Applied Mathematics

A Contemporary Approach

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APPLIED MATHEMATICS

A CONTEMPORARY APPROACH

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To
Tess
and our little Aquarian
David R.

PREFACE

This is a textbook designed to introduce applied mathematics to seniors and beginning graduate students majoring in mathematics, engineering, and the physical sciences. Prerequisites include a good command of calculus, elementary linear algebra (matrices), and postcalculus differential equations, as well as familiarity with a few of the concepts presented in an elementary physics course. It differs from other books in that an attempt is made to present some of the more current topics in applied mathematics in an elementary format. These include singular perturbation, nonlinear waves, similarity methods, and bifurcation phenomena. An effort was made to write in a style that makes the topics accessible to students with widely varying backgrounds and interests but who have a common need to know the rudiments of these subjects. Some of the more standard topics are covered as well, such as the calculus of variations, dimensional analysis, Fourier methods, integral equations, and the numerical solution of partial differential equations. Because many of the chapters are independent, there is considerable flexibility for the instructor in using this book as a text for either a one-year or one-semester course.

The text was spawned from a two-semester three-credit-hour sequence in applied mathematics at the University of Nebraska. The course tries to strike a balance between the mathematical aspects of a subject and its origins in empirics, and to teach a way of thinking about problems that emphasizes the interplay between mathematics and science. The insight gained can be of benefit later when students try to apply the methods or explore new concepts on their own.

I took the task of writing this book because of a belief that there is a need for a survey of applied mathematics at this level, particularly one that incorporates the current topics just mentioned. Just as in algebra, for example, where an introductory course includes the study of groups, rings, etc., a beginning course in applied mathematics should introduce some of the basic areas of study. Such a course can be of tremendous benefit to students in a terminal Masters Degree program or to seniors who may be considering graduate study. At the University of Nebraska this course is followed by specialized courses covering each of the topics at a more advanced level. The guiding principle in the exposition in the text was to take a classical approach that would be accessible to students with a wide range of interests and previous mathematical training.

Applied mathematics is a broad field of study and every applied mathematician will view its role and content differently. This presentation is one practitioner's view. The topics covered herein are ones classically associated with mathematics applied to physical sciences and do not include important topics like control theory, optimization, combinatorics, or such. In this sense the scope of the text is limited.

Scaling and dimensional analysis are topics usually ignored in treatments of applied mathematics. These subjects are often left to the folklore in which the student is supposed to pick up as needed. Yet a good understanding of scaling is essential for perturbation calculations, and dimensional analysis is required in the mathematical modeling of physical phenomena. In Chapter 1 a short introduction to the basic concepts is presented, including an elementary proof of the Buckingham Pi theorem.

In Chapter 2 the underlying ideas of regular and singular perturbation theory are offered in the context of ordinary differential equations. Chapter 3 introduces the classical techniques of the calculus of variations in a functional analytic setting.

In Chapter 4 begins a study of the fundamental equations of applied mathematics, partial differential equations and integral equations. Classical techniques involving Fourier series and transform methods are illustrated on the diffusion equation. In Chapter 5 the study of evolution equations continues with emphasis on wave propagation. An approach is taken in which model equations are developed to illustrate basic physical processes such as convection, diffusion, dispersion, distortion, and so on. The differences between linear and nonlinear phenomena become apparent.

Fluid dynamics in one and three dimensions is discussed in Chapter 5 as well. Wave propagation in continuous media is a nonsterile example of wave phenomena and provides the correct context for developing the wave equation and for understanding some of the origins of singular perturbation and bifurcation theory. For these reasons and others there seems to be an increasing demand among mathematicians to learn fluid dynamics. What was once a standard part of the applied mathematics curriculum appears to be undergoing a renaissance; it offers a rich context for illustrating mathematical modeling and analysis.

Chapter 6 contains an introduction to stability and bifurcation. The latter has become a popular area of research in applied analysis. One dimensional problems are presented in a fair amount of detail within the context of singularity theory. Nonlinear systems and phase plane phenomena are discussed as well as hydrodynamic stability.

Chapter 7 on similarity methods shows how one can take advantage of the symmetry or invariance properties of a problem to obtain a significant simplification or a solution. This is carried out in problems in the calculus of variations where the famous E. Noether theorem on conservation laws is presented and in partial differential equations where it is shown how symmetries permit a reduction to an ordinary differential equation. Similarity methods are not often taught in elementary partial differential equation courses as a technique, but their wide applicability is firmly established and there is indication that the method should be introduced in elementary contexts.

The final chapter on finite difference methods for partial differential equations contains some of the basic numerical algorithms for solving the diffusion equation, the wave equation, Laplace's equation, and hyperbolic systems. Concepts of convergence and stability are introduced and programs in BASIC are presented for some of the algorithms. The idea is to indicate the logical structure and the ease with which the calculations can be performed, even on a microcomputer. The student is encouraged to make the programs more efficient and applicable to more general problems.

The exercises form an essential element of the text and the course. They range from routine problems designed to build confidence and test basic technique to more challenging problems that build technique.

The bibliography has been selected to suit the needs of an introductory text. At the end of each chapter are listed a few standard, and in most cases classical, references to the material. In these the reader can find parallel discussions or extended coverage of the topics.

Equations are numbered consecutively starting anew at the beginning of each section. Theorems, definitions, and examples are numbered within each section as well. For example, Theorem 3.2 refers to the second theorem in Section 3 of the current chapter.

This work was influenced either directly or indirectly by many individuals. The University of Nebraska supported my efforts during the summers of 1984 through 1987 by relieving me of some of my duties as Chairman of the Department. Special thanks go to Mr. Kevin TeBeest, who carefully proofread the manuscript and made many suggestions and corrections, and to my colleague Dr. Steven Dunbar who was frequently a sounding board. The comments of Professors Ivar Stakgold at Delaware, Bernard J. Matkowsky at Northwestern, and Gunter H. Meyer at Georgia Tech also led to many improvements. Maria Taylor, the editor at Wiley-Interscience, shared my enthusiasm for this project and skillfully managed its development and production. Several versions of the manuscript were typed with skill and dedication by Rhonda Bordeaux at the University. Three colleagues with whom I have worked during the last ten years deserve a strong acknowledgment for their influence; these are Dr. John Bdzil at Los Alamos, Dr. Robert Krueger at Iowa State Ames Research Laboratory, and Dr. Kane Yee at Livermore.

Finally, it is rare that one gets to thank in such a permanent, public form those who have made the quality of one's life so high. On this occasion I thank my mother Dorothy for her ideals, devotion, and for having a vision for herself and for me. To my wife Tess goes my deepest gratitude for her support; without her steady encouragement this book may never have been completed.

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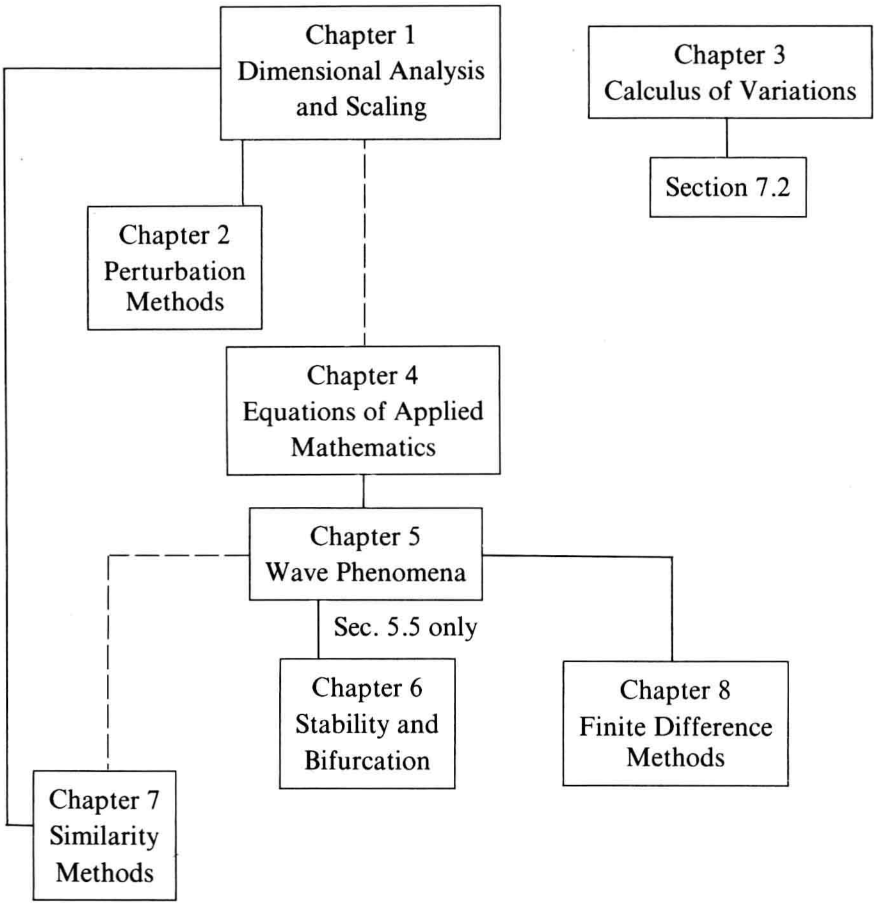
Lincoln, Nebraska
September 1987.

Suggestions for Use of the Text. The following diagram shows the dependence of the Chapters. The dashed lines indicate only a weak dependence and the earlier material can be referred to only as needed.



For a one-year course: First Semester (Chapters 1, 2, 3)
Second Semester (Chapters 4, 5, 6)

Chapters 1, 4 (except integral equations), 5, and 8 have been used for a one-semester introduction to partial differential equations, which includes numerical methods; Chapter 7 can be substituted for Chapter 8. If the section on hydrodynamic stability is omitted, Chapter 6 is independent of the remaining parts of the book.

CONTENTS

1

DIMENSIONAL ANALYSIS AND SCALING

The techniques of dimensional analysis and scaling are basic in the theory and practice of mathematical modeling. In every physical setting a good grasp of the possible relationships and comparative magnitudes among the various dimensioned parameters nearly always leads to a better understanding of the problem and sometimes points the way toward approximations and solutions. In this chapter we briefly introduce some of the basic concepts from these two topics. Along with several examples, a statement and proof of the fundamental result in dimensional analysis, the Buckingham Pi theorem, is presented, and scaling is discussed in the context of reducing problems to dimensionless form. The notion of scaling also points the way toward a proper treatment of perturbation methods, especially boundary layer phenomena in singular perturbation theory.

1.1 DIMENSIONAL ANALYSIS

The Program of Applied Mathematics

There are many phases to the solution of a problem that arises in a physical context and that requires careful mathematical analysis. One way to view the attack on such a physical problem is as follows. When a problem arises in empirics, the first stage is to formulate a mathematical model of the situation. This step includes defining the relevant quantities and formulating a set of governing equations that describe the process involved in detail. We can regard the mathematical problem represented by these model equations as a pure mathematics problem. Its solution by some mathematical technique is the second stage of analysis. Once the solution is obtained, the third stage is to go back and verify that the analytical results are consistent with the experimental observations in the original physical problem. If indeed there is consistency, and if the solution is predictive of other similar physical results, then we can conclude that the devised mathematical equations do in fact represent a realistic model.

It would be a limited view, in fact an incorrect one, to believe that applied mathematics consists only of developing techniques and algorithms to solve problems that arise in a physical or applied context. Applied mathematics deals with all these stages, not merely the formal solution as represented in stage two. It is true that an important aspect of applied mathematics consists of studying, investigating, and developing procedures that are useful in solving such mathematical problems: these include analytic and approximation techniques, numerical analysis, and methods for solving differential and integral equations. It is more the case, however, that applied mathematics deals with every phase of the problem. Formulating the model and understanding its origin in empirics are crucial steps. Because there is a constant interplay between the various stages, the scientist, engineer, or mathematician must understand each phase. For example, in the second stage the solution to a problem sometimes involves making approximations that lead to a simplification. The approximations often come from a careful examination of the physical reality, which in turn suggests what terms may be neglected, what quantities (if any) are small, and so on. Finally, inaccurate predictions may suggest refinements in the model that lead to even better descriptions of reality. All of this is the practice of applied mathematics; heuristic reasoning, manipulative skills, and physical insight are all essential elements.

In this chapter our aim is to focus upon the first stage, or modeling process. We carry this out by formulating models for various physical systems while emphasizing the interdependence of mathematics and the physical world. Through study of the modeling process we gain insight into the equations themselves. For example, it is possible to study the diffusion equation, a partial differential equation of the form

$$u_t(x,t) - u_{xx}(x,t) = 0$$

without regard to its origin. We can investigate it mathematically by asking questions regarding the existence of solutions, methods of solution, and so on. Such an endeavor, however, is sterile from the point of view of applied mathematics; the origins and analysis are equally important. Indeed, physical insight forces us toward the right questions and at times leads us to the theorems and their proofs.

In addition to presenting some concrete examples of modeling, we also discuss two techniques that are useful in developing and interpreting the model equations. One technique is dimensional analysis and the other is