Lecture Notes in Mathematics

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Paris-Princeton Lectures on Mathematical Finance 2010

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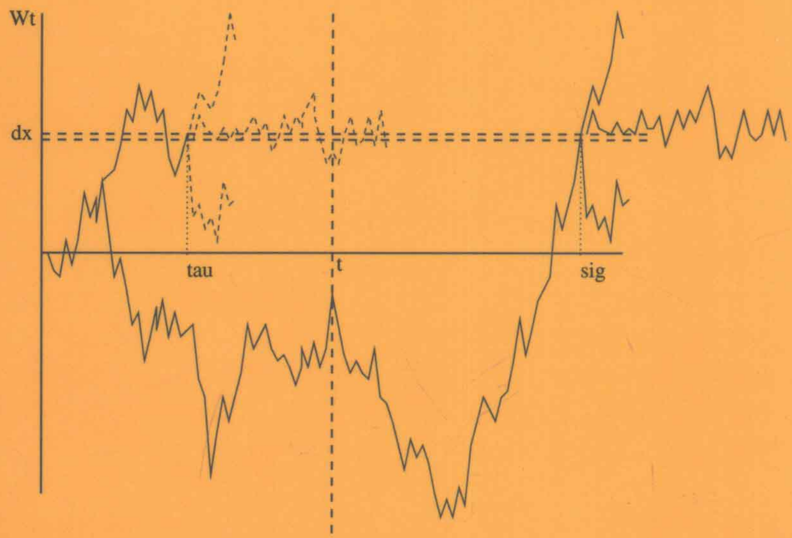

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Preface

This is the fourth volume of the Paris-Princeton Lectures in Mathematical Finance. The goal of this series is to publish cutting edge research in self-contained articles prepared by established academics or promising young researchers invited by the editors. Contributions are refereed and particular attention is paid to the quality of the exposition, the goal being to publish articles that can serve as introductory references for research.

The series is a result of frequent exchanges between researchers in finance and financial mathematics in Paris and Princeton. Many of us felt that the field would benefit from timely exposés of topics in which there is important progress. René Carmona, Erhan Cinlar, Ivar Ekeland, Elyes Jouini, José Scheinkman and Nizar Touzi serve in the first editorial board of the Paris-Princeton Lectures in Financial Mathematics. Although many of the chapters involve lectures given in Paris or Princeton, we also invite other contributions. Springer Verlag kindly offered to host the initiative under the umbrella of the Lecture Notes in Mathematics series, and we are thankful to Catriona Byrne for her encouragement and her help.

This fourth volume contains five chapters. In the first chapter, Areski Cousin, Monique Jeanblanc, and Jean-Paul Laurent discuss risk management and hedging of credit derivatives. The latter are over-the-counter (OTC) financial instruments designed to transfer credit risk associated to a reference entity from one counterparty to another. The agreement involves a seller and a buyer of protection, the seller being committed to cover the losses induced by the default. The popularity of these instruments lead a runaway market of complex derivatives whose risk management did not develop as fast. This first chapter fills the gap by providing rigorous tools for quantifying and hedging counterparty risk in some of these markets.

In the second chapter, Stéphane Crépey reviews the general theory of forward backward stochastic differential equations and their associated systems of partial integro-differential obstacle problems and applies it to pricing and hedging financial derivatives. Motivated by the optimal stopping and optimal stopping game formulations of American option and convertible bond pricing, he discusses the well-posedness and sensitivities of reflected and doubly reflected Markovian Backward Stochastic Differential Equations. The third part of the paper is devoted to the variational inequality formulation of these problems and to a detailed discussion of viscosity solutions. Finally he also considers discrete path-dependence issues such as dividend payments.

The third chapter written by Olivier Guéant Jean-Michel Lasry and Pierre-Louis Lions presents an original and unified account of the theory and the applications of the mean field games as introduced and developed by Lasry and Lions in a series of lectures and scattered papers. This chapter provides systematic studies illustrating the application of the theory to domains as diverse as population behavior (the so-called Mexican wave), or economics (management of exhaustible resources). Some of the applications concern optimization of individual behavior when interacting with a large population of individuals with similar and possibly competing objectives. The analysis is also shown to apply to growth models and for example, to their application to salary distributions.

The fourth chapter is contributed by David Hobson. It is concerned with the applications of the famous Skorohod embedding theorem to the proofs of model independent bounds on the prices of options. Beyond the obvious importance of the financial application, the value of this chapter lies in the insightful and extremely pedagogical presentation of the Skorohod embedding problem and its application to the analysis of martingales with given one-dimensional marginals, providing a one-to-one correspondence between candidate price processes which are consistent with observed call option prices and solutions of the Skorokhod embedding problem, extremal solutions leading to robust model independent prices and hedges for exotic options.

The final chapter is concerned with pricing and hedging in exponential Lévy models. Peter Tankov discusses three aspects of exponential Lévy models: absence of arbitrage, including more recent results on the absence of arbitrage in multidimensional models, properties of implied volatility, and modern approaches to hedging in these models. It is a self contained introduction surveying all the results and techniques that need to be known to be able to handle exponential Lévy models in finance.

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Contents

Hedging CDO Tranches in a Markovian Environment

Areski Cousin, Monique Jeanblanc, and Jean-Paul Laurent

Abstract In this first chapter, we show that a CDO tranche payoff can be perfectly replicated with a self-financed strategy based on the underlying credit default swaps. This extends to any payoff which depends only upon default arrivals, such as basket default swaps. Clearly, the replication result is model dependent and relies on two critical assumptions. First, we preclude the possibility of simultaneous defaults. The other assumption is that credit default swap premiums are adapted to the filtration of default times which therefore can be seen as the relevant information set on economic grounds. Our framework corresponds to a pure contagion model, where the arrivals of defaults lead to jumps in the credit spreads of survived names, the magnitude of which depending upon the names in question, and the whole history of defaults up to the current time. These jumps can be related to the derivatives of the joint survival function of default times. The dynamics of replicating prices of CDO tranches follows the same way. In other words, we only deal with default risks and not with spread risks.

Unsurprisingly, the possibility of perfect hedging is associated with a martingale representation theorem under the filtration of default times. Subsequently, we exhibit a new probability measure under which the short term credit spreads (up to some scaling factor due to positive recovery rates) are the intensities associated with the corresponding default times. For ease of presentation, we introduced first some instantaneous default swaps as a convenient basis of hedging instruments. Eventually, we can exhibit a replicating strategy of a CDO tranche payoff with respect

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A.R. Carmona et al. (eds.), *Paris-Princeton Lectures on Mathematical Finance 2010*, Lecture Notes in Mathematics 2003, DOI 10.1007/978-3-642-14660-2_1,

to actually traded credit default swaps, for instance, with the same maturity as the CDO tranche. Let us note that no Markovian assumption is required for the existence of such a replicating strategy.

However, the practical implementation of actual hedging strategies requires some extra assumptions. We assume that all pre-default intensities are equal and only depend upon the current number of defaults. We also assume that all recovery rates are constant across names and time. In that framework, it can be shown that the aggregate loss process is a homogeneous Markov chain, more precisely a pure death process. Thanks to these restrictions, the model involves as many unknown parameters as the number of underlying names. Such Markovian model is also known as a local intensity model, the simplest form of aggregate loss models. As in local volatility models in the equity derivatives world, there is a perfect match of unknown parameters from a complete set of CDO tranches quotes. Numerical implementation can be achieved through a binomial tree, well-known to finance people, or by means of Markov chain techniques. We provide some examples and show that the market quotes of CDOs are associated with pronounced contagion effects. We can therefore explain the dynamics of the amount of hedging CDS and relate them to deltas computed by market practitioners. The figures are hopefully roughly the same, the discrepancies being mainly explained by contagion effects leading to an increase of dependence between default times after some defaults.

1 Introduction

The risk management and the hedging of credit derivatives and related products are topics of tremendous importance, especially given the recent credit turmoil. The risks at hand are usually split into different categories, which may sometimes overlap, such as credit spread and default risks, correlation and contagion risks.

The credit crisis also drove attention to counterparty risk and related issues such as collateral management, downgrading of guarantors and of course liquidity issues. For simplicity, these will not be dealt within this part.[1]

Credit derivatives are over-the-counter (OTC) financial instruments designed to transfer credit risk of a reference entity between two counterparties by way of a bilateral agreement. The agreement involves a seller of protection and a buyer of protection. The seller of protection is committed to cover the losses induced by the default of a reference entity, typically a corporate. In return, the buyer of protection has to pay at some fixed dates a premium to the seller of protection. By the default, we mean that the entity goes bankrupt or fails to pay a coupon on time, for some of its issued bonds. Even though credit derivatives are traded over-the-counter, credit events are standardized by the International Swap and Derivative Association (ISDA).[2]

[1] See [33] for a discussion of the issues involved.

[2] Although ISDA reports a list of six admissible credit events, most of the contracts only include bankruptcy and failure to pay as credit events. This is the case of contracts referencing companies settled in developed countries. The definitions have been last updated in 2003. An overview of these standardized definitions can be found in [54]. However, these are likely to be updated, for instance due to the ISDA big bang protocol.

Since credit derivatives involve some counterparty risk, the protection seller may be asked to post some collateral. Also, depending on the market value of the contract, the amount of collateral may be dynamically adjusted. However, after the recent credit crisis and subsequent defaults, settlement procedures had to be updated. Various projects including the ISDA, tend to standardize the cash-flows of credit default swaps (CDS), netting and settlement procedures. It is likely that some market features will change. Nevertheless, the main ideas expressed here will still be valid with some minor adaptation.

Financial institutions such as banks, mutual funds, pension funds, insurance and reinsurance companies, monoline insurance companies, corporations or sovereign wealth funds have a natural incentive to use credit derivatives in order to assume, reduce or manage credit exposures.

Surprisingly enough, since pricing at the cost of the hedge is the cornerstone of the derivatives modelling field, models that actually connect pricing and hedging issues for CDOs have been studied after the one factor Gaussian copula model became a pricing standard. This discrepancy with the equity or interest derivatives fields can actually be seen as a weakness and one can reasonably think that further researches in the credit area will aim at closing the gap between pricing and hedging.

Before proceeding further, let us recall the main features in a hedging and risk management problem, which come to light whatever the underlying risks:

- A first issue is related to the choice and the liquidity of the hedging instruments: typically, one could think of credit index default swaps, CDS on names with possibly different maturities, standardized synthetic single tranche CDOs and even other products such as equity put options, though this will not be detailed in this part. We reckon that the use of equity products to mitigate risks can be useful in the high yield market, but this is seemingly not the case for CDO tranches related to investment grade portfolios.
- A second issue is related to the products to be hedged. In the remainder, we will focus either on single name CDS or basket credit derivatives, such as First to Default Swaps, CDO tranches, bespoke CDOs or tranchelets. We will leave aside interest rate or foreign exchange hybrid products, credit spread options and exotic basket derivatives such as leveraged tranches, forward starting CDOs or tranche options.
- A third issue relies on the choice of the hedging method. The mainstream theoretical approach in mathematical finance favors the notion of replication of complex products through dynamic hedging strategies based on plain underlying instruments. However, it is clear that in many cases, risk can be mitigated by offsetting long and short positions, providing either a complete clearing or more usually leaving the dealer exposed to some basis albeit small risk. Moreover, such an approach is obviously quite robust to model risk. Unfortunately, there are some imbalances in customer demand and investment banks can be left with rather large outstanding positions on parts of the capital structure that must be managed up to maturity.

2 Hedging Instruments

This section is a primer about hedging of defaultable securities. It aims at presenting a general model of prices and hedging of defaultable claims, in a pure jump setting (there is no Brownian motion involved in our presentation). It also introduces the main hedging instruments we will consider throughout this part. We will particularly describe the cash-flows of CDS and derive the dynamics of their price. We also stress the impact of a credit event on the price dynamics of the surviving names.

2.1 Credit Default Swap

A CDS is a bilateral over-the-counter agreement which transfers the credit risk of a defined reference entity from a buyer of protection to a seller of protection up to a fixed maturity time T. The reference entity denoted **C** is typically a corporate or a sovereign obligor.

We assume that **C** may default at a particular time τ which is a non negative random variable constructed on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. The default time τ corresponds to a credit event leading to payment to the protection buyer. Moreover, if **C** defaults, only a fraction R (the recovery rate) of the initial investment is recovered. Figure 1 illustrates the structure of a CDS.

2.1.1 Cash-Flow Description

Let us consider a CDS initiated at time $t = 0$ with maturity T and nominal value E. The cash-flows of a CDS can be divided in two parts (or legs): the default leg which corresponds to the cash-flows generated by the seller of protection and the premium leg which is the set of cash-flows generated by the protection buyer. For simplicity, we will assume that nor the protection seller, neither the protection buyer can default.

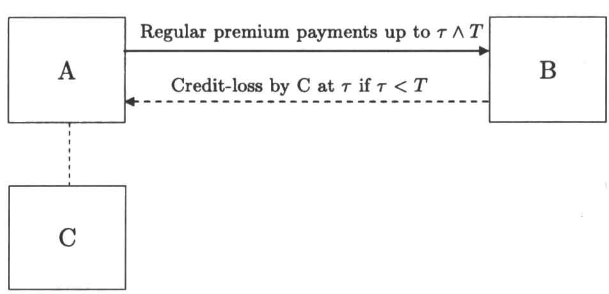

Fig. 1 Structure of a credit default swap

Default Leg

The seller of credit protection (denoted **B** in Fig. 1) agrees to cover losses induced by the default of the obligor **C** at time τ if the latter occurs before maturity ($\tau < T$). In that case, the payment is exactly equal to the fraction of the loss that is not recovered, i.e., the loss given default $E(1 - R)$. The settlement procedures in order to determine the recovery rate are not detailed here. The contract is worthless after the default of **C**.

Premium Leg or Fee Leg

In return, the buyer of protection (denoted **A** in Fig. 1) pays a periodic fee to **B** up to default time τ or until maturity T, whichever comes first. Each premium payment is proportional to a contractual credit spread[3] κ and to the nominal value E. More precisely, the protection buyer pays $\kappa \cdot \Delta_i \cdot E$ to the protection seller **B**, at every premium payment date $0 < T_1 < \cdots < T_p = T$ or until $\tau < T$, where $\Delta_i = T_i - T_{i-1}$, $i = 1, \ldots, p$ are the time intervals between two premium payment dates.[4] Let us remark that premium payments are made in arrears and begin at the end of the fist period (at T_1). If default happens between two premium payment dates, say $\tau \in]T_{i-1}, T_i[$, the protection fee has not been paid yet for the period $]T_{i-1}, \tau]$. In that case **A** will pay **B** an accrued premium equal to $\kappa \cdot (\tau - T_{i-1}) \cdot E$. The accrued premium payment is usually made at time τ. After default of C ($t > \tau$), there are no more cash-flows on the premium leg which is worthless.

It is noteworthy that the contractual spread κ is fixed at inception (at $t = 0$) and remains the same until maturity. It is determined so that the expected discounted cash-flows (under a proper pricing measure to be detailed below) between **A** and **B** are the same when the CDS contract is settled.

Due to the credit turmoil, some major market participants encourage a change in market convention for single name CDS quotes. In the proposal, the contractual spread will be fixed at $\kappa = 100$ bps or $\kappa = 500$ bps depending on the quality of the credit. The buyer of protection will have to make an immediate premium payment (upfront payment) to enter the contract (see [5] for more details).

2.2 *Theoretical Framework*

2.2.1 Default Times

In what follows, we consider n default times τ_i, $i = 1, \ldots, n$, that is, non-negative and finite random variables constructed on the same probability space $(\Omega, \mathcal{G}, \mathbb{P})$. For

[3] The contractual spread is quoted in basis points per annum.

[4] With the convention that $T_0 = 0$.

any $i = 1, \dots, n$, we denote by $(N_t^i = 1\!\!1_{\tau_i \le t}, t \ge 0)$ the ith default process, and by $\mathcal{H}_t^i = \sigma(N_s^i, s \le t)$ the natural filtration of N^i (after completion and regularization on right). We introduce $\mathbb{H}$, the filtration generated by the processes $N^i, i = 1, \dots, n$, defined as $\mathbb{H} = \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$, i.e., $\mathcal{H}_t = \vee_{i=1}^n \mathcal{H}_t^i$.

We denote by $\tau_{(1)}, \dots, \tau_{(n)}$ the ordered default times.

Hypothesis 1. We assume that no simultaneous defaults can occur, i.e., $\mathbb{P}(\tau_i = \tau_j) = 0, \ \forall i \neq j$. This assumption is important with respect to the completeness of the market. As shown below, it allows to dynamically hedge credit derivatives referencing a pool of defaultable entities with n credit default swaps.[5]

Hypothesis 2. We assume that, for any $i = 1, \dots, n$, there exists a non-negative $\mathbb{H}$-adapted process $(\alpha_t^{i,\mathbb{P}}, t \ge 0)$ such that the process

$$M_t^{i,\mathbb{P}} := N_t^i - \int_0^t \alpha_s^{i,\mathbb{P}} ds \tag{1}$$

is a $(\mathbb{P}, \mathbb{H})$-martingale. The process $\alpha^{i,\mathbb{P}}$ is called the $(\mathbb{P}, \mathbb{H})$-intensity of τ_i (Note that the value of the intensity depends strongly of the underlying probability). This process vanishes after τ_i (otherwise, after τ_i, the martingale $M^{i,\mathbb{P}}$ would be continuous and strictly decreasing, which is impossible) and can be written $\alpha_t^{i,\mathbb{P}} = (1 - N_t^i)\widetilde{\alpha}_t^{i,\mathbb{P}}$ for some $\mathbb{H}^1 \vee \cdots \vee \mathbb{H}^{i-1} \vee \mathbb{H}^{i+1} \vee \cdots \vee \mathbb{H}^n$-adapted process $\widetilde{\alpha}^{i,\mathbb{P}}$ (see [6] for more details). In particular, for $n = 1$, the process $\widetilde{\alpha}^{1,\mathbb{P}}$ is deterministic. In terms of the process $\widetilde{\alpha}^{i,\mathbb{P}}$, one has

$$M_t^{i,\mathbb{P}} = N_t^i - \int_0^{t \wedge \tau_i} \alpha_s^{i,\mathbb{P}} ds = N_t^i - \int_0^t (1 - N_s^i)\widetilde{\alpha}_s^{i,\mathbb{P}} ds.$$

Comments. (a) Let us remark that the latter hypothesis is not as strong as it seems to be. Indeed, the process N^i is an increasing $\mathbb{H}$-adapted process, hence an $\mathbb{H}$-submartingale. The Doob–Meyer decomposition implies that there exists a unique increasing $\mathbb{H}$-predictable process Λ^i such that $(N_t^i - \Lambda_t^i, t \ge 0)$ is an $\mathbb{H}$-martingale. We do not enter into details here,[6] it's enough to know that a left-continuous adapted process is predictable. It is also well known that the process Λ^i is continuous if and only if τ_i is totally inaccessible.[7] Here, we restrict our attention to processes Λ^i which are absolutely continuous with respect to Lebesgue measure. (b) It will be important to keep in mind that the martingale $M^{i,\mathbb{P}}$ has only one jump of size 1 at time τ_i.

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[5] In the general case where multiple defaults could occur, we have to consider possibly 2^n states, and we would require non standard credit default swaps with default payments conditionally on all sets of multiple defaults to hedge multiname credit derivatives.

[6] The reader is referred to [56] for the definition of a predictable process. A stopping time ϑ is predictable if there exists a sequence of stopping times ϑ_n such that $\vartheta_n < \vartheta$ and ϑ_n converges to ϑ as n goes to infinity.

[7] A stopping time τ is totally inaccessible if $\mathbb{P}(\tau = \vartheta) = 0$ for any predictable stopping time ϑ.

2.2.2 Market Assumptions

For the sake of simplicity, let us assume that instantaneous digital default swaps are traded on the names. An instantaneous digital credit default swap on name i traded at time t is a stylized bilateral agreement between a buyer and a seller of protection. More precisely, the protection buyer receives one monetary unit at time $t + dt$ if name i defaults between t and $t + dt$. If α_t^i denotes the contractual spread of this stylized CDS, the seller of protection receives in return a fee equal to $\alpha_t^i dt$ which is paid at time $t + dt$ by the buyer of protection. The *cash-flows* associated with a buy protection position on an instantaneous digital default swaps on name i traded at time t are summarized in Fig. 2.

Let us also remark that there is no charge at inception (at time t) to enter an instantaneous digital credit default swap trade. Then, its payoff is equal to $dN_t^i - \alpha_t^i dt$ at $t + dt$ where dN_t^i is the payment on the default leg and $\alpha_t^i dt$ is the (short term) premium on the default swap.

Hypothesis 3. We assume that contractual spreads $\alpha^1, \dots, \alpha^n$ are adapted to the filtration $\mathbb{H}$ of default times. The natural filtration of default times can thus be seen as the relevant information on economic grounds.

Moreover, since the instantaneous digital credit default swap is worthless after default of name i, credit spreads must vanish after τ_i, i.e., $\alpha_t^i = 0$ on the set $\{t > \tau_i\}$.

Note that considering such instantaneous digital default swaps rather than actually traded credit default swaps is not a limitation of our purpose. This can rather be seen as a convenient choice of basis from a theoretical point of view.

For simplicity, we further assume that (continuously compounded) default-free interest rates are constant and equal to r. Given some initial investment V_0 and some $\mathbb{H}$-predictable bounded processes $\delta^1, \dots, \delta^n$ associated with some self-financed trading strategy in instantaneous digital credit default swaps, we attain at time T the payoff:

$$V_0 e^{rT} + \sum_{i=1}^{n} \int_0^T \delta_s^i e^{r(T-s)} \left(dN_s^i - \alpha_s^i ds \right).$$

By definition, δ_s^i is the nominal amount of instantaneous digital credit default swap on name i held at time s. This induces a net cash-flow of $\delta_s^i \cdot \left(dN_s^i - \alpha_s^i ds \right)$ at time $s + ds$, which has to be invested in the default-free savings account up to time T.

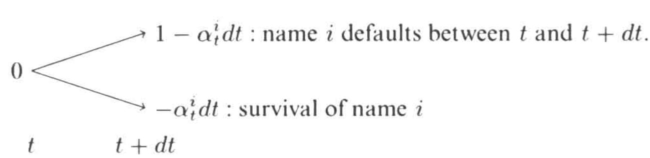

Fig. 2 *Cash-flows* of an instantaneous digital credit default swap (buy protection position)

2.2.3 Hedging and Martingale Representation Theorem

In our framework (we do not have any extra noise in our model, and the intensities do no depend on an exogenous factor), individual default intensities are not driven by a specific spread risk but by the arrival of new defaults: default intensities $\alpha^{i,\mathbb{P}}$, $i = 1, \ldots, n$ are deterministic functions of the past default times between two default dates. More precisely, as we shall prove later on, the intensity of τ_i on the set $\{t; \tau_{(j)} \leq t < \tau_{(j+1)}\}$ is a deterministic function of $\tau_{(1)}, \ldots, \tau_{(j)}$.

The main mathematical result of the study derives from the predictable representation theorem (see Theorem 9 in [10], Chap. III or [42]).

Theorem 1. *Let $A \in \mathcal{H}_T$ be a $\mathbb{P}$-integrable random variable. Then, there exists $\mathbb{H}$-predictable processes $\theta^i, i = 1, \ldots, n$ such that*

$$A = \mathbb{E}_{\mathbb{P}}[A] + \sum_{i=1}^{n} \int_0^T \theta_s^i \left(dN_s^i - \alpha_s^{i,\mathbb{P}} ds\right) = \mathbb{E}_{\mathbb{P}}[A] + \sum_{i=1}^{n} \int_0^T \theta_s^i dM_s^{i,\mathbb{P}}, \quad (2)$$

and $\mathbb{E}_{\mathbb{P}}\left(\int_0^T |\theta_s^i|\alpha_s^{i,\mathbb{P}} ds\right) < \infty$.

Proof. We do not enter into details. The idea is to prove that the set of random variables

$$Y = \exp\left(\sum_{i=1}^{n} \int_0^T \varphi_s^i dM_s^i - \int_0^T (e^{\varphi_s^i} - 1)\alpha_s^{i,\mathbb{P}} ds\right)$$

where φ^i are deterministic functions, is total in $L^2(\mathcal{H}_T)$ and to note that Y satisfies (2): indeed,

$$Y = 1 + \sum_{i=1}^{n} \int_0^T \varphi_s^i Y_{s-}^i dM_s^i .$$

Due to the integrability assumption on the r.v. A, and the predictable property of the θ's, the processes $\int_0^t \theta_s^i dM_s^i$, $i = 1, \ldots, n$ are $(\mathbb{P}, \mathbb{H})$-martingales. □

Let us remark that relation (2) implies that the predictable representation theorem (PRT) holds: any $(\mathbb{P}, \mathbb{H})$-martingale can be written in terms of the fundamental martingales $M^{i,\mathbb{P}}$. Indeed, if $M^{\mathbb{P}}$ is a $(\mathbb{P}, \mathbb{H})$-martingale, applying (2) to $A = M_T^{\mathbb{P}}$ and using the fact that $\int_0^t \theta_s^i dM_s^{i,\mathbb{P}}$ are martingales,

$$M_t^{\mathbb{P}} = \mathbb{E}_{\mathbb{P}}\left[M_T^{\mathbb{P}} \mid \mathcal{H}_t\right] = \mathbb{E}_{\mathbb{P}}\left[M_T^{\mathbb{P}}\right] + \sum_{i=1}^{n} \int_0^t \theta_s^i dM_s^{i,\mathbb{P}}. \quad (3)$$

From the PRT, any strictly positive $(\mathbb{P}, \mathbb{H})$-martingale ζ with expectation equal to 1 (as any Radon–Nikodym density) can be written as

$$d\zeta_t = \zeta_{t-} \sum_{i=1}^{n} \theta_t^i dM_t^{i,\mathbb{P}}, \; \zeta_0 = 1. \quad (4)$$

Indeed, as any martingale, ζ admits a representation as

$$d\zeta_t = \sum_{i=1}^{n} \hat{\theta}_t^i dM_t^{i,\mathbb{P}}, \ \zeta_0 = 1$$

Since ζ is assumed to be strictly positive, introducing the predictable processes θ^i as $\theta_s^i = \frac{1}{\zeta_{s-}^i}\hat{\theta}_s^i$ allows to obtain the equality (4). We emphasize that the predictable property of θ is essential to guarantee that the processes $\int \theta_s dM_s^i$ are (local) martingales.

Conversely, the Doléans–Dade exponential, (unique) solution of

$$d\zeta_t = \zeta_{t-} \sum_{i=1}^{n} \theta_t^i dM_t^{i,\mathbb{P}}, \ \zeta_0 = 1$$

is a (local) martingale. Note that, in order that ζ is indeed a non-negative local martingale, one needs that $\theta_t^i > -1$. Indeed, the solution of (4) is

$$\zeta_t = \exp\left(-\int_0^t \sum_{i=1}^{n} \theta_s^i \alpha_s^{i,\mathbb{P}} ds\right) \prod_{i=1}^{n} (1+\theta_{\tau_i}^i)^{N_t^i}.$$

The process ζ is a true martingale under some integrability conditions on θ (e.g., θ bounded) or if $\mathbb{E}^{\mathbb{P}}[\zeta_t] = 1$ for any t. Note that the jump of ζ at time $t = \tau_i$ is $\Delta\zeta_t = \zeta_t - \zeta_{t-} = \zeta_{t-}\theta_t^i$ (so that $\zeta_t = \zeta_{t-}(1+\theta_t^i)$ at time τ_i, hence the condition on θ to preserve non-negativity of ζ).

Theorem 2. *Let ζ satisfying (4) with $\theta_t^i > -1$ and $\mathbb{E}^{\mathbb{P}}[\zeta_t] = 1$, and define the probability measure $\mathbb{Q}$ as*

$$d\mathbb{Q}|_{\mathcal{H}_t} = \zeta_t d\mathbb{P}|_{\mathcal{H}_t}.$$

Then, the process

$$M_t^i := M_t^{i,\mathbb{P}} - \int_0^t \theta_s^i \alpha_s^{i,\mathbb{P}} ds = N_t^i - \int_0^t (1+\theta_s^i)\alpha_s^{i,\mathbb{P}} ds$$

is a $\mathbb{Q}$-martingale. In particular, the $(\mathbb{Q},\mathbb{H})$-intensity of τ_i is $\alpha_t^i = (1+\theta_t^i)\alpha_t^{i,\mathbb{P}}$.

Proof. The process M^i is an $(\mathbb{Q},\mathbb{H})$-martingale if and only if the process $M^i\zeta$ is a $(\mathbb{P},\mathbb{H})$-martingale. Using integration by parts formula

$$\begin{aligned}
d(M_t^i\zeta_t) &= M_{t-}^i d\zeta_t + \zeta_{t-}dM_t^i + \Delta M_t^i \Delta\zeta_t \\
&= M_{t-}^i d\zeta_t + \zeta_{t-}dM_t^{i,\mathbb{P}} - \zeta_{t-}\theta_t^i\alpha_t^{i,\mathbb{P}}dt + \zeta_{t-}\theta_t^i dN_t^i \\
&= M_{t-}^i d\zeta_t + \zeta_{t-}dM_t^{i,\mathbb{P}} - \zeta_{t-}\theta_t^i\alpha_t^{i,\mathbb{P}}dt + \zeta_{t-}\theta_t^i(dM_t^{i,\mathbb{P}} + \alpha^{i,\mathbb{P}}dt) \\
&= M_{t-}^i d\zeta_t + \zeta_{t-}dM_t^{i,\mathbb{P}} - \zeta_{t-}\theta_t^i\alpha_t^{i,\mathbb{P}}dt + \zeta_{t-}\theta_t^i dM_t^{i,\mathbb{P}} + \zeta_{t-}\theta_t^i\alpha_t^{i,\mathbb{P}}dt \\
&= M_{t-}^i d\zeta_t + \zeta_{t-}\left(1+\theta_t^i\right) dM_t^{i,\mathbb{P}}.
\end{aligned}$$

□