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PREFACE

This volume contains eight lectures resulting from papers delivered at the conference "Journées mathématiques sur les Fibrés vectoriels et Equations différentielles" held in Nice, France from June 12 through June 17, 1979.

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TABLE OF CONTENTS

COUNTING SINGULARITIES OF QUADRATIC FORMS ON VECTOR BUNDLES

Wolf BARTH

0. INTRODUCTION.

The study of surfaces in $\mathbb{P}_3$ with many nodes (= ordinary double points) is a beautiful classical topic, which recently found much attention again [3, 4]. All systematic ways to produce such surfaces seem related to symmetric matrices of homogeneous polynomials or, more generally, to quadratic forms on vector bundles :
If the form q on the bundle E is generic, then q is of maximal rank on an open set. The rank of q is one less on the discriminant hypersurface $\{\det q = o\}$, which represents the class $2c_1(E^*)$. This hypersurface is nonsingular in codimension one, but has ordinary double points in codimension two exactly where rank q drops one more step.

The aim of this paper is to show that the (rational homology class of the) singular variety of the discriminant is given by

(o) $$4(c_1c_2 - c_3), \qquad c_i = c_i(E^*).$$

If the base space has dimension three, the number of nodes of the discriminant surface is computed in this way.

Although I do not know of any place in the literature, where this formula can be found, I do not claim originality. If rank E = 2 for example, then $q \in \Gamma(S^2E^*)$, and the problem comes down to show that $c_3(S^2E^*) = 4c_1(E^*)c_2(E^*)$, which is well-known. Also, for morphisms $E \longrightarrow F$ there is Porteous' formula [9] expressing the loci of degeneration in terms of Chern classes. This formula does not apply directly to quadratic forms however, because they are selfadjoint, hence not generic as morphisms.

Formula (o) of course is some intersection number on the bundle space $\mathbb{P}(S^2E^*)$. Formulas for the higher-order degeneracies of quadratic forms analogous to Porteous' formula are to be expected as results of some computations in the intersection ring of $\mathbb{P}(S^2E^*)$.

I do not use here intersection theory on $\mathbb{P}(S^2E^*)$, partly because I had some trouble to identify the cycle on $\mathbb{P}(S^2E^*)$ of forms which on every fibre of E have a fixed given rank. My method is to associate with a quadratic form q a sheaf $\mathscr{C}$ on the discriminant hypersurface and to compute $ch(\mathscr{C})$. This sheaf $\mathscr{C}$ is closely related to the theory of "even nodes" [4]

1. PRELIMINARIES.

Let S be the vector space of complex symmetric r x r matrices and define the following subvarieties

$$D := \{s \in S : \text{rank } s \leq r - 1\}$$

$$C := \{s \in S : \text{rank } s \leq r - 2\}$$

$$B := \{s \in S : \text{rank } s \leq r - 3\}$$

Then D is the zero-set of the determinant function, hence a hypersurface in S.

Lemma 1 : D <u>is nonsingular outside of</u> C <u>and</u> C <u>is nonsingular outside of</u> B. <u>One has</u>

$$\text{codim}_S \; C = 3, \quad \text{codim}_S \; B = 6.$$

Proof : Put

$$s' = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad s'' = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Then each $s \in D \setminus C$ (resp. $s \in C \setminus B$) is of the form $as'a^t$ (resp. $as''a^t$) with $a \in GL(n)$. This shows that $D \setminus C$ (resp. $C \setminus B$) is homogeneous under $GL(n)$, hence smooth. Also, the dimension of $C \setminus B$ is

$$n^2 - \dim \{a \in GL(n) : as''a^t = s''\} .$$

Any $a \in GL(n)$ leaving s" invariant is of the form

$$a = \begin{pmatrix} a_1 & o \\ a_2 & a_3 \end{pmatrix} \qquad \begin{matrix} (a_1, a_2) \in \mathbb{C}^{2n} \\ a_3 \in O(n-2) \end{matrix} .$$

So the dimension to be subtracted is

$$2n + \dim O(n-2) = \frac{1}{2} n\, (n-1) + 3$$

and $\dim C \setminus B = \frac{1}{2} n(n+1) - 3$. The same argument gives the dimension of B. ┘

Lemma 2 : D <u>has ordinary quadratic singularities along</u> C, <u>i.e. any nonsingular (local) threefold meeting</u> C <u>transversally in a point</u> $s_o \notin B$ <u>intersects</u> S <u>in a surface with an ordinary doublepoint at</u> s_o.

Proof : We may assume $s_o = s''$. Parametrize the threefold as $s(u_1,u_2,u_3) = (s_{ij}(u_1,u_2,u_3))$ with $s(o,o,o) = s''$. The intersection with S has the equation

$$f(u_1,u_2,u_3) = \det\,(s_{ij}(u_1,u_2,u_3)) = \sum_{i \neq j} s_{1i}\, s_{2j}\, s^{1i,2j},$$

where $s^{1i,2j}$ is the corresponding minor. So

$$\left.\frac{\partial^2 f}{\partial u_m\, \partial u_n}\right|_{o,o,o} = \frac{\partial s_{11}}{\partial u^m} \frac{\partial s_{22}}{\partial u^n} + \frac{\partial s_{11}}{\partial u^n} \frac{\partial s_{22}}{\partial u^m} - 2\, \frac{\partial s_{12}}{\partial u^m} \frac{\partial s_{12}}{\partial u^n}$$

and the hessian of f at (o,o,o) will be

$$(\partial s/\partial u)^{t} \begin{pmatrix} o & 1 & o \\ 1 & o & o \\ o & o & -2 \end{pmatrix} (\partial s/\partial u)$$

with

$$(\partial s/\partial u) := \begin{pmatrix} \partial s_{11}/\partial u_1 & \partial s_{11}/\partial u_2 & \partial s_{11}/\partial u_3 \\ \partial s_{22}/\partial u_1 & \partial s_{22}/\partial u_2 & \partial s_{22}/\partial u_3 \\ \partial s_{12}/\partial u_1 & \partial s_{12}/\partial u_2 & \partial s_{12}/\partial u_3 \end{pmatrix}$$

But the assumption that the threefold meets C transversally means rank $(\partial s/\partial u) = 3$.┘

2. QUADRATIC FORMS ON VECTOR BUNDLES

A quadratic form on $\mathbb{C}^r$ can be thought of as a linear map $q : \mathbb{C}^r \longrightarrow (\mathbb{C}^r)^*$ with $q^t = q$. This is the viewpoint for the study of quadratic forms on vector bundles to be used in the sequel.

So let X be a smooth projective threefold over $\mathbb{C}$ and E some rank-r vector bundle on X.

Definition : A <u>quadratic form</u> on E is linear morphism $q : E \longrightarrow E^*$ with $q^t = q$. The set

$$\Delta := \{x \in E \; ; \; q(x) \text{ is not bijective}\}$$

is called the <u>discriminant</u> of q.

The quadratic forms on E form the vector space of sections in $S^2(E^*)$. If q is degenerate everywhere , Δ equals X, but in general Δ will be a surface. Δ can be empty only if $E = E^*$ and q is constant. The vector bundle $S^2(E^*)$ with typical fibre the space of symmetric r x r matrices contains as sub-fibre bundles the bundles

D(E), C(E), and B(E),

the associated bundles with typical fibre D,C, and B. $s \in S^2(E^*)$ belongs to these subvarieties if rank $s \leq$ r-1, r-2, and r-3 respectively.

For a quadratic form $q \in \Gamma(S^2E^*)$, Δ is the projection into X of $q \cap D(E)$.

Lemma 3 : (transversality) : <u>Assume that</u> q

- <u>does not intersect</u> B(E),
- <u>intersects</u> C(E) <u>transversally (in finitely many points)</u>,
- <u>intersects</u> D(E) <u>transversally outside of</u> C(E).

<u>Then</u> Δ <u>is a surface representing the class</u> $2c_1(E^*)$. <u>It is nonsingular except for finitely many nodes, the points x where rank</u> q(x) = r-2.

Proof : a) Let $x_o \in X$ be a point with rank $q(x_o)$ = r-1. By assumption q intersects D(E) transversally near x_o, so $q \cap D(E)$ is nonsingular there. The projection $q \longrightarrow X$ being biregular, Δ will also be nonsingular near x_o. The equation det q=o vanishes to the first order on Δ in all but the finitely many point $x \in \Delta$ with rank $q(x) = r-2$. So the surface Δ represents in Pic X the class $\det E^* - \det E = 2c_1(E^*)$.

b) Let $x_o \in X$ be a point with rank $q(x_o) = r-2$. There is a neighborhood $U \subset X$ of x_o and a trivialization $E|U = U \times \mathbb{C}^r$ inducing trivializations

$$S^2(E^*)|U = U \times S, \; D(E)|U = U \times D, \; C(E)|U = U \times C.$$

Let $\pi = S \times U \longrightarrow S$ and $\rho = S \times U \longrightarrow U$ be the projections. The trivialization can be chosen such that $q(x_o) = s''$.

Now the equation for Δ near x_o is

$$\det(\pi\, q(x)) = 0.$$

By assumption, q intersects C(E) transversally at $q(x_o)$, so $\pi q(U)$ intersects C transversally at s" and $\pi : q(U) \longrightarrow \pi q(U)$ is biregular near $q(x_o)$. Lemma 2 shows that $q(U) \cap D(E)$ is a surface with an ordinary node at $q(x_o)$. So Δ , the biregular image of this surface under ρ , will have an ordinary node at x_o. ┘

Lemma 4 (Bertini): <u>Assume that</u> $S^2(E^*)$ <u>is generated by global sections</u>. <u>Then there is some Zariski-open subset of sections</u> $q \in \Gamma(S^2(E^*))$ <u>satisfying the conditions in Lemma 3.</u>

Proof : Put $\Gamma := \Gamma(S^2(E^*))$ and consider the evaluation map $\gamma : X \times \Gamma \longrightarrow S^2(E^*)$. This map γ is regular everywhere, so the subvarieties

$$\tilde{D} = \gamma^{-1}D(E), \quad \tilde{C} = \gamma^{-1}C(E), \; \tilde{B} = \gamma^{-1}B(E)$$

of $X \times \Gamma$ have codimension 1,3 and 6 respectively. Denote by $\pi : X \times \Gamma \longrightarrow \Gamma$ the (proper) projection and define subvarieties of Γ as follows :

i) $\Gamma_1 := \pi(\tilde{B})$. This is a subvariety of Γ with codimension ≥ 3, because $\dim X = 3$.

ii) Let $C' \subset \tilde{C}$ be the subvariety of points where $d(\pi|\tilde{C})$ is not surjective. Then $\Gamma_2 := \pi(C')$ is a subvariety of codimension ≥ 1.

iii) Let $D' \subset \tilde{D}\setminus\tilde{C}$ be the subvariety of points where $d(\pi|\tilde{D}\setminus\tilde{C})$ is not surjective and let $\bar{D}$ be its closure in $X \times \Gamma$. Then $\Gamma_3 := \pi(\bar{D})$ again is a subvariety of Γ of codimension ≥ 1.

Now the Zariski-open subset can be taken as the complement of $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. ┘

Combining lemmas 3 and 4 one obtains :

Proposition 1 : If $S^2(E^*)$ is spanned by global sections, then for general $q \in \Gamma(S^2(E^*))$ the discriminant $\Delta \subset X$ is a nonsingular surface except for finitely many nodes. It represents the class $2c_1(E^*)$.

3. THE COKERNEL OF q

Now let $q = E \longrightarrow E^*$ be a quadratic form which is general in the sense of proposition 1. Outside of Δ, the morphism q is an isomorphism.

So there is an exact sequence

$$o \longrightarrow E \xrightarrow{q} E^* \longrightarrow \mathscr{C} \longrightarrow o$$

with an $\mathcal{O}_X$-sheaf $\mathscr{C}$ supported on Δ. Next we shall analyze the cokernel $\mathscr{C}$.

Denote by $\{x_i\}$ the finite set of nodes of Δ.

Lemma 5 : Outside of $\{x_i\}$, the sheaf $\mathscr{C}$ is an invertible $\mathcal{O}_\Delta$-sheaf.

Proof : Fix some point $x_o \in \Delta \setminus \{x_i\}$ and let f be a local equation for Δ near x_o. Since rank $(q|\Delta) = r-1$ near x_o, there is a section e_1 in E, without zeroes, such that $q(e_1)|\Delta = o$. This section e_1 can be extented to a basis $e_1,\dots,e_r$ for E near x_o. In the basis for E and an arbitrary one for E^* write

$$q = \begin{pmatrix} q_{11} & \cdots & q_{1r} \\ \vdots & & \vdots \\ q_{r1} & & q_{rr} \end{pmatrix}$$

then $q(e_1)$ is the vector $(q_{11},\dots,q_{r1})$. Since it vanishes on Δ, we can write $q_{j1} = f.q'_{j1}$ and $q = q' \circ \varphi$ with

$$q' = \begin{pmatrix} q'_{11} & \cdots & q_{1r} \\ \vdots & & \vdots \\ q'_{r1} & \cdots & q_{rr} \end{pmatrix} \qquad \varphi = \begin{pmatrix} f & & & o \\ & 1 & & \\ & & \ddots & \\ o & & & 1 \end{pmatrix}$$

Since det q = f det q' vanishes on Δ to the first order only, det q' cannot vanish near x_o and q' is an isomorphism of E onto E^*.

The diagram

$$\begin{array}{ccccccccc} o & \longrightarrow & E & \xrightarrow{\varphi} & E & \longrightarrow & \mathcal{O}_\Delta & \longrightarrow & o \\ & & \| & & \downarrow q' & & & & \\ o & \longrightarrow & E & \xrightarrow{q} & E^* & \longrightarrow & \mathcal{C} & \longrightarrow & o \end{array}$$

then shows that $\mathcal{C} = \mathcal{O}_\Delta$ near x_o.

To understand the situation near the singularities x_i, we shall blow them up : Fix some x_i and let $\sigma_i : \tilde{X}_i \longrightarrow X$ be the monoidal transform with center x_i. The surface $\Sigma_i := \sigma_i^{-1} x_i$ then is a copy of $\mathbb{P}_2$ with self-intersection $\Sigma_i^2 = -h_i$, h_i the positive generator of $H^2(\mathbb{P}_2,\mathbb{Z})$. Since x_i was an ordinary node of Δ, the proper transform $\tilde{\Delta}_i \subset \tilde{X}_i$ of Δ is nonsingular near Σ_i. It intersects Σ_i in a curve C_i, which is a non-degenerate conic on Σ_i and has on $\tilde{\Delta}_i$ self-intersection -2.

Additionally, it is no loss of generality to assume $r = 2$ (locally near x_i). In fact, there is a basis for E_{x_i} such that $q(x_i)$ looks like s". There is a rank-(r-2) subbundle $G \subset E$ near x_i restricting in x_i to the subspace of E_{x_i} spanned by the last r-2 basis vectors. So $q|G$ is non-degenerate near x_i. Define $F \subset E$ as the subbundle $G^\perp$, i.e. the kernel of $E \xrightarrow{q} E^* \longrightarrow G^*$. Then locally near x_i, E is an orthogonal direct sum $F \oplus G$ and

$$q = \begin{pmatrix} q_F & o \\ o & q_G \end{pmatrix}$$

with respect to this decomposition. q_G being an isomorphism, the original cokernel $\mathcal{C}$ is isomorphic to the cokernel of $q_F : F \longrightarrow F^*$, with rank F = 2.

So replace E by $F = 2\mathcal{O}$ and write

$$q = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

with functions a,b,c vanishing at x_i. Let $\tilde{a},\tilde{b},\tilde{c},\tilde{q}$ be the pullbacks of a,b,c,q to X_i. On $\tilde{X}_i$ there is near Σ_i a diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2\mathcal{O}_{\tilde{X}_i} & \xrightarrow{p} & 2\mathcal{J}_{\Sigma_i} & \longrightarrow & M_i & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2\mathcal{O}_{\tilde{X}_i} & \xrightarrow{\tilde{q}} & 2\mathcal{O}_{\tilde{X}_i} & \longrightarrow & \tilde{\mathcal{C}}_i & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 2\mathcal{O}_{\Sigma_i} & = & 2\mathcal{O}_{\Sigma_i} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now $2\mathcal{J}_{\Sigma_i}$ is locally free, and p, the map induced by $\tilde{q}$, is given by a matrix

$$\begin{pmatrix} a/g & c/g \\ c/g & b/g \end{pmatrix},$$

g a local equation for Σ_i. Also det $\tilde{q}$ vanishes on Σ_i outside of C_i only to order 2. So det p does not vanish there at all. Since det p vanishes only along $\tilde{\Delta}_i$, and there of order one, a modification of lemma 5 shows that M_i is an invertible $\mathcal{O}_{\tilde{\Delta}_i}$-sheaf. Outside of C_i, the morphism $M_i \longrightarrow \mathcal{C}_i = \sigma^*\mathcal{C}$ is an isomorphism.

To formulate the result, let $\sigma : \tilde{X} \longrightarrow X$ be the simultaneous blow up of all x_i, let $\tilde{q} : \tilde{E} \longrightarrow \tilde{E}^*$ be the pullback of q, let $\Sigma = \cup \Sigma_i$ be the union of the exceptional planes, $\tilde{\Delta}$ the proper transform of Δ, and $C = \cup C_i = \Sigma \cap \tilde{\Delta}$ the union of the conics.

Proposition 2 : On $\tilde{X}$ the quadratic form q induces an exact sequence

(1) $$0 \longrightarrow \tilde{E} \xrightarrow{\tilde{q}} \tilde{E}^* \longrightarrow \tilde{\mathcal{C}} \longrightarrow 0,$$

and the cokernel $\tilde{\mathcal{C}}$ is an extension

(2) $$0 \longrightarrow M \longrightarrow \tilde{\mathcal{C}} \longrightarrow 2\mathcal{O}_\Sigma \longrightarrow 0$$

with an invertible $\mathcal{O}_{\tilde{\Delta}}$-sheaf M.

Proof : Outside of Σ , M is the sheaf $q^*\mathscr{C}$. Over each conic C_i, it extends by the sheaf M_i constructed above. ┘

The next proposition is a reformulation of the symmetry of q.

Proposition 3 : <u>There is an isomorphism</u>

$$\varepsilon : \mathscr{C} \to \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathscr{C}},\mathcal{O}_{\tilde{X}})$$

<u>inducing an isomorphism</u>

(3) $$M^{\otimes 2} = \mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta} - \Sigma).$$

Proof : By virtue of $\tilde{q} = \tilde{q}^t$, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{E} & \xrightarrow{\tilde{q}} & \tilde{E}^* & \longrightarrow & \tilde{\mathscr{C}} & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ 0 & \longrightarrow & \tilde{E} & \xrightarrow{\tilde{q}^t} & \tilde{E}^* & \longrightarrow & \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathscr{C}},\mathcal{O}_{\tilde{X}}) & \longrightarrow & 0 \end{array}$$

inducing the isomorphism ε .

Now the dual sequence of (2) is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(2\mathcal{O}_\Sigma,\mathcal{O}_{\tilde{X}}) & \longrightarrow & \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathscr{C}},\mathcal{O}_{\tilde{X}}) & \longrightarrow & \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(M,\mathcal{O}_{\tilde{X}}) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & 2\mathcal{O}_\Sigma(\Sigma) & \longrightarrow & \tilde{\mathscr{C}} & \longrightarrow & M^* \otimes \mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta}) & \longrightarrow & 0 \end{array}$$

Since the map

$$2\mathcal{O}_\Sigma(\Sigma) \longrightarrow \tilde{\mathscr{C}} \longrightarrow 2\mathcal{O}_\Sigma$$

is injective on Σ outside of C, it induces an exact sequence

$$0 \longrightarrow 2\mathcal{O}_\Sigma(\Sigma) \longrightarrow 2\mathcal{O}_\Sigma \longrightarrow \mathcal{O}_C \longrightarrow 0$$

and a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M & = & M & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2\mathcal{O}_{\Sigma}(\Sigma) & \longrightarrow & \tilde{\mathscr{C}} & \longrightarrow & M^{*}\otimes\mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2\mathcal{O}_{\Sigma}(\Sigma) & \longrightarrow & 2\mathcal{O}_{\Sigma} & \longrightarrow & \mathcal{O}_{C} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The righthand column implies

$$M^{*} \otimes \mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta}) = M \otimes \mathcal{O}_{\tilde{\Delta}}(C). \qquad \lrcorner$$

Corollary : For $N := M \otimes \sigma^{*}(\det E) \otimes \mathcal{O}_{\tilde{\Delta}}(2C)$ one has

$$N^{\otimes 2} = \mathcal{O}_{\tilde{\Delta}}(C),$$

i.e., the divisor class C on $\tilde{\Delta}$ is divisible by 2.

Proof : $\mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta}) = \sigma^{*}(2c_1E^{*}) \otimes \mathcal{O}_{\tilde{\Delta}}(-2C)$ and $M^{\otimes 2} = \sigma^{*}(\det E^{*})^{\otimes 2} \otimes \mathcal{O}_{\tilde{\Delta}}(-3C)$. $\lrcorner$

4. APPLICATION OF GROTHENDIECK-RIEMANN-ROCH.

This section gives the number of singularities x_i in terms of the Chern-classes $c_i = c_i(E^{*})$. In fact, all one has to do is to compute Σ^3, because of

$$\Sigma^3 = \sum_i \Sigma_i^3$$

and $\Sigma_i^3 = 1$ for all i. (We shall denote by c_i also the pull-back $\sigma^{*} c_i \in H^2(\tilde{X},\mathbb{Z})$). Consider the two exact sequences (1) and (2). The additivity of the Chern-character [7, Appendix A] shows.

$$\text{(4)} \qquad ch(E^{*}) - ch(E) = ch(\tilde{\mathscr{C}}) = ch(M) + ch(2\mathcal{O}_{\Sigma}).$$

E^{*} being locally free, one has the well-known formula [7, p. 432]

$$ch(E^{*}) = r + c_1 + \frac{1}{2}(c_1^2 - c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3).$$

Because of $c_i(E) = (-1)^i\, c_i(E^*)$ we find for the codimension -3 component of $ch(\tilde{\mathcal{C}})$

(5) $$ch_3(\tilde{\mathcal{C}}) = \frac{1}{3} c_1^3 - c_1 c_2 + c_3.$$

The computation of $ch(2\mathcal{O}_\Sigma) = 2\, ch(\mathcal{O}_\Sigma)$ is easy too : Because of the exact sequence

$$o \longrightarrow \mathcal{O}_{\tilde{X}}(-\Sigma) \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_\Sigma \longrightarrow o$$

one has

$$ch(\mathcal{O}_\Sigma) = ch(\mathcal{O}_{\tilde{X}}) - ch(\mathcal{O}_{\tilde{X}}(-\Sigma)) = 1-(1-\Sigma + \frac{1}{2}\Sigma^2 - \frac{1}{6}\Sigma^3)$$

and

(6) $$ch_3(2\mathcal{O}_\Sigma) = \frac{1}{3}\Sigma^3.$$

The complicated part is to compute $ch(M)$ by $\mathcal{G}r$ RR for the embedding $i : \tilde{\Delta} \longrightarrow \tilde{X}$,

$$ch(M) = i_* \left[ch(M|\tilde{\Delta}).\ td(-N_{\tilde{\Delta}/\tilde{X}})\right].$$

Here (intersections are taken in $H^*(\tilde{\Delta}, \mathbb{Q})$)

$$ch(M|\tilde{\Delta}) = 1 + M + \frac{1}{2}M^2$$

$$c(N_{\tilde{\Delta}/\tilde{X}}) = 1 + \mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta})$$

$$c(-N_{\tilde{\Delta}/\tilde{X}}) = 1 - \mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta}) + \mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta})^2$$

$$td(-N_{\tilde{\Delta}/\tilde{X}}) = 1 + \frac{1}{2}c_1(-N_{\tilde{\Delta}/\tilde{X}}) + \frac{1}{12}(c_1^2 + c_2)(-N_{\tilde{\Delta}/\tilde{X}}) =$$

$$= 1 - \frac{1}{2}\mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta}) + \frac{1}{6}\mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta})^2$$

$$ch_3(M) = i_*\left[ch(M|\tilde{\Delta})\ .\ td(-N_{\tilde{\Delta}/\tilde{X}})\right]_2 =$$

$$= \frac{1}{6}\tilde{\Delta}^3 - \frac{1}{2}(M.\mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta})) + \frac{1}{2}(M.M).$$

Now using $\mathcal{O}_{\tilde{X}}(\tilde{\Delta}) = 2c_1 - 2\mathcal{O}_{\tilde{X}}(\Sigma)$ and

$$M = \frac{1}{2}(2c_1 - 3\mathcal{O}_{\tilde{\Delta}}(\Sigma)) = c_1 - \frac{3}{2}\mathcal{O}_{\tilde{\Delta}}(\Sigma)$$

by formula (3), we compute

$$\tilde{\Delta}^3 = 8\, c_1^3 - 8\, \Sigma^3$$

$$(M.\mathcal{O}_{\tilde{\Delta}}(\tilde{\Delta})) = 4\, c_1^3 - 6\, \Sigma^3$$

$$(M.M) = 2\, c_1^3 - \frac{9}{2}\, \Sigma^3$$

(7) $$ch_3(M) = \frac{1}{3}\, c_1^3 - \frac{7}{12}\, \Sigma^3.$$

Then substituting (5), (6), and (7) in formula (4), one obtains finally

$$\frac{1}{3}\, c_1^3 - c_1c_2 + c_3 = \frac{1}{3}\, c_1^3 - \frac{7}{12}\, \Sigma^3 + \frac{1}{3}\, \overset{3}{\Sigma},$$

or

$$\Sigma^3 = 4(c_1c_2 - c_3) \quad .$$

Theorem : In the situation of proposition 1, the number of nodes of Δ equals $4(c_1c_2 - c_3)$. Here $c_i = c_i(E^*)$, $i = 1,2,3$.

Remark : If rank $E = 2$, then $c_3 = o$ and the formula gives $4\, c_1c_2$ for the number of nodes. In fact, in this case the nodes are exactly the points where $q \in \Gamma(S^2E^*)$ vanishes and $c_3(S^2E^*) = 4c_1(E^*)c_2(E^*)$ is well-known.

5. GENERALISATIONS.

Next two possible generalisations of the formula in the theorem above are given without proofs.

a) Of course, it is not necessary to assume that $\dim X = 3$. Also if $\dim X > 3$, the discriminant hypersurface of a sufficiently general quadratic form $q : E \longrightarrow E^*$ is nonsingular in codimension 1, it represents the class $2c_1(E^*)$, and its singularities form a cycle representing the class $4(c_1(E^*)c_2(E^*)-c_3(E^*))$.

b) Very often one meets twisted quadratic forms : A quadratic form on E with values in L, some line bundle on X, is a linear morphism

$$q = E \longrightarrow E^* \otimes L,$$

which is symmetric in the sense that

$$q = q^{\dagger} \otimes id_L.$$