FOUNDATIONS OF DIFFERENTIAL GEOMETRY

VOLUME 11

FOUNDATIONS OF DIFFERENTIAL GEOMETRY

VOLUME II

SHOSHICHI KOBAYASHI
University of California, Berkeley, California

and

KATSUMI NOMIZU
Brown University, Providence, Rhode Island

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PREFACE

This is a continuation of Volume I of the *Foundations of Differential Geometry*. The chapter numbers are continued from Volume I and the same notations are preserved as much as possible. The main text, Chapters VII–XII, deals with the topics that have been promised in the Preface of Volume I. The Notes include material supplementary to Volume I as well. The Bibliography duplicates, for the sake of convenience of readers, all the references in the Bibliography of Volume I in the same numbering and continues to references for Volume II.

The content of each chapter is now briefly described.

Chapter VII gives the fundamental results and some classical theorems concerning geometry of an n-dimensional submanifold M immersed in an $(n + p)$-dimensional Riemannian manifold N, in particular, $\mathbf{R}^{n+p}$. In §1, the natural connections in the orthogonal bundle and the normal bundle over M are derived from the Riemannian connection in the orthogonal bundle over N. In §2, where $N = \mathbf{R}^{n+p}$, we show that these connections are induced from the canonical connections in the Stiefel manifolds $V(n, p)$ and $V(p, n)$, both over the Grassmann manifold $G(n, p)$, respectively, by means of the bundle maps associated to the generalized Gauss map of M into $G(n, p)$. In §§3 and 4, we use the formalism of covariant differentiation $\nabla_X Y$ to study the relationship between the invariants of M and N and obtain the classical formulas of Weingarten, Gauss and Codazzi. We prove a result of Chern-Kuiper which generalizes the theorem of Tompkins. §§5, 6, and 7 are concerned with the classical notions and theorems on hypersurfaces in a Euclidean space, including a result of Thomas-Cartan-Fialkow on Einstein hypersurfaces and results on the type number and the so-called fundamental theorem. In the last §8, we discuss auto-parallel submanifolds and totally geodesic submanifolds of a manifold with an affine connection and prove, in particular, that these two notions coincide in the case where the connection in the ambient space has no torsion. The content of Chapter VII is supplemented by Notes 14, 15, 16, 17, 18, 21, and 27.

Chapter VIII is devoted to the study of variational problems on geodesics. In §1, we define Jacobi fields and conjugate points for a manifold with an affine connection and discuss their geometric meaning. In §2 and 3, we make a further study of these notions in a Riemannian manifold and prove the classical result on the distance between consecutive conjugate points on a geodesic when the sectional curvature (or, more generally, each of the eigenvalues of the Ricci tensor) is greater than a certain positive number everywhere. In §4, we prove Rauch's comparison theorem. In §5, we study the first and second variations of the length integral, considered as a function on the space of all piecewise differentiable curves, and obtain, among others, a proof of Myers's Theorem. The Index Theorem of Morse is proved in §6. In §7, we prove basic properties of cut loci. Although the results of §7 are not used elsewhere in this book, they are basic in the study of manifolds with positive curvature. In §8, we prove a theorem of Hadamard and Cartan which says that for a complete Riemannian space with non-positive curvature the exponential map is a covering map. Applications are made to a homogeneous Riemannian manifold with non-positive sectional curvature and negative definite Ricci tensor. In §9, we prove a theorem to the effect that on a simply connected complete Riemannian manifold with non-positive sectional curvature every compact group of isometries has a fixed point. Applications are given to the case of a homogeneous Riemannian space. Results of §§8 and 9 are used in §11 of Chapter XI. Note 22 supplements the content of this chapter.

In Chapter IX, we provide differential geometric foundations for almost complex manifolds and Hermitian metrics, in particular, complex manifolds and Kaehler metrics. The results in this chapter are essentially of local character. After purely algebraic preliminaries in §1, we discuss in §2 the notion of an almost complex structure, its torsion and integrability as well as complex tangent spaces, operators ∂ and $\bar{\partial}$ for complex differential forms on an almost complex manifold. Many examples are given, including complex Lie groups, complex parallelizable spaces, complex Grassmann manifolds, Hopf manifolds and their generalizations, and a result of Kirchhoff on almost complex structures on spheres. In §3, we discuss connections in the bundle of complex linear frames of an almost complex manifold and relate their torsions with the torsion of the almost complex structure. In §4, Hermitian metrics and the bundle of unitary frames are discussed. The most interesting case is that of a Kaehler metric, whose basic properties are proved here. In §5, we build a bridge between intrinsic notations and complex tensor notations for Kaehlerian geometry. In §6, many examples of Kaehler manifolds are discussed, including the Fubini-Study metric in the complex projective space and the Bergman metric in the open unit ball in $\mathbf{C}^n$. In §7, we give basic local properties of holomorphic sectional curvature and prove that a simply connected and complete Kaehler manifold of constant holomorphic sectional curvature c is a complex projective space, a complex Euclidean space or an open unit ball in a complex Euclidean space according as $c > 0$, $= 0$ or < 0. In §8, we discuss the de Rham decomposition of a Kaehler manifold and the notion of non-degeneracy. §9 is concerned with holomorphic sectional curvature and the Ricci tensor of a complex submanifold of a Kaehler manifold. In the last §10, we study the existence and properties of Hermitian connection in a Hermitian vector bundle following Chern, Nakano, and Singer. This chapter is supplemented by §6 of Chapter X, §10 of Chapter XI (where examples are discussed from the viewpoint of symmetric spaces), and Notes 13, 18, 23, 24, and 26.

In Chapter X, we discuss the existence and properties of invariant affine connections and invariant almost complex structures on homogeneous spaces (especially, reductive homogeneous spaces). In §1, the results of Wang in §11 of Chapter II are specialized to the situation where P is a K-invariant G-structure on a homogeneous space $M = K/H$, and K-invariant connections in P are studied. In §2, we specialize further to the case where K/H is reductive and obtain the canonical connection and the natural torsion-free connection of Nomizu. In §3, we study homogeneous spaces with invariant (possibly indefinite) Riemannian metrics. As an example we provide a differential geometric proof of Weyl's theorem that a Lie group G is compact if the Killing-Cartan form of its Lie algebra is negative-definite. In §§4 and 5, results of Nomizu and Kostant on the holonomy group and reducibility of an invariant affine connection are proved. In §6, following Koszul we give algebraic formulations for an invariant almost complex structure on a homogeneous space and for its integrability. This chapter serves as a basis for Chapter XI and is supplemented by Notes 24 and 25.

In Chapter XI, we present the basic results in the theory of affine, Riemannian, and Hermitian symmetric spaces. We lay emphasis on the affine case a little more than the standard treatment of the subject. In §1, we consider affine symmetric spaces, thus giving a geometric motivation to the group-theoretic notion of symmetric space which is introduced in §2. In §3, we reverse the process in §1; thus we begin with a symmetric space G/H and introduce the canonical affine connection on G/H, making G/H an affine symmetric space. The curvature of the canonical connection is given an algebraic expression. In §4, we study totally geodesic submanifolds of a symmetric space G/H (with canonical connection) from both geometric and algebraic viewpoints. The symmetric Lie algebra introduced in §3 is to a symmetric space what the Lie algebra is to a Lie group. In §5, two results on Lie algebras, namely, Levi's theorem and the decomposition of a semi-simple Lie algebra into a direct sum of simple ideals, are extended to the case of symmetric Lie algebras. The global versions of these results are also given. In §6, we consider Riemannian symmetric spaces and the corresponding symmetric spaces. The symmetric Lie algebra corresponding to a Riemannian symmetric space is called an orthogonal symmetric Lie algebra. In §7, where orthogonal symmetric Lie algebras are studied, the decomposition theorems proved in §5 are made more precise. In §8, the duality between the orthogonal symmetric Lie algebras of compact type and those of non-compact type are studied together with geometric interpretations. In §9, we discuss geometric properties and an algebraic characterization of Hermitian symmetric spaces. Many examples of classical spaces are studied in §10 from viewpoints of symmetric spaces, including real space forms originally defined in Chapter V and complex space forms discussed in Chapter IX. In the last §11, we show, assuming Weyl's existence theorem of a compact real form of a complex simple Lie algebra, that the classification of irreducible orthogonal symmetric Lie algebras is equivalent to the classification of real simple Lie algebras.

In Chapter XII, we present differential geometric aspects of characteristic classes. If G is the structure group of a principal bundle P over M, then using the curvature of a connection in P we can associate to each Ad(G)-invariant homogeneous polynomial f of degree k on the Lie algebra of G a closed $2k$-form on the base space M in a natural manner. The cohomology class represented by this closed $2k$-form is independent of the choice of connection and is called the characteristic class determined by f. In §1, following Chern we prove this basic result of Weil. In §2, we study the algebra of Ad(G)-invariant polynomials on the Lie algebra of G and determine the algebra explicitly when G is a classical group. In §3, adopting the axiomatic definition of Chern classes by Hirzebruch, we express the Chern classes of a complex vector bundle in terms of the curvature form of a connection in the bundle. The formula for the Chern character in terms of the curvature form is also given. In §4, using Hirzebruch's definition of the Pontrjagin classes of a real vector bundle, we derive differential geometric formulas for the Pontrjagin classes. In §5, we characterize the real Euler class of a vector bundle in a simple axiomatic manner and derive the general Gauss-Bonnet formula. This chapter, particularly §5, is supplemented by Notes 20 and 21.

We wish to note specifically that we do not go into the following subjects: the theory of (2-dimensional) minimal surfaces; the theory of global convex surfaces developed by A. D. Aleksandrov and his school; Finsler geometry and its generalizations; the general theory of conformal and projective connections; a deeper study of differential systems. On the subjects of complex manifolds, homogeneous spaces (especially symmetric homogeneous spaces), vector bundles, G-structures and so on, our treatment is limited to the foundational material in differential geometric aspects that does not require deeper knowledge from algebra, analysis or topology. Neither do we treat the harmonic theory nor a generalized Morse theory, although these theories have many important applications to Riemannian geometry. The Bibliography of Volume II contains some basic references in these areas. In particular, for the global theory of compact Kaehler manifolds which requires the theory of harmonic integrals, the reader is advised to read Weil's book: *Introduction à l'Étude des Variétés Kählériennes*.

During the preparations of this volume, we have been most encouraged by the reactions to Volume I of many readers who wanted to find self-contained and complete proofs of the standard results in the field. We sincerely hope that the present volume will continue to meet the needs of these readers.

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SHOSHICHI KOBAYASHI
KATSUMI NOMIZU

September, 1968

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CHAPTER VII

Submanifolds

1. *Frame bundles of a submanifold*

Let $e_1, \ldots, e_{n+p}$ be the natural basis for $\mathbf{R}^{n+p}$. We shall denote by $\mathbf{R}^n$ and $\mathbf{R}^p$ the subspaces of $\mathbf{R}^{n+p}$ spanned by $e_1, \ldots, e_n$ and $e_{n+1}, \ldots, e_{n+p}$, respectively. Similarly, we identify $O(n)$ (resp. $O(p)$) with the subgroup of $O(n+p)$ consisting of all elements which induce the identity transformation on the subspace $\mathbf{R}^p$ (resp. $\mathbf{R}^n$) of $\mathbf{R}^{n+p}$. In other words,

$$O(n) \approx \begin{pmatrix} O(n) & 0 \\ 0 & I_p \end{pmatrix} \quad \text{and} \quad O(p) \approx \begin{pmatrix} I_n & 0 \\ 0 & O(p) \end{pmatrix},$$

where I_n and I_p denote the identity matrices of order n and p, respectively. Let $\mathfrak{o}(n+p)$, $\mathfrak{o}(n)$ and $\mathfrak{o}(p)$ be the Lie algebras of $O(n+p)$, $O(n)$ and $O(p)$, respectively, and let $\mathfrak{g}(n, p)$ be the orthogonal complement to $\mathfrak{o}(n) + \mathfrak{o}(p)$ in $\mathfrak{o}(n+p)$ with respect to the Killing-Cartan form of $\mathfrak{o}(n+p)$ (cf. Volume I, p. 155 and also Appendix 9). Then $\mathfrak{g}(n, p)$ consists of matrices of the form

$$\begin{pmatrix} 0 & A \\ -{}^tA & 0 \end{pmatrix},$$

where A is a matrix with n rows and p columns and ${}^tA$ denotes the transpose of A.

Let N be a Riemannian manifold of dimension $n + p$ and let f be an immersion of an n-dimensional differentiable manifold M into N. We denote by g the metric of N as well as the metric induced on M (cf. Example 1.2 of Chapter IV). For any point x of M we shall denote $f(x) \in N$ by the same letter x if there is no danger of confusion. Thus the tangent space $T_x(M)$ is a subspace of the tangent space $T_x(N)$. Let $T_x(M)^\perp$ be the orthogonal complement of $T_x(M)$ in $T_x(N)$; it is called the *normal space* to M at x.

Let $O(M)$ and $O(N)$ be the bundles of orthonormal frames over M and N, respectively. Then $O(N) \mid M = \{v \in O(N); \pi(v) \in M\}$, where $\pi: O(N) \to N$ is the projection, is a principal fibre bundle over M with structure group $O(n + p)$. A frame $v \in O(N) \mid M$ at $x \in M$ is said to be *adapted* if v is of the form $(Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{n+p})$ with $Y_1, \ldots, Y_n$ tangent to M (and hence $Y_{n+1}, \ldots, Y_{n+p}$ normal to M). Thus, considered as a linear isomorphism $\mathbf{R}^{n+p} \to T_x(N)$, v is adapted if and only if v maps the subspace $\mathbf{R}^n$ onto $T_x(M)$ (and hence the subspace $\mathbf{R}^p$ onto $T_x(M)^\perp$). It is easy to verify that the set of adapted frames forms a principal fibre bundle over M with group $O(n) \times O(p)$; it is a subbundle of $O(N) \mid M$ in a natural manner. We shall denote the bundle of adapted frames by $O(N, M)$. We define a homomorphism $h': O(N, M) \to O(M)$ corresponding to the natural homomorphism $O(n) \times O(p) \to O(n)$ as follows:

$$h'(v) = (Y_1, \ldots, Y_n) \qquad \text{for } v = (Y_1, \ldots, Y_{n+p}) \in O(N, M).$$

If we consider v as a linear transformation $\mathbf{R}^{n+p} \to T_x(N)$, then $h'(v)$ is the restriction of v to the subspace $\mathbf{R}^n$. Hence, $O(M)$ is naturally isomorphic to $O(N, M)/O(p)$. Similarly, denoting by $h''(v)$ the restriction of $v \in O(N, M)$ to the subspace $\mathbf{R}^p$ of $\mathbf{R}^{n+p}$, we obtain a homomorphism $h'': O(N, M) \to O(N, M)/O(n)$ corresponding to the natural homomorphism $O(n) \times O(p) \to O(p)$. By a *normal frame* at $x \in M$, we mean an orthonormal basis $(Z_1, \ldots, Z_p)$ for the normal space $T_x(M)^\perp$. If $(Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{n+p})$ is an adapted frame at x, then $(Y_{n+1}, \ldots, Y_{n+p})$ is a normal frame at x. Since every normal frame is thus obtained and since two adapted frames give rise to the same normal frame if and only if they are congruent modulo $O(n)$, the bundle $O(N, M)/O(n)$ can be considered as the bundle of normal frames over M. Then $h'': O(N, M) \to O(N, M)/O(n)$ maps an adapted frame $v = (Y_1, \ldots, Y_{n+p})$ upon the normal frame $(Y_{n+1}, \ldots, Y_{n+p})$. We denote by $T(M)^\perp$ the set $\bigcup_{x \in M} T_x(M)^\perp$. It is then a vector bundle over M associated to the bundle of normal frames $O(N, M)/O(n)$ by letting the structure group $O(p)$ act naturally on the standard fibre $\mathbf{R}^p$ (cf. §1 of Chapter III). We shall call this vector bundle the *normal bundle* of M (for the given immersion f into N). The following diagrams illustrate these bundles:

$$\begin{array}{ccccc}
O(M) = O(N, M)/O(p) & \overset{h'}{\longleftarrow} & O(N, M) & \overset{h''}{\longrightarrow} & O(N, M)/O(n) \\
{\scriptstyle O(n)}\big\downarrow{\scriptstyle \pi'} & & {\scriptstyle O(n) \times O(p)}\big\downarrow{\scriptstyle \pi} & & {\scriptstyle O(p)}\big\downarrow{\scriptstyle \pi''} \\
M & \leftrightarrow & M & \leftrightarrow & M
\end{array}$$

$$\begin{array}{ccccc}
O(N, M) & \overset{i}{\longrightarrow} & O(N) \mid M & \overset{j}{\longrightarrow} & O(N) \\
{\scriptstyle O(n) \times O(p)}\big\downarrow{\scriptstyle \pi} & & {\scriptstyle O(n+p)}\big\downarrow{\scriptstyle \pi} & & {\scriptstyle O(n+p)}\big\downarrow{\scriptstyle \pi} \\
M & \leftrightarrow & M & \longrightarrow & N,
\end{array}$$

where both i and j are injections.

Let θ and φ be the canonical forms of M and N, respectively (cf. §2 of Chapter III); θ is an $\mathbf{R}^n$-valued 1-form on $O(M)$ and φ is an $\mathbf{R}^{n+p}$-valued 1-form on $O(N)$. Then we have

PROPOSITION 1.1. *$h'^*(\theta)$ coincides with the restriction of φ to $O(N, M)$. In particular, the restriction of the $\mathbf{R}^{n+p}$-valued form φ to $O(N, M)$ is $\mathbf{R}^n$-valued.*

Proof. By definition of φ we have

$$\varphi(Y) = v^{-1}(\pi(Y)) \qquad \text{for } Y \in T_v(O(N, M)).$$

Since $\pi(Y) \in T_x(M)$, where $x = \pi(v)$, and since v^{-1} maps $T_x(M)$ onto $\mathbf{R}^n$, $\varphi(Y)$ is in $\mathbf{R}^n$. Since $h'(v) = v \mid \mathbf{R}^n$ and since $\pi' \circ h'(Y) = \pi(Y)$, we have

$$\begin{aligned}\varphi(Y) &= v^{-1}(\pi(Y)) = h'(v)^{-1}(\pi' \circ h'(Y)) = \theta(h'(Y)) \\ &= (h'^*(\theta))(Y).\end{aligned}$$

QED.

Let ψ be the Riemannian connection form on $O(N)$. Its restriction to $O(N) \mid M$, that is, $j^*\psi$, defines a connection in the bundle $O(N) \mid M$. But its restriction to $O(N, M)$, that is, $i^*j^*\psi$, is not, in general, a connection form on $O(N, M)$.

PROPOSITION 1.2. *Let ψ be the Riemannian connection form on $O(N)$ and let ω be the $\mathfrak{o}(n) + \mathfrak{o}(p)$-component of $i^*j^*\psi$ with respect to the decomposition $\mathfrak{o}(n + p) = \mathfrak{o}(n) + \mathfrak{o}(p) + \mathfrak{g}(n, p)$. Then ω defines a connection in the bundle $O(N, M)$.*

Proof. Since $\mathrm{ad}(O(n) \times O(p))$ maps $\mathfrak{g}(n, p)$ onto itself, we see from Proposition 6.4 of Chapter II that the form ω defines a connection in $O(N, M)$. QED.

PROPOSITION 1.3. *The homomorphism $h'\colon O(N, M) \to O(M) = O(N, M)/O(p)$ maps the connection in $O(N, M)$ defined by ω into the Riemannian connection of M. The Riemannian connection form ω' on $O(M)$ is determined by*

$$h'^*(\omega') = \omega_{\mathfrak{o}(n)},$$

where $\omega_{\mathfrak{o}(n)}$ denotes the $\mathfrak{o}(n)$-component of the $\mathfrak{o}(n) + \mathfrak{o}(p)$-valued form ω.

Proof. By Proposition 6.1 of Chapter II we know that h' maps the connection defined by ω into the connection in $O(M)$ defined by a form ω' such that $h'^*(\omega') = \omega_{\mathfrak{o}(n)}$. To show that ω' defines the Riemannian connection of M we have only to show that the torsion form of ω' is zero. Restricting the first structure equation of ψ to $O(N, M)$, we obtain

$$d(i^*j^*\varphi) = -(i^*j^*\psi) \wedge (i^*j^*\varphi).$$

Since $i^*j^*\varphi$ is equal to $h'^*(\theta)$ and is $\mathbf{R}^n$-valued by Proposition 1.1, comparing the $\mathbf{R}^n$-components of the both sides we obtain

$$d(h'^*(\theta)) = -h'^*(\omega') \wedge h'^*(\theta).$$

Since h' maps $O(N, M)$ *onto* $O(M)$, this implies $d\theta = -\omega' \wedge \theta$. QED.

Similarly, by Proposition 6.1 of Chapter II we see that there is a unique connection form ω'' on the bundle $O(N, M)/O(n)$ such that

$$h''^*(\omega'') = \omega_{\mathfrak{o}(p)},$$

where $\omega_{\mathfrak{o}(p)}$ denotes the $\mathfrak{o}(p)$-component of the $\mathfrak{o}(n) + \mathfrak{o}(p)$-valued form ω. Geometrically speaking, ω'' defines the parallel displacement of the normal space $T_x(M)^\perp$ onto the normal space $T_y(M)^\perp$ along any curve τ in M from x to y.

The bundles $O(N, M)$, $O(M) = O(N, M)/O(p)$, and $O(N, M)/O(n)$, and their connection form ω, ω', and ω'' are related as follows:

PROPOSITION 1.4. *The mapping $(h', h'')\colon O(N, M) \to O(M) \times (O(N, M)/O(n))$ induces a bundle isomorphism $O(N, M) \approx O(M) + (O(N, M)/O(n))$. The connection form ω coincides with $h'^*(\omega') + h''^*(\omega'')$.*

The proof is trivial (see p. 82 of Volume I).

Finally, we say a few words about the special case of a hypersurface. By a hypersurface in an $(n + 1)$-dimensional manifold N we mean a (generally connected) n-dimensional manifold M with an immersion f. For each $x \in M$, there is a coordinate neighborhood U of x in M and a differentiable field, say, ξ, of unit normal vectors defined on U. Such a ξ can be easily constructed by choosing a coordinate system $x^1, \ldots, x^n$ around x in U and a coordinate system $y^1, \ldots, y^{n+1}$ around x $(= f(x))$ in N; in fact, a unit normal vector field on U is determined uniquely up to sign. For a fixed choice of ξ on U, it is obvious that ξ is parallel along all closed curves in U (with respect to the connection in the normal bundle). Assume that N is orientable and is oriented. Then we can choose a differentiable field of unit normal vectors over M if and only if M is also orientable. Indeed, for a fixed orientation on M, there is a unique choice of the field of unit normal vectors ξ such that, for an oriented basis $\{X_1, \ldots, X_n\}$ of $T_x(M)$ at each $x \in M$, $\{\xi_x, X_1, \ldots, X_n\}$ is an oriented basis of $T_x(N)$. Conversely, if a field ξ of unit normal vectors exists globally on M, then a basis $\{X_1, \ldots, X_n\}$ of $T_x(M)$ such that $\{\xi_x, X_1, \ldots, X_n\}$ is an oriented basis of $T_x(N)$ determines an orientation of M. If we forget about the particular orientations of N and M, then again a differentiable field of unit vectors on M is unique up to sign. For a choice of ξ, it is obvious that ξ is parallel along all curves in M.

Without assuming that N and M are orientable, let us choose a unit normal vector ξ_0 at a point x_0 on M. The parallel displacement along all closed curves at x_0 on M will map ξ_0 either upon ξ_0 or upon $-\xi_0$. In other words, the holonomy group of the linear connection in the normal bundle is a subgroup of the group $\{1, -1\}$. (This is also clear from the fact that the bundle $O(N, M)/O(n)$ of normal frames over M has structure group $O(1) = \{1, -1\}$.) If the holonomy group is trivial (and this is the case if M is simply connected), then ξ_0 is invariant by parallel displacement along all closed curves at x_0. In this case, we may define a differentiable unit normal vector field ξ on M by translating ξ_0 parallelly to each point x of M, the result being independent of the choice of a curve from x_0 to x in M. We may thus conclude that *if M is a simply connected, connected hypersurface immersed in a Riemannian manifold N, then M admits a differentiable field of unit normal vectors defined on M.*