_1^3 + s_2 t_1^2 t_2 + s_2 t_1^4 + e_1^2 k_4 + e_1 t_2 L_3.$$

The Poincaré Series for $H^*(H)$ is given by $p_H(t) = 1/(1-t)^3$. ∎

Similarly, we determine $H^*(W)$, $H^*(W')$ and their intersection in $H^*(H)$. We now describe $H^*(M_{12})$:

DIMENSION	GENERATOR
2	α
3	x, y, z
4	β
5	γ
6	δ
7	ϵ

RELATIONS

$\alpha(x+y+z) = 0$	$x^3 = \alpha^3 x + \alpha\beta x + x\delta$
$xy = \alpha^3 + x^2 + y^2$	$xz = \alpha^3 + y^2$
$x^2 y = \alpha^3 z + \alpha\beta z + y\delta + \alpha\epsilon$	$yz = \alpha^3 + x^2$
$\epsilon x = \beta x^2 + \alpha^2 x^2$	$\alpha\gamma = \alpha^2 y$
$\epsilon y = \alpha^2\delta + \alpha^2 y^2 + \beta x^2 + \beta y^2$	$y\gamma = \alpha y^2$
$\epsilon z = \gamma^2 + \alpha^2\delta + \alpha^2 x^2 + \beta x^2 + \beta z^2$	$x\gamma = \alpha^4 + \alpha z^2$
$z^4 = \gamma\epsilon + x^4 + \alpha^4\beta + z^2\delta$	$\epsilon^2 = z^3\gamma + \alpha^2\beta\delta + \alpha^5\beta + z\beta\epsilon$
	$+ z\delta(\gamma+\alpha z) + \beta^2(\alpha^3 + xz + yz)$

The Poincaré Series for $H^*(M_{12})$ is given by

$$P_{M_{12}}(t) = \frac{1 + t^2 + 3t^3 + t^4 + 3t^5 + 4t^6 + 2t^7 + 4t^8 + 3t^9 + t^{10} + 3t^{11} + t^{12} + t^{14}}{(1-t^4)(1-t^6)(1-t^7)}.$$

$$(2.4)$$

§3. AN INTERPRETATION OF THE RESULTS

We now provide a geometric analysis of our calculation of $H^*(M_{12})$. First, note that $\mathbf{F}_2[\beta_4, \delta_6, \epsilon_7] \subseteq H^*(M_{12})$ is a polynomial subalgebra of maximal rank, and in fact the cohomology ring is free and finitely generated over it (this implies that it is Cohen-Macaulay). One can also verify that $Sq^2\beta = \delta$, $Sq^1\delta = \epsilon$, $Sq^4\epsilon = \beta\epsilon$, $Sq^6\epsilon = \delta\epsilon$. On the other hand, let $\mathbf{G}_2$ denote the usual exceptional compact Lie group of automorphisms of the Cayley octaves. A result due to Borel is that

$$H^*(B\mathbf{G}_2, \mathbf{F}_2) \cong \mathbf{F}_2[\beta_4, \delta_6, \epsilon_7]$$

with Steenrod operations as above. $\mathbf{G}_2$ is also a 14-dimensional closed manifold. From (2.4) we see that the numerator of $p_{M_{12}}(t)$ looks like the Poincaré Series of a 14-dimensional Poincaré Duality Complex. How does $\mathbf{G}_2$ relate to the group M_{12}?

At this point it is pertinent to mention a technique used by Borel to compute the cohomology of $B\mathbf{G}_2$. Although $\mathbf{G}_2$ has rank 2, it contains a 2-torus of rank 3, denoted by V. Then Borel [Bo] analyzed the spectral sequence for the fibration

$$\begin{array}{ccc} \mathbf{G}_2/V & \hookrightarrow & BV \\ & & \downarrow \\ & & B\mathbf{G}_2 \end{array},$$