Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Besov Spaces and Applications to Difference Methods for Initial Value Problems

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PREFACE

The purpose of these notes is to present certain Fourier techniques for analyzing finite difference approximations to initial value problems for linear partial differential equations with constant coefficients. In particular, we shall be concerned with stability and convergence estimates in the L_p norm of such approximations; the main theme is to determine the degree of approximation of different methods and the precise dependence of this degree upon the smoothness of the initial data as measured in L_p. In L_2 the analysis generally depends on Parseval's relation and is simple; it is to overcome the difficulties present in order to obtain estimates in the maximum-norm, or more generally in L_p with $p \neq 2$, which is the aim of this study.

The main tools which we shall use are some simple results on Fourier multipliers based on inequalities by Carlson and Beurling and by van der Corput. Many results are expressed in terms of norms in Besov spaces $B_p^{s,q}$ where s essentially describes the degree of smoothness with respect to L_p.

The first two chapters contain the prerequisits on Fourier multipliers and on Besov spaces, respectively, needed for our applications. The purpose of these two chapters is only to make these notes self-contained and not to give an extensive treatment of their topics. Chapters 3 through 6 then form the main part of the notes. In Chapter 3 we present preliminary material on initial value problems and finite difference schemes for such problems. In particular, the concepts of well-posedness in L_p of an initial value problem and stability in L_p and accuracy of a finite difference approximation are defined and expressed in terms of Fourier transforms, and estimates which are based on simple analysis in L_2 are derived. The remaining chapters are then devoted to the more refined results in L_p with $p \neq 2$ for the heat equation, first order hyperbolic equations and the Schrödinger equation, respectively.

Except for some results in Chapter 6, the material in these notes can be found in papers published by the authors and others. Rather than striving for generality we have chosen, for the purpose of making the techniques transparent, to treat only simple cases.

The results and formulae are numbered by chapter, section, and order within each section so that, for instance, Theorem 1.2.3 means the third theorem of Chapter 1, Section 2 (or Section 1.2). For reference within a chapter the first number is dropped so that the above theorem within Chapter 1 is referred to as Theorem 2.3. The references to the literature are listed at the end of each chapter.

Throughout these notes, C and c will denote large and small positive constants, respectively, not necessarily the same at different occurrences.

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TABLE OF CONTENTS

CHAPTER 1. FOURIER MULTIPLIERS ON L_p.

In this chapter we develop the theory of Fourier multipliers on L_p to the extent needed for the applications in later chapters. Since our applications are quantitative rather than qualitative, we shall define the L_p multiplier norm $M_p(a)$ for smooth a only, and our efforts will then be to describe some techniques to estimate this norm. In Section 1 we introduce the necessary definitions and in Section 2 we then collect a number of basic properties of the multiplier norms. In Section 3 we derive an inequality for $M_\infty(a)$ by Carlson and Beurling which will be one of our main tools later. In Section 4 we reduce the problem of estimating periodic multipliers to the corresponding problem for multipliers with compact support, and in Section 5, finally, we prove a lemma by van der Corput and some consequences relevant to the present context.

1.1. Preliminaries and definition.

For $x = (x_1,\ldots,x_d) \in R^d$ and $\xi = (\xi_1,\ldots,\xi_d) \in R^d$, let $\langle x,\xi\rangle = x_1\xi_1+\ldots+x_d\xi_d$ and $|x| = \langle x,x\rangle^{1/2}$. We shall use the Fourier transform normalized so that for functions $u \in L_1$,

$$\mathcal{F}u(\xi) \equiv \hat{u}(\xi) = \int e^{-i\langle x,\xi\rangle}u(x)dx .$$

Its inverse is then formally

$$\mathcal{F}^{-1}v(x) \equiv \check{v}(x) = (2\pi)^{-d}\int e^{i\langle x,\xi\rangle}v(\xi)d\xi ,$$

and the Fourier inversion formula $\mathcal{F}^{-1}\hat{u} = u$ holds if u and $\hat{u}$ both belong to L_1. Parseval's formula now reads

$$\int u\bar{v}dx = (2\pi)^{-d}\int \hat{u}\bar{\hat{v}}d\xi .$$

(Unless specified to the contrary all functions considered will be complex-valued.)

For $\alpha = (\alpha_1,\ldots,\alpha_d)$ a non-negative multi-integer we define

$$\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_d^{\alpha_d}, \quad D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d},$$

and have then

$$D^\alpha u(x) = \mathcal{F}^{-1}((i\xi)^\alpha \hat{u})(x).$$

Further, for $y \in R^d$,

$$(1.1) \qquad u(x+y) = \mathcal{F}^{-1}(e^{i\langle y,\xi\rangle}\hat{u})(x),$$

and

$$(u * v)(x) = \int u(x-y)v(y)dy = \mathcal{F}^{-1}(\hat{u}\hat{v})(x).$$

Let $\hat{C}_0^\infty = \hat{C}_0^\infty(R^d)$ denote the class of functions which have Fourier transforms in $C_0^\infty = C_0^\infty(R^d)$, the set of functions $v \in C^\infty(R^d)$ with $\text{supp}(v)$ (the support of v) compact. In this chapter we shall mainly work with the Fourier transform and its inverse acting on functions in $\hat{C}_0^\infty$ and C_0^∞, respectively, but in later chapters the Fourier transform (and differentiation) will be applied more generally to elements in the space S' of tempered distributions, in particular to functions in L_p, $1 \le p \le \infty$.

The norms in the spaces L_p, $1 \le p \le \infty$, are given as usual by

$$\|u\|_p = \begin{cases} \left(\int |u(x)|^p dx\right)^{1/p} & \text{for } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x \in R^d} |u(x)| & \text{for } p = \infty. \end{cases}$$

We shall denote by W_p the closure in the L_p norm of $\hat{C}_0^\infty$ (or C_0^∞, or S, the class of functions which together with all their derivatives tend to zero faster than any negative power of $|x|$, as $|x|$ tends to infinity). For $1 \le p < \infty$ we have $W_p = L_p$ whereas W_∞ is the space of continuous functions which vanish at infinity and is properly contained in L_∞.

Let now for $a \in C^\infty(R^d)$ the operator A from $\hat{C}_0^\infty$ into itself be defined by

$$(1.2) \qquad Au = \mathcal{F}^{-1}(a\hat{u}).$$

It is easily seen from (1.1) that A is translation invariant so that for any $y \in R^d$,

$$A(u(\cdot - y))(x) = Au(x-y).$$

If A is a given translation invariant operator of the form (1.2), the function a is referred to as its symbol and is often denoted by $\hat{A}$.

In Chapter 2 ff. we shall mainly work with functions a in the class of slowly increasing functions, i.e. functions which together with all their derivatives have at most polynomial growth. In this case we may consider A defined by (1.2) as a continuous operator in S'.

We say that a is a Fourier multiplier on L_p. or that $a \in M_p = M_p(R^d)$ if

$$M_p(a) = \sup\{\|Au\|_p : u \in \hat{C}_0^\infty, \|u\|_p \leq 1\} < \infty.$$

The operator A on $\hat{C}_0^\infty$ defined by (1.2) may then be extended by completion from $\hat{C}_0^\infty$ to W_p. Since L_p is continuously embedded in S' this extension is consistent with the distribution interpretation of (1.2) when a is slowly increasing. We shall see later (Theorem 2.3) that in the case $p = \infty$ we can similarly extend A to a bounded linear operator not only on W_∞ but on L_∞.

Occasionally we shall use the notation $M_p^{(d)}$ and $M_p^{(d)}(\cdot)$ for the multipliers and their norms in order to emphasize the dimension of the underlying space R^d. Notice that in this presentation multipliers are always C^∞ functions.

1.2. Basic properties.

We first show that M_p is symmetric with respect to conjugate indices.

Theorem 2.1. Let $1 \leq p, p' \leq \infty$, $1/p + 1/p' = 1$. Then $M_p = M_{p'}$ and for $a \in C^\infty$,

$$M_p(a) = M_{p'}(a).$$

Proof. Let $u_-(x) = u(-x)$. Using (1.2) we have by Parseval's formula, Hölder's inequality, and the fact that $(\hat{u})_- = \widehat{(u_-)}$, that for $u,v \in \hat{C}_0^\infty$,

$$\left|\int Av \cdot u\,dx\right| = \left|\int \mathcal{F}^{-1}(a\hat{v})\overline{\mathcal{F}^{-1}\hat{\bar{u}}}\,dx\right| = (2\pi)^{-d}\left|\int a\hat{v}(\hat{u})_-\,d\xi\right|$$

$$= \left|\int \mathcal{F}^{-1}(a\widehat{(u_-)})v_-\,dx\right| \le \|Au_-\|_p\|v\|_{p'} \le M_p(a)\|u\|_p\|v\|_{p'} .$$

Hence by the converse of Hölder's inequality,

$$\|Av\|_{p'} \le M_p(a)\|v\|_{p'} ,$$

that is, $M_{p'}(a) \le M_p(a)$. Reversing the roles of p and p' we obtain the desired result.

We next give characterizations of M_2 and M_∞ (= M_1 by Theorem 2.1).

<u>Theorem 2.2</u>. M_2 consists of the uniformly bounded functions in C^∞ and for $a \in M_2$,

$$M_2(a) = \|a\|_\infty .$$

<u>Proof</u>. Assume first that a is bounded. Then with A defined by (1.2) we have for $u \in \hat{C}_0^\infty$,

$$\|Au\|_2 = (2\pi)^{-d/2}\|a\hat{u}\|_2 \le (2\pi)^{-d/2}\|a\|_\infty\|\hat{u}\|_2 = \|a\|_\infty\|u\|_2 ,$$

and hence $a \in M_2$ and

(2.1) $\quad M_2(a) \le \|a\|_\infty .$

Conversely, let $a \in M_2$ and let $\xi_0 \in R^d$ and $\varepsilon > 0$ be arbitrary. Then there exists a sphere B with center at ξ_0 such that

(2.2) $\quad |a(\xi)| \ge |a(\xi_0)|(1-\varepsilon)$ for $\xi \in B$.

Let $u_0 \in \hat{C}_0^\infty$, $u_0 \not\equiv 0$, be such that $\text{supp}(\hat{u}_0) \subset B$. Then Parseval's formula and (2.2) give

$$\|Au_0\|_2 = (2\pi)^{-d/2}\|a\hat{u}_0\|_2 \ge (2\pi)^{-d/2}|a(\xi_0)|(1-\varepsilon)\|\hat{u}_0\|_2 = |a(\xi_0)|(1-\varepsilon)\|u_0\|_2 .$$

Since ξ_0 and ε are arbitrary, we conclude that a is bounded and that

$$\|a\|_\infty \leq M_2(a).$$

Together with (2.1) this completes the proof of the theorem.

For the characterization of M_∞, let B denote the set of bounded complex regular measures on R^d with the total variation norm $V(\cdot)$. Recall that for $\mu(x) = f(x)dx$ with $f \in L_1$, $V(\mu) = \|f\|_1$, and that the convolution between a function and a measure is defined by

$$u * \mu(x) = \int u(x-y)d\mu(y).$$

Our next result shows that the elements of M_∞ (or M_1) are Fourier transforms of measures in B.

<u>Theorem 2.3</u>. Let $a \in M_\infty$. Then there exists $\mu \in B$ such that

$$(2.3) \qquad a(\xi) = \int e^{-i\langle x,\xi\rangle} d\mu(x),$$

$$(2.4) \qquad M_\infty(a) = V(\mu),$$

$$(2.5) \qquad Au \equiv \mathcal{F}^{-1}(a\hat{u}) = u * \mu \quad \text{for} \quad u \in \hat{C}_0^\infty.$$

Conversely, let $a \in C^\infty$ and assume that (2.3) holds with $\mu \in B$. Then $a \in M_\infty$ and (2.4), (2.5) hold true.

<u>Proof</u>. Assume first that $a \in M_\infty$. We have for the operator A,

$$|Au(0)| \leq M_\infty(a)\|u\|_\infty.$$

Hence the linear form $u \rightsquigarrow Au(0)$ may be extended to a bounded linear functional on W_∞. By the Riesz representation theorem there exists a measure μ in B such that

$$Au(0) = \int u(-y)d\mu(y) \quad \text{for} \quad u \in W_\infty.$$

Since the operator A is translation invariant it follows that

$$Au(x) = A(u(\cdot + x))(0) = \int u(x-y)d\mu(y) = u * \mu(x),$$

which proves (2.5).

By the Riesz representation theorem we also have for each fixed x that

$$\sup_{u \in \hat{C}_0^\infty} \frac{|Au(x)|}{\|u\|_\infty} = V(\mu),$$

and hence the norm equality (2.4) follows easily.

It remains to prove (2.3). Fourier transformation of (2.5) gives for $u \in \hat{C}_0^\infty$,

(2.6) $$a\hat{u} = \mathcal{F}(u * \mu).$$

The right hand side may be calculated as follows:

$$\mathcal{F}(u * \mu)(\xi) = \int e^{-i\langle x,\xi\rangle} \int u(x-y)d\mu(y)dx$$

$$= \int \left(\int e^{-i\langle x,\xi\rangle} u(x-y)dx \right) d\mu(y) = \hat{u}(\xi) \int e^{-i\langle y,\xi\rangle} d\mu(y).$$

Here the change in the order of integration is justified by the Fubini-Tonelli theorem since

$$\int \left(\int |e^{-i\langle x,\xi\rangle} u(x-y)| dx \right) d|\mu|(y) \leq \|u\|_1 V(\mu).$$

Hence it follows from (2.6) that

$$a(\xi)\hat{u}(\xi) = \hat{u}(\xi) \int e^{-i\langle y,\xi\rangle} d\mu(y),$$

which proves (2.3).

For the converse we find for $u \in \hat{C}_0^\infty$, using again the Fubini-Tonelli theorem to justify the interchange in the order of integration,

$$Au(x) = \mathcal{F}^{-1}(a(\xi)\hat{u}(\xi))(x) = (2\pi)^{-d} \int e^{i\langle x,\xi\rangle} \int e^{-i\langle y,\xi\rangle} d\mu(y)\hat{u}(\xi)d\xi$$

$$= \int (2\pi)^{-d} \int e^{i\langle x-y,\xi\rangle} \hat{u}(\xi)d\xi d\mu(y) = \int u(x-y)d\mu(y) = u * \mu(x).$$

This proves (2.5), and

$$\|Au\|_\infty \le V(\mu)\|u\|_\infty .$$

Hence $a \in M_\infty$, and the equality (2.4) follows as before.

This completes the proof of the theorem.

In particular, if $a \in M_\infty$, and if μ is as in the theorem, we may define a bounded linear operator A on L_∞ with norm $M_\infty(a)$ by $Au = u * \mu$ for $u \in L_\infty$. It is easily seen that if a is slowly increasing, then we have in the sense of distributions, $Au = \mathcal{F}^{-1}(a\hat{u})$, for $u \in L_\infty$, so that A coincides on L_∞ with the extension to S' of the operator in (1.2) on $\hat{C}_0^\infty$.

Our next two results describe inclusions and norm relations among different spaces of multipliers. The proofs will be based on the following well known lemma.

<u>Lemma 2.1</u>. (The Riesz-Thorin interpolation theorem.) Let $1 \le p_0, p_1, r_0, r_1 \le \infty$ and let T be a linear operator from $L_{p_0} \cap L_{p_1}$ into $L_{r_0} \cap L_{r_1}$ such that there exist constants N_0 and N_1 such that

$$\|Tf\|_{r_i} \le N_i \|f\|_{p_i}, \text{ for } f \in L_{p_i}, \ i = 0,1.$$

Let $0 < \theta < 1$ and let p and r be defined by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} .$$

Then T may be extended to a bounded linear operator from L_p to L_r with

$$\|Tf\|_r \le N_0^{1-\theta} N_1^\theta \|f\|_p, \text{ for } f \in L_p.$$

<u>Theorem 2.4</u>. Let $1/p + 1/p' = 1$ with $1 \le p \le p' \le \infty$ and assume that $a \in M_p$. Then $a \in M_q$ for all q with $p \le q \le p'$ and

(2.7) $\quad M_q(a) \le M_p(a).$

In particular, if $a \in M_p$ for some p with $1 \leq p \leq \infty$, then a is bounded and

$$\|a\|_\infty \leq M_1(a).$$

Proof. By Theorem 2.1 we have $a \in M_{p'}$, with $M_{p'}(a) = M_p(a)$, and hence the operator A in (1.2) is bounded in both L_p and $L_{p'}$. Writing $1/q = (1-\theta)/p + \theta/p'$ we therefore obtain by Lemma 2.1, for $u \in \hat{C}_0^\infty$,

$$\|Au\|_q \leq M_p(a)^{1-\theta} M_{p'}(a)^\theta \|u\|_q = M_p(a)\|u\|_q ,$$

which proves (2.7). The last statement is now an immediate consequence of Theorem 2.2.

Theorem 2.5. Assume that $a \in M_\infty$. Then $a \in M_p$ for $1 \leq p \leq \infty$ and

$$M_p(a) \leq M_2(a)^{2-2/p} M_\infty(a)^{2/p-1}, \quad \text{for } p \leq 2,$$

$$M_p(a) \leq M_2(a)^{2/p} M_\infty(a)^{1-2/p}, \quad \text{for } p \geq 2 .$$

Proof. The fact that $a \in M_p$ for $1 \leq p \leq \infty$ is contained in Theorem 2.4. Since $M_\infty(a) = M_1(a)$, the inequalities now follow by applying Lemma 2.1 to the operator A in (1.2).

We shall now prove that under certain conditions, limits of multipliers are multipliers.

Theorem 2.6. Let $a_n \in M_p$, $n = 1,2,\ldots$ be such that for some constant K,

(2.8) $\quad M_p(a_n) \leq K, \; n = 1,2,\ldots$.

Assume further that there exists a function $a \in C^\infty$ such that for every $v \in C_0^\infty$,

(2.9) $$\lim_{n \to \infty} \int a_n v \, d\xi = \int a v \, d\xi .$$

Then $a \in M_p$ and $M_p(a) \leq K$.

<u>Proof</u>. Setting $A_n u = \mathcal{F}^{-1}(a_n \hat{u})$ we obtain by (2.9) for each $u \in \hat{C}_0^\infty$ and $x \in R^d$,

$$\lim_{n\to\infty} A_n u(x) = (2\pi)^{-d} \lim_{n\to\infty} \int a_n(\xi) e^{i\langle x,\xi\rangle} \hat{u}(\xi) d\xi = \mathcal{F}^{-1}(a\hat{u})(x) = Au(x).$$

Further, since by Theorems 2.2 and 2.4,

$$\|a_n\|_\infty \le M_p(a_n) \le K,$$

we have

$$|A_n u(x)| \le (2\pi)^{-d} \|a_n\|_\infty \|\hat{u}\|_1 \le (2\pi)^{-d} K \|\hat{u}\|_1,$$

and hence by dominated convergence, for $v \in C_0^\infty$,

$$(2.10) \qquad \lim_{n\to\infty} \int A_n u \cdot v\,dx = \int Au \cdot v\,dx.$$

On the other hand, using Hölder's inequality, we have for p and p' conjugate indices,

$$\left| \int A_n u \cdot v dx \right| \le \|A_n u\|_p \|v\|_{p'},$$

so that by (2.8) and (2.10),

$$\left| \int Au \cdot v dx \right| \le K \|u\|_p \|v\|_{p'}.$$

The converse of Hölder's inequality then proves that

$$\|Au\|_p \le K \|u\|_p,$$

which completes the proof of the theorem.

We next show that M_p is closed under multiplication.

<u>Theorem 2.7</u>. Let $a, b \in M_p$. Then $ab \in M_p$ and

$$M_p(ab) \le M_p(a) M_p(b).$$

<u>Proof</u>. Let $u \in \hat{C}_0^\infty$. Then $Bu = \mathcal{F}^{-1}(b\hat{u}) \in \hat{C}_0^\infty$ and hence with the notation (1.2),

$$\|A(Bu)\|_p \leq M_p(a)\|Bu\|_p \leq M_p(a)M_p(b)\|u\|_p .$$

Since $A(Bu) = \mathcal{F}^{-1}(ab\,\hat{u})$, this proves the theorem.

In the next two theorems we shall study the behavior of multipliers under affine transformations. It will be convenient to prove first the following lemma, in which we denote $(a \otimes b)(\xi,\eta) = a(\xi)b(\eta)$.

Lemma 2.2. Let $a \in M_p^{(l)}$, $b \in M_p^{(n)}$. Then $a \otimes b \in M_p^{(l+n)}$ and

$$M_p^{(l+n)}(a \otimes b) = M_p^{(l)}(a)M_p^{(n)}(b).$$

In particular, if $a \in M_p^{(l)}$, the natural extension $a \otimes 1$ of a to R^{l+n} is in $M_p^{(l+n)}$ and

$$M_p^{(l+n)}(a \otimes 1) = M_p^{(l)}(a).$$

Proof. Let (x,y) denote the variable in R^{l+n} and (ξ,η) its dual variable. Consider first the extension $\tilde{a}(\xi,\eta) \equiv a(\xi) \otimes 1$ of a to a function on R^{l+n}. We then have, with obvious notation, for $u \in \hat{C}_0^\infty(R^{l+n})$,

$$\tilde{A}u(x,y) = \mathcal{F}^{-1}_{\xi,\eta}(\tilde{a}\mathcal{F}_{x,y}u)(x,y) = \mathcal{F}^{-1}_{\xi}(a(\xi)\mathcal{F}^{-1}_{\eta}\mathcal{F}_y\mathcal{F}_x u)(x,y)$$

$$= \mathcal{F}^{-1}_{\xi}(a(\xi)\mathcal{F}_x u)(x,y).$$

Integration with respect to x gives

$$\int |\tilde{A}u(x,y)|^p dx \leq M_p^{(l)}(a)^p \int |u(x,y)|^p dx ,$$

and after integration also with respect to y we conclude that $\tilde{a} \in M_p^{(l+n)}$ and

$$M_p^{(l+n)}(\tilde{a}) \leq M_p^{(l)}(a).$$

Denoting similarly $\tilde{b}(\xi,\eta) = 1 \otimes b(\eta)$ we conclude by Theorem 2.7 that