World Scientific Lecture Notes in Physics Vol. 16

RIEMANNIAN GEOMETRY FIBER BUNDLES KALUZA-KLEIN THEORIES AND ALL THAT....

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World Scientific
Singapore • New Jersey • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.
P.O. Box 128, Farrer Road, Singapore 9128

U. S. A. office: World Scientific Publishing Co., Inc.
687 Hartwell Street, Teaneck NJ 07666, USA

RIEMANNIAN GEOMETRY, FIBER BUNDLES, KALUZA-KLEIN THEORIES AND ALL THAT

ISBN 9971-50-426-X
9971-50-427-8 pbk

Printed in Singapore by Kim Hup Lee Printing Co. Pte. Ltd.

Pour Anne-Marie.

Dla Anny.

Acknowledgments

We are grateful to several institutes for gaving us the opportunity to meet and to work on this book, in particular the CERN Theory division, in Geneva, the Centre de Physique Théorique in Marseille, IHES in Bures sur Yvette, the University in Hamburg and the Centrum Fizyki Teoretycnej in Wroclaw.

We would also like to thank Professors R. Haag and D. Kastler for their interest and encouragement.

Most of the writing (and typing) of this book took place between 9 P.M. and 2 A.M. We are grateful to Anne-Marie, Valérie, Eric, Anna and Agneczka for putting up with it, and for their love.

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I

GENERAL REMARKS AND PREREQUISITES

1-1 Introduction

Physical motivations

It is nowadays believed that it is possible and useful to describe physics in an "extended" space-time. Events that we see and that we measure are usually described by 3+1 numbers labelling the position and the time. Forces acting on objects and influencing their trajectories have also been described in the past in term of various tensor fields defined on a four dimensional manifold modelising the "space of events". It happens that, in many cases,the theory takes a simpler form if we assume that what we observe is just a shadow (projection) of something that takes place in space-time which has more than 4 dimensions. As an example, it has been recognised long ago that coupled gravitational and electromagnetic fields respectively described by a (4 dimensional) hyperbolic metric and a Maxwell field (a U(1) connection), could be also described by a U(1) invariant metric on a five-dimensional space. It is very possible (and it is the belief of the authors) that a correct formalization of the physics of "our universe" should involve an infinite dimensional manifold, and that for reasons which are still unknown, what we see classically looks four -dimensional. The fact that we do not see the extra dimensions (those of the so called "internal space") can be described, if not explained,by the fact that the metric of our multidimensional universe singles out some directions along which it is invariant or at least equivariant (in some sense).

Many papers have been published recently in the physical literature presenting many different constructions,sometimes under the same headings (Kaluza-Klein theories, Dimensional reduction, Symmetries of gravitational and gauge fields , etc.); very often the generality of the described situation was not studied. One of the aims of the present book is to present the geometrical and analytical aspects of "dimensional reduction" and to discuss with more generality several situations which have been considered in the past.

Content of the book

What the book is really about is Riemannian geometry of those spaces on which a group action is given (with a view on applications to physical theories -"unified theories"-). This study involves in particular the geometry of group manifolds, homogeneous spaces, principal bundles, non principal bundles (with group action),.., but also, in order to study the different kinds of "fields" defined on those spaces, it requires an appropriate generalization of (non abelian) harmonic analysis.

Each chapter of the book begins with a summary section which stresses the main ideas in plain terms ; the reader willing to make his knowledge more precise should then read the rest of the chapter where a more detailed discussion (using a more precise mathematical language) is given. The summary introductions do not usually require any knowledge of fiber bundles; however,we use freely the corresponding terminology and results in the core of each chapter. Indeed, although the summary section usually describes everything in a "local" way (e.g. using coordinates), we always want to render our considerations global.

The remaining sections of this first chapter recall some standard definitions of differential (Riemannian) geometry and has also the purpose of setting our conventions. Most of the results discussed here will be used freely in all chapters of the book; however, we should mention that the reader who wants to recast Riemannian geometry in the general framework of the theory of connections should jump directly to Ch.6, where these notions are developed from scratch.

Anybody willing to construct physical models generalizing the "old" Kaluza-Klein ideas should be first acquainted with some basic facts about the Riemannian structure(s) of Lie groups (Ch.2) and homogeneous spaces (Ch.3). The study of G-invariant metrics on groups and homogeneous spaces is also compulsory if one wants to analyse the situation when the space is a (generally only local) product of some manifold **M** times a group **G** or a homogeneous space **G/H**. G-invariant metrics on principal bundles and non-principal bundles carrying a **G** action are discussed respectively in Ch.4 and Ch.5. A general study of the Riemannian geometry of "matter fields", i.e., vector valued functions (or forms) defined on a manifold (in particular the covariant derivative acting on tensors, spinors, p-forms valued in some vector space...) is made in Ch.6 (this chapter could be read independently of the rest of the book). It is well known that, when a real (or complex) valued function is defined on a group or on a homogeneous space, it is possible to "expand" it (think of the usual spherical harmonics); however, when the underlying space is only a (local) product of some manifold **M** by **G** or **G/H**, the formalism has to be generalised and this is done in Ch.7. The particular case where such matter fields are usual tensors or spinors is studied in Ch.8 (G-spin-structures are naturally obtained there as a result of a process of "dimensional reduction"). The techniques described in particular in chapters 5, 7 and 8 provide us with a "general-purpose-tool" that we may use in several situations; as an example of such a use, we study in Ch.9 the dimensional reduction of Einstein-Yang-Mills systems i.e. analyse the geometry of a manifold on which both metric and connection are given, along with the action of a symmetry group. We will study this case by showing how it can be reduced to the situation studied in Ch.5. Finally, in Ch.10, we consider a more general situation where there is no global action of a finite dimensional Lie group G but where we can nevertheless define "interesting metrics" which are invariant under a "bundle of groups" (infinite dimensional groups of automorphisms of bundles are defined and studied in section 4.11).

Each main section of the book ends with a paragraph entitled "Pointers to the literature"; indeed, references are usually not given within each chapter but collected at the end.

New results

Before ending this introduction, we should maybe mention what is "original" in this book and what can be found elsewhere. It is clear that the whole discussion of homogeneous metrics on Lie groups and coset spaces can be found in a scattered way inside many papers of the mathematical literature. However the general discussion of metrics leading to "dimensionnal reduction", and in particular the general study of metrics on bundles with homogeneous fibers is probably new: although known "in principle", many explicit constructions and calculations carried out here do not seem to have been discussed elsewhere in the mathematical or in the physical literature (but by the authors themselves).

Also, let us mention a few other mathematical (or physical) constructions hardly to be found elsewhere: generalization of Frobenius and Peter-Weyl theorems (in Ch.7), intrinsic definition of the Lichnerowicz operator (in Ch.6), non-standard discussion of Einstein-Cartan theory with spinors (in Ch.6), link between G-spin structures and dimensional reduction (in Ch.8), generalization of the Wang theorem on G-invariant connections (in Ch.9), definition and study of "local" action of groups (bundle of groups, in Ch.4.11 and Ch.10).

How to read the book?

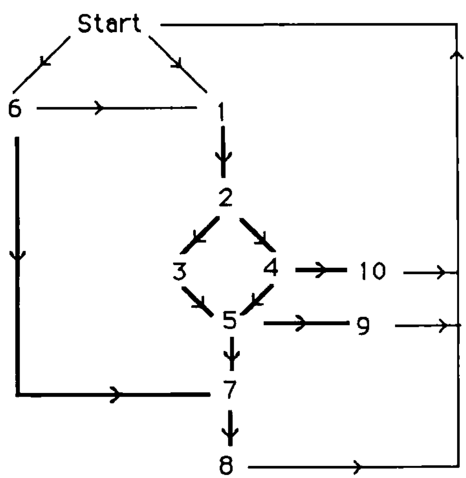
Method 1: from the beginning to the end.

Method 2: read only the summary sections.

The following diagram illustrates the interdependency of the chapters:

⟶ denotes a compulsory logical link

→ denotes an optional logical link



1.2 Differentiable manifolds

For this,we refer to the standard literature. Notice that a topological space is not necessarily a differentiable manifold (it has to be smooth !). Also, a given topological manifold may be endowed with none or several differentiable structures. For example, the number of inequivalent differentiable structures for spheres is 1 for S^p (p<6), 28 for S^7, 8 for S^9, 2 for S^{10}, 992 for S^{11}, while $\mathbf{R}^4$, being truly exceptional, is believed to have even uncountably many different differentiable structures. The concept of differentiable structure should not be confused with that of metric structure (the later beeing defined after the former). Unless otherwise specified, by "manifold" we will always mean here "differentiable manifold with a given smooth structure". Many concrete manifolds we shall deal with will be homogeneous spaces; unless otherwise specified, by a homogeneous space we will always mean here homogeneous space with the smooth structure induced by the quotient space definition. In most examples of low dimensionality, the smooth structure is unique, anyway. Notice in particular that when we discuss "non standard" metrics on some spheres -as in sect.3.4-3.6-, we will assume that the sphere has a fixed differentiable structure.

Since there exists several conventions in the definitions of exterior deivative and exterior products, we give here those that we will follow.

f being a function on the manifold M, (a zero-form), we write its differential as $df = \partial_\mu f \, dx^\mu$ in the coordinate basis $\{dx^\mu\}$. Let now ω be a k-form, we write it as $\omega = 1/k! \, \omega_{i1.....ik} \, dx^{i1} \wedge .... \wedge dx^{ik}$, and its differential $d\omega$ (a k+1-form) is given by

$$d\omega = 1/k! \, d\omega_{i1.....ik} \, dx^{i1} \wedge .... \wedge dx^{ik}.$$

Observe that we know what $d\omega_{i1.....ik}$ is since $\omega_{i1.....ik}$ is a function. The exterior differential **d** is the unique operator such that, for all forms ω_1, ω_2,

$d(\omega_1+\omega_2) = d\omega_1 + d\omega_2$,

$d(\omega_1\wedge\omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$ (ω_1 being a k-form)

$d^2 = 0$

Moreover, if $\xi_1, \xi_2,\ldots,\xi_{k+1}$ are vector fields on **M**, we have

$$d\omega(\xi_1, \xi_2,\ldots,\xi_{k+1}) = \Sigma_{i=1..k+1} (-1)^{i+1} \xi_i(\omega(\xi_1,\ldots,\hat{\xi}_i,\ldots,\xi_{k+1}) + \Sigma_{1\le i\le j\le n} (-1)^{i+j} \omega([\xi_i,\xi_j],\xi_1, \ldots,\hat{\xi}_i,\ldots,\hat{\xi}_j,\ldots,\xi_{k+1})$$

The group of permutations Σ_k acts as follows on k-upples of vectors

$\sigma\in\Sigma_k$: $\sigma(v_1,v_2,\ldots,v_k)=(v_{\sigma(1)},v_{\sigma(2)},\ldots,v_{\sigma(k)})$

Let T be a covariant tensor of rank k; then we get a fully antisymmetrised tensor **Alt T** via the equation

$\text{Alt } T = 1/k!\ \Sigma_{\sigma\in\Sigma k}\ \epsilon_{\sigma.} T.\sigma$

This equation also defines the operator **Alt**. Notice that thanks to the presence of 1/k!, we have $\text{Alt}((\text{Alt } \omega\otimes\varphi)\otimes\beta) = \text{Alt}(\omega\otimes\varphi\otimes\beta)$

$= \text{Alt}(\omega\otimes\text{Alt}(\varphi\otimes\beta))$,

ω,φ and β being covariant tensors.

Let us now take ω a k-form (completely antisymmetric tensor of rank k) and α a l-form; then we define their exterior product as follows

$\omega\wedge\alpha = (k+l)!/k!\ l!\quad \text{Alt}(\omega\otimes\alpha)$

In particular, if ω and α are 1-forms, we get $\omega\wedge\alpha = \omega\otimes\alpha-\alpha\otimes\omega$. Notice that if ω is a k-form, then $\text{Alt}(\omega)=\omega$ and that $\wedge$ has the following properties (call Ω^k the space of k-forms)

i) $\wedge$ is distributive over + from the left and from the right

ii) $a(\omega\wedge\alpha) = a\omega\wedge\alpha = \omega\wedge a\alpha$ with $a\in\mathbb{R}$.

iii) $\omega\wedge\alpha = (-1)^{kl}\alpha\wedge\omega$ where $\omega \in \Omega^k$, $\alpha \in \Omega^l$, (in particular, if ω is odd $\omega\wedge\omega = 0$).

If φ denotes a smooth map from a manifold M to a manifold N, we will write φ_* (sometimes $d\varphi$) for the tangent map (locally it can be written as the Jacobian matrix of partial derivatives) and φ^* for the cotangent map. Notice that vectors can be pushed forward in the direction of the map but forms are pulled back (if v^μ are the components of a vector of M, then $\partial_\mu\varphi^i . v^\mu$ are those of its push-forward in N, whereas if α_i are the components of a 1-form in N, then $\partial_\mu\varphi^i . \alpha_i$ are those of its pull back in M). Finally, if ω and τ are forms in N, we have $\varphi^*(\omega\wedge\tau)=\varphi^*(\omega)\wedge\varphi^*(\tau)$.

1.3 Riemannian manifolds

We now give some notations and state without much discussion some well known formulae of Riemannian geometry. This serves mainly the purpose of setting our conventions (for a more precise definition of the concepts involved, the reader may consult chapt. 6).

1.3.1 Metrics, connections and curvatures

Metrics

The metric $\mathbf{g(y)}$ at the point $\mathbf{y}$ describes a scalar product $\mathbf{g}(\ ,\ )$ in the tangent space at y and can be represented as a matrix $(g_{\mu\nu}(y))$ or as

$$g(y)=g_{\mu\nu}(y)\ \ dy^\mu \otimes dy^\nu$$

in a coordinate basis $\{dy^\mu\}$. It can also be represented as a matrix g_{ij} or as

$$g(y)=g_{ij}(y)\ \ \omega^i(y) \otimes \omega^j(y)$$

in a moving frame of forms $\{\omega^i(y)\}$.

Calling $(g^{\mu\nu}(y))$ the inverse of the matrix $(g_{\mu\nu}(y))$, we may consider the following object which defines a scalar product in the cotangent space at $\mathbf{y}$