Structural Analysis Of Laminated Anisotropic Plates

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To my wife Phyllis and to my children for their patience, understanding, and encouragement

Preface

This book is a revision and major expansion of *Theory of Laminated Plates* by J. E. Ashton and J. M. Whitney published in 1970. In the original book both the theoretical development and pertinent solutions for plates fabricated of thin layers of anisotropic material were presented. With the expanded structural use of advanced composite materials comes a continued need for a textbook which addresses the structural behavior of laminated plates. Five of the original seven chapters are contained in the present book with minor revision. The subject matter of the remaining two chapters is contained in the new book with major revision. In addition, the new book contains four additional chapters which include material on laminated beams, expansional strain effects, curved plates, and free-edge effects.

The objective of this book is to provide a clear foundation in the theory of laminated anisotropic plates, including the problems of bending under transverse load, stability, and free-vibration. Although the theoretical development is complete, the principal demonstration of the behavior of laminated plates is made through the presentation of a large number of actual solutions. In particular, the effects of bending anisotropy, stacking sequence, and bending-stretching coupling are illustrated through numerous solutions with comparison to the simpler cases of orthotropic plates. The solutions presented by J. E. Ashton in Chapters 4 and 5 of the original book are contained in Chapters 5 and 6 of the new book with some revision, including new material. These solutions have become a classic in laminated plate analysis and form an important part of the new book.

The book contains eleven chapters. Chapter 1 presents fundamental information from anisotropic elasticity; Chapters 2 and 3 provide a development of the governing partial differential equations and boundary conditions, including variational forms, for thin laminated anisotropic plates subject to the assumption of non-deformable normals. Chapter 4 treats one-dimensional theories associated with cylindrical bending and laminated beams. Chapter 5 treats the simplified form of the laminated plate equations equivalent to homogeneous orthotropic plates. This form of the equations is rarely applicable to real laminated plates except as an approximation, and Chapters 6 and 7 indicate the effect of an assumption of orthotropic behavior by comparing solutions, including bending anisotropy (Chapter 6) and bending-stretching coupling (Chapter 7) to these orthotropic solutions. In Chapter 8 the effect of expansional strains on the behavior of laminated plates is presented. Example problems include the effects of thermal expansion and dimensional changes induced by matrix swelling associated with moisture absorption. The basic theory is extended to cylindrical plates in Chapter 9. In Chapter 10 a higher order theory applicable to laminated anisotropic plates which includes the effects of transverse shear deformation is developed. Solutions involving the higher order theory are compared to results obtained from classical laminated plate theory in which transverse shear deformation is neglected. A discussion of sandwich plates is also included in Chapter 10. Free-edge effects are discussed in Chapter 11 along with the development of a higher order laminated plate theory which includes a thickness-stretch mode in addition to transverse shear deformation. The new theory is then applied to an approximate free-edge analysis of cross-ply laminates.

This book is intended to combine theoretical development with solutions to the governing equations in order to indicate the importance of stacking sequence, degree of bending anisotropy, bending-extensional coupling, expansional strains, transverse shear deformation, and free-edge effects. A software program called LAMPCAL is available with the book to perform many of these calculations. The appendix of this book provides a full description of LAMPCAL and can serve as the users' guide. It is hoped that engineers and materials scientists will find both the book and software useful in developing an understanding of laminated structural elements.

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Dayton, Ohio
March 1987

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CHAPTER 1

Theory of an Anisotropic Elastic Continuum

1.1 INTRODUCTION

WITH THE INCREASED use of composite materials in structural applications has come a need for the analysis of laminated anisotropic plates. This chapter provides the fundamental principles of anisotropic elasticity from which laminated plate theory is developed in the following two chapters. Much of the presentation on anisotropic elasticity is based on the works of Lekhnitskii [1] and Hearmon [2]. A detailed derivation of the theory of finite deformations can be found in Fung [3].

1.2 STRESS AND STRAIN IN AN ANISOTROPIC CONTINUUM

Figure 1.1 shows the stress nomenclature in cartesian coordinates. In linear mechanics little or no distinction is made between the stresses with respect to the deformed and undeformed coordinates since the difference is a second order effect. However in the development of a plate theory which includes inplane force effects it is useful to relate stresses on the deformed body to the initial configuration.

Consider a force vector $d\bar{F}$ acting on a deformed surface dS and a corresponding force vector $d\bar{F}_o$ acting on the same surface in the undeformed state $d\bar{S}_o$. The stress components in the deformed state are given by the Cauchy relationship:

$$dF_i = \sum_{j=1}^{3} \tau_{ij} n_j dS \tag{1.1}$$

where τ_{ij} are components of the Eulerian stress tensor and n_j are direction cosines of the outward normal to the deformed surface. The Kirchhoff stress tensor refers to the original configuration and its components are defined as follows:

$$dF_{oi} = \sum_{j=1}^{3} \sigma_{ij} n_{oj} dS_o = \sum_{j=1}^{3} \frac{\partial x_i}{\partial \bar{x}_j} dF_j \tag{1.2}$$

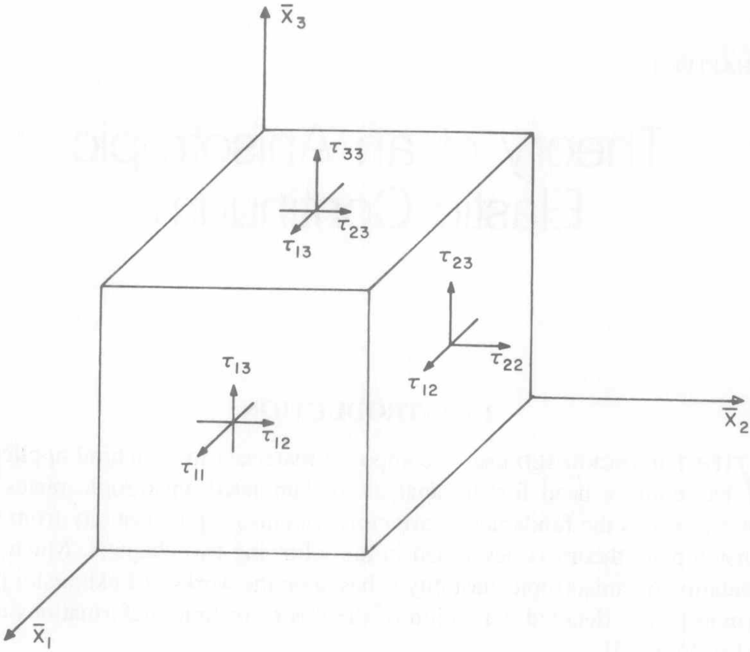

Figure 1.1. *Stress nomenclature on body in deformed state.*

where σ_{ij} are the components of the Kirchhoff stress tensor, n_{oj} are direction cosines of the outward normal to the undeformed surface, x_i are coordinates of the undeformed surface, and $\bar{x}_j$ are coordinates of the deformed surface.

In order to determine the principal stresses in a plate it is often necessary to consider the stresses with respect to an axis system rotated in the plane of the plate. Consider a rotation through an angle θ from x_1 and x_2 about the x_3 axis (see Figure 1.2). The rotated axes are denoted by $x_1{}'$ and $x_2{}'$. The transformed stresses $\sigma_{ij}{}'$ are given by

$$\begin{bmatrix} \sigma_{11}{}' \\ \sigma_{22}{}' \\ \sigma_{33}{}' \\ \sigma_{23}{}' \\ \sigma_{13}{}' \\ \sigma_{12}{}' \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & +n & m & 0 \\ -mn & mn & 0 & 0 & 0 & (m^2 - n^2) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{1.3}$$

where $m = \cos\theta$, $n = \sin\theta$.

For finite deformation the Green strain tensor is used. The strain displacement relations are given by

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] \tag{1.4}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_2}\right)^2\right] \tag{1.5}$$

$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_3}\right)^2 + \left(\frac{\partial u_2}{\partial x_3}\right)^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right] \tag{1.6}$$

$$\epsilon_{23} = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3} \tag{1.7}$$

$$\epsilon_{13} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3} \tag{1.8}$$

$$\epsilon_{12} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2} \tag{1.9}$$

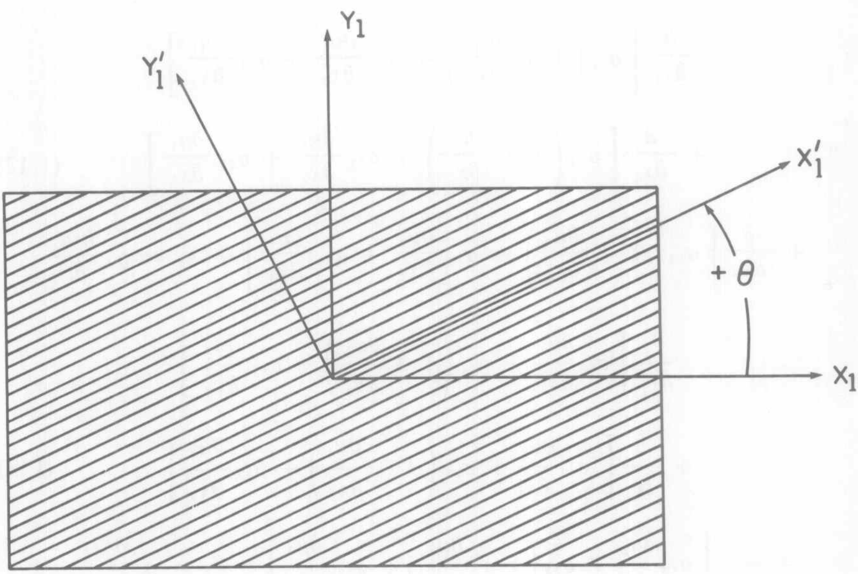

Figure 1.2. *Rotation of axes.*

where ϵ_{ij} are engineering strains and u_i are displacements along the x_i coordinate. For small displacement theory Equations (1.4–1.9) become

$$\left.\begin{aligned} \epsilon_{11} &= \frac{\partial u_1}{\partial x_1} & \epsilon_{22} &= \frac{\partial u_2}{\partial x_2} & \epsilon_{33} &= \frac{\partial u_3}{\partial x_3} \\ \epsilon_{23} &= \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} & \epsilon_{13} &= \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} & \epsilon_{12} &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \end{aligned}\right\} \tag{1.10}$$

For a rotation about the x_3 axis we have the following transformation:

$$\begin{bmatrix} \epsilon'_{11} \\ \epsilon'_{22} \\ \epsilon'_{33} \\ \epsilon'_{23} \\ \epsilon'_{13} \\ \epsilon'_{12} \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & mn \\ n^2 & m^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -2mn & 2mn & 0 & 0 & 0 & (m^2 - n^2) \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix} \tag{1.11}$$

1.3 EQUATIONS OF MOTION AND COMPATIBILITY

The Kirchhoff stress tensor must satisfy the following nonlinear equations of motion:

$$\begin{aligned} &\frac{\partial}{\partial x_1}\left[\sigma_{11}\left(1 + \frac{\partial u_1}{\partial x_1}\right) + \sigma_{12}\frac{\partial u_1}{\partial x_2} + \sigma_{13}\frac{\partial u_1}{\partial x_3}\right] \\ &+ \frac{\partial}{\partial x_2}\left[\sigma_{12}\left(1 + \frac{\partial u_1}{\partial x_1}\right) + \sigma_{22}\frac{\partial u_1}{\partial x_2} + \sigma_{23}\frac{\partial u_1}{\partial x_3}\right] \\ &+ \frac{\partial}{\partial x_3}\left[\sigma_{13}\left(1 + \frac{\partial u_1}{\partial x_1}\right) + \sigma_{23}\frac{\partial u_1}{\partial x_2} + \sigma_{33}\frac{\partial u_1}{\partial x_3}\right] + X_1 = \varrho_o\frac{\partial^2 u_1}{\partial t^2} \end{aligned} \tag{1.12}$$

$$\begin{aligned} &\frac{\partial}{\partial x_1}\left[\sigma_{11}\frac{\partial u_2}{\partial x_1} + \sigma_{12}\left(1 + \frac{\partial u_2}{\partial x_2}\right) + \sigma_{13}\frac{\partial u_2}{\partial x_3}\right] \\ &+ \frac{\partial}{\partial x_2}\left[\sigma_{12}\frac{\partial u_2}{\partial x_1} + \sigma_{22}\left(1 + \frac{\partial u_2}{\partial x_2}\right) + \sigma_{23}\frac{\partial u_2}{\partial x_3}\right] \\ &+ \frac{\partial}{\partial x_3}\left[\sigma_{13}\frac{\partial u_2}{\partial x_1} + \sigma_{23}\left(1 + \frac{\partial u_2}{\partial x_2}\right) + \sigma_{33}\frac{\partial u_2}{\partial x_3}\right] + X_2 = \varrho_o\frac{\partial^2 u_2}{\partial t^2} \end{aligned} \tag{1.13}$$

$$\frac{\partial}{\partial x_1}\left[\sigma_{11}\frac{\partial u_3}{\partial x_1}+\sigma_{12}\frac{\partial u_3}{\partial x_2}+\sigma_{13}\left(1+\frac{\partial u_3}{\partial x_3}\right)\right]$$

$$+\frac{\partial}{\partial x_2}\left[\sigma_{12}\frac{\partial u_3}{\partial x_1}+\sigma_{22}\frac{\partial u_3}{\partial x_2}+\sigma_{23}\left(1+\frac{\partial u_3}{\partial x_3}\right)\right] \tag{1.14}$$

$$+\frac{\partial}{\partial x_3}\left[\sigma_{13}\frac{\partial u_3}{\partial x_1}+\sigma_{23}\frac{\partial u_3}{\partial x_2}+\sigma_{33}\left(1+\frac{\partial u_3}{\partial x_3}\right)\right]+X_3=\varrho_o\frac{\partial^2 u_3}{\partial t^2}$$

where t denotes time, ϱ_o is the density, and X_i are body forces. For linear small deformation theory, Equations (1.12–1.14) become

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}+X_1=\varrho_o\frac{\partial^2 u_1}{\partial t^2}$$

$$\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3}+X_2=\varrho_o\frac{\partial^2 u_2}{\partial t^2} \tag{1.15}$$

$$\frac{\partial\sigma_{13}}{\partial x_1}+\frac{\partial\sigma_{23}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}+X_3=\varrho_o\frac{\partial^2 u_3}{\partial t^2}$$

Given a strain field the question arises as to how Equations (1.10) can be integrated to determine the displacements. Since there are six strain equations in three unknown displacements, solutions will not be single-valued or continuous unless certain relations are satisfied. The following compatibility equations from linear theory of elasticity are well known.

$$\frac{\partial^2\epsilon_{12}}{\partial x_1\partial x_2}=\frac{\partial^2\epsilon_{11}}{\partial x_2^2}+\frac{\partial^2\epsilon_{22}}{\partial x_1^2} \tag{1.16}$$

$$\frac{\partial^2\epsilon_{23}}{\partial x_2\partial x_3}=\frac{\partial^2\epsilon_{22}}{\partial x_3^2}+\frac{\partial^2\epsilon_{33}}{\partial x_2^2} \tag{1.17}$$

$$\frac{\partial^2\epsilon_{13}}{\partial x_1\partial x_3}=\frac{\partial^2\epsilon_{11}}{\partial x_3^2}+\frac{\partial^2\epsilon_{33}}{\partial x_1^2} \tag{1.18}$$

$$2\,\frac{\partial^2\epsilon_{11}}{\partial x_2\partial x_3}=\frac{\partial}{\partial x_1}\left(-\frac{\partial\epsilon_{23}}{\partial x_1}+\frac{\partial\epsilon_{13}}{\partial x_2}+\frac{\partial\epsilon_{12}}{\partial x_3}\right) \tag{1.19}$$

$$2\frac{\partial^2\epsilon_{22}}{\partial x_1\partial x_3}=\frac{\partial}{\partial x_2}\left(\frac{\partial\epsilon_{23}}{\partial x_1}-\frac{\partial\epsilon_{13}}{\partial x_2}+\frac{\partial\epsilon_{12}}{\partial x_3}\right)\tag{1.20}$$

$$2\frac{\partial^2\epsilon_{33}}{\partial x_1\partial x_2}=\frac{\partial}{\partial x_3}\left(\frac{\partial\epsilon_{23}}{\partial x_1}+\frac{\partial\epsilon_{13}}{\partial x_2}-\frac{\partial\epsilon_{12}}{\partial x_3}\right)\tag{1.21}$$

1.4 GENERALIZED HOOKE'S LAW

Consider the following contracted stresses and strains:

$$\sigma_1=\sigma_{11}\ \sigma_2=\sigma_{22}\ \sigma_3=\sigma_{33}\ \sigma_4=\sigma_{23}\ \sigma_5=\sigma_{13}\ \sigma_6=\sigma_{12}\tag{1.22}$$

$$\epsilon_1=\epsilon_{11}\ \epsilon_2=\epsilon_{22}\ \epsilon_3=\epsilon_{33}\ \epsilon_4=\epsilon_{23}\ \epsilon_5=\epsilon_{13}\ \epsilon_6=\epsilon_{12}\tag{1.23}$$

Using Equations (1.22) and (1.23), the generalized Hooke's law can be written in the following matrix form:

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&c_{14}&c_{15}&c_{16}\\c_{12}&c_{22}&c_{23}&c_{24}&c_{25}&c_{26}\\c_{13}&c_{23}&c_{33}&c_{34}&c_{35}&c_{36}\\c_{14}&c_{24}&c_{34}&c_{44}&c_{45}&c_{46}\\c_{15}&c_{25}&c_{35}&c_{45}&c_{55}&c_{56}\\c_{16}&c_{26}&c_{36}&c_{46}&c_{56}&c_{66}\end{bmatrix}\begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\\\epsilon_4\\\epsilon_5\\\epsilon_6\end{bmatrix}\tag{1.24}$$

where c_{ij} is the stiffness matrix. Equation (1.24) can be written in the inverted form

$$\epsilon_i=\sum_{j=1}^{6}S_{ij}\sigma_j\tag{1.25}$$

where s_{ij} is the compliance matrix. Obviously the compliance matrix is the inverse of the stiffness matrix.

For the general case there are 21 independent elastic constants. If, however, there are any planes of elastic symmetry this number is reduced. Assume that the x_3-axis is perpendicular to a plane of elastic symmetry. Then

$$c_{14}=c_{15}=c_{24}=c_{25}=c_{34}=c_{35}=c_{46}=c_{56}=0$$