Lecture Notes in Mathematics

Constantin Năstăsescu
Freddy Van Oystaeyen

Methods of Graded Rings

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Authors

Constantin Năstăsescu

Faculty of Mathematics
University of Bucharest
Str. Academiei 14
70609 Bucharest, Romania

e-mail: nastases@racai.ro

Freddy Van Oystaeyen

Department of Mathematics and Computer Science
University of Antwerp
Middelheimlaan 1
2020 Antwerp, Belgium

e-mail: fred.vanoystaeyen@ua.ac.be

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Dedication

We dedicate this book our dear wives : Petruta and Danielle ! They allowed us to spend some "quality time" on this book, and even supported our efforts in this.

Introduction

The aim of this book is to provide a survey of algebraic methods useful in the investigation of the structure of graded rings and their modules. The concept of gradation is strongly linked to the notion of degree; for example, graded rings may be viewed as rings generated by certain elements having a "degree" and this degree is a natural number or an integer.

The mother of all graded rings, a polynomial ring in one variable over a field, is graded by $\mathbb{Z}$. But when considering polynomials in more than one, say n variables one has the choice of using the total degree in $\mathbb{Z}$, or a new multi-degree in $\mathbb{Z}^n$ extending the idea of exponent of a variable. In modern language this multi-degree is an element of the ordered group $\mathbb{Z}^n$ with the lexicographical order; this concept plays an essential role in the proof of the fundamental theorem for symmetric polynomials. The concept of degree then extends to other classes of rings, commutative but even noncommutative, and in fact it also underlies the modern treatment of projective varieties in Algebraic Geometry. Another important generalization consists in allowing the degree to take values in abstract groups, for example finite groups, usually not embeddeable into groups of numbers. Then group theory and representation theory of groups enter the picture.

You do not have to know a group to see it act. Perhaps this statement summarizes adequately one of the basic principles of representation theory of groups. Indeed, even before the definition of abstract group had been given, arguments in classical geometry often referred to actions of specific groups, usually symmetry groups of a geometric configuration. The best testament of the faith mathematicians had in the power of group theory within geometry is worded in Klein's "Erlangen Programm". But group actions continued to be successful even on the more abstract side, e.g. Galois Theory featured group actions on abstract algebraic structures in terms of automorphism just to solve down to earth problems related to polynomial equations. The success of the abstract approach induced new ideas concerning continuous groups or Lie groups, and since then the devil of abstract algebra, as in A. Weil's famous quote, is indeed invading the soul of many disciplines in mathematics, most often using group theory to open the door ! The formal treatment of group theory and group representations allowed other algebraic methods to enter the picture, e.g. Ring Theory. Indeed the representation theory of finite groups amounts to the study of modules over the group algebra KG of the group G over the field K. The KG-module structure allows to encode precisely most properties of a G-action on a K-vector space in ring theoretical data. The fact that KG is graded by the group G may be omnipresent in representation theory, the basic theory can be developed completely without stressing that point ... up to a point. That happens when a subgroup H of G is being considered and representations of H and G have to be compared. Hence, in Clifford Theory, certain aspects of graded algebra are more dominantly present. In particular, when H is not normal in G and passing from G to G/H does not really fit into the framework of group theory, then the graded objects e.g. gradation by G-sets etc ..., turn out to be useful exactly because of their generality. It is even possible to keep the philosophy of "action" because a gradation by G may be viewed as an action of the dual algebra $(KG)^*$, a Hopf algebra dual to the Hopf algebra KG for a finite group G. Hence a G-action will be a KG-action and a G-gradation will be a $(KG)^*$-action, or equivalently, a KG-coaction. In the shadow of "Quantum Groups", abstract Hopf algebras also gained popularity in recent years and therefore several techniques developed in Hopf algebra theory, independent of their equivalent in graded ring theory. A clear case is presented by the use of smash products, originating in a Hopf algebra setting but most effective for the graded theory because the presence of a group allows more concrete interpretations.

Chapter 7 is devoted to the introduction of smash product constructions inspired by the general concept of smash product associated to a Hopf module algebra. In our case the Hopf algebra considered will be the group algebra $k[G]$ over a commutative ring k with respect to an arbitrary group G. Several constructions of the smash product exist, for example depending on the fact whether G is a finite group or not. We have chosen to follow the approach of M. Cohen, S. Montgomery in the first case and D. Quin's in the second, cf. [43], [174] resp. The main idea behind the introduction of smash products associated to graded rings is that the smash product defines a new ring such that the category of graded modules over the graded ring becomes isomorphic to a closed subcategory of modules over the smash product.

At this point let us state that the philosophy underlying graded ring theory is almost contrary to the one of representation theory. Indeed a polynomial ring over a field, is graded by $\mathbb{Z}$ but what will this tell us about $\mathbb{Z}$? The graded methods are not aiming to obtain information about the grading group G, on the contrary the existence of a gradation is used to relate R and R_e, e the unit element of G, or graded to ungraded properties of R. Where possible, knowledge about the structure of G is used.

We may formulate the basic problems in graded algebra as the relational problems between a trio of categories. Indeed, for a graded ring R with respect to a group G we consider the three important categories

i. R-gr, the category of graded left modules

ii. R-mod, the category of left modules

iii. R_e-mod, the category of left modules over the subring R_e of R consisting of the homogeneous elements of degree e where e is the neutral element of G.

These categories are connected by several functors :

Ind : R_e-mod $\rightarrow$ R-gr, the induction

Coind : R-gr $\rightarrow$ R_e-mod, the coindiction

U : R-gr $\rightarrow$ R-mod, the forgetful functor

F : R-mod $\rightarrow$ R-gr, the right adjoint of U

Observe that the functors Ind and Coind stem from representation theory; the induction functor Ind is a left adjoint of the functor $(-)_e : R\text{-gr} \rightarrow R_e$-mod, associating to a graded odule M then R_e-module M_e which is the part of M consisting of elements of degree e, the coinduction functor is a right adjoint adjoint to the same functor. A large part of this book deals with problems relating to the "transfer of structure" via the functors introduced alone. A typical example is presented by the problem of identifying properties of ungraded nature implied by similar (or slightly modified) graded properties.

The material in this book is aimed to have a general applicability and therefore we stress "methods" and avoid specific technical structure theory. Since we strove to make the presentation as self contained as possible, this should make the text particularly useful for graduate students or beginning researchers; however, it should be an asset if some knowledge of a graduate course on general algebra is present e.g. several chapters in P. Cohn's book [45]. For full detail on the category theoretical aspects of Ring Theory we refer to the book by B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, 1975, [181]. For classical notions concerning riings and modules we recommend the reading of F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, Berlin, 1992, [6].

We have chosen to present these methods in suitable category theoretical settings, more specific applications have often been listed as exercises (but with extensive hints of how to solve them, or even with complete solution included). Typical topics include: category of graded rings and graded modules, the structure of simple or injective objects in the category of graded modules, Green theory for strongly graded rings, graded Clifford theory, internal and external homogenization, smash products and related functors, localization theory for graded rings, ...

For more detail on the contents of each separate chapter, we refer to the extra section "Comments and References" at the end of each chapter.

Contents

Chapter 1

The Category of Graded Rings

1.1 Graded Rings

Unless otherwise stated, all rings are assumed to be associative rings and any ring R has an identity $1 \in R$. If X and Y are nonempty subsets of a ring R then XY denotes the set of all finite sums of elements of the form xy with $x \in X$ and $y \in Y$. The group of multiplication invertible elements of R will be denoted by $U(R)$.

Consider a multiplicatively written group G with identity element $e \in G$. A ring R is **graded of type** G **or** R, is G-graded, if there is a family $\{R_\sigma, \sigma \in G\}$ of additive subgroups R_σ of R such that $R = \oplus_{\sigma\in G} R_\sigma$ and $R_\sigma R_\tau \subset R_{\sigma\tau}$, for every $\sigma, \tau \in G$. For a G-graded ring R such that $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$, we say that R is **strongly graded** by G.

The set $h(R) = \cup_{\sigma\in G} R_\sigma$ is the set of **homogeneous elements** of R; a nonzero element $x \in R_\sigma$ is said to be **homogeneous of degree** σ and we write : $\deg(x) = \sigma$. An element r of R has a unique decomposition as $r = \sum_{\sigma\in G} r_\sigma$ with $r_\sigma \in R_\sigma$ for all $\sigma \in G$, but the sum being a finite sum i.e. almost all r_σ zero. The set $\sup(r) = \{\sigma \in G, r_\sigma \neq 0\}$ is the **support of** r **in** G. By $\sup(R) = \{\sigma \in G, R_\sigma \neq 0\}$ we denote the **support of the graded ring** R. In case $\sup(R)$ is a finite set we will write $\sup(R) < \infty$ and then R is said to be a G**-graded ring of finite support**.

If X is a nontrivial additive subgroup of R then we write $X_\sigma = X \cap R_\sigma$ for $\sigma \in G$. We say that X is graded (or homogeneous) if : $X = \sum_{\sigma\in G} X_\sigma$. In particular, when X is a subring, respectively : a left ideal, a right ideal, an ideal, then we obtain the notions of graded subring, respectively : a graded left ideal, a graded right ideal, graded ideal. In case I is a graded ideal of R then the factor ring R/I is a graded ring with gradation defined by : $(R/I)_\sigma = R_\sigma + I/I$, $R/I = \oplus_{\sigma\in G}(R/I)_\sigma$.

1.1.1 Proposition

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a G-graded ring. Then the following assertions hold :

1. $1 \in R_e$ and R_e is a subring of R.

2. The inverse r^{-1} of a homogeneous element $r \in U(R)$ is also homogeneous.

3. R is a strongly graded ring if and only if $1 \in R_\sigma R_{\sigma^{-1}}$ for any $\sigma \in G$.

Proof

1. Since $R_e R_e \subseteq R_e$, we only have to prove that $1 \in R_e$. Let $1 = \sum r_\sigma$ be the decomposition of 1 with $r_\sigma \in R_\sigma$. Then for any $s_\lambda \in R_\lambda (\lambda \in G)$, we have that $s_\lambda = s_\lambda .1 = \sum_{\sigma \in G} s_\lambda r_\sigma$, and $s_\lambda r_\sigma \in R_{\lambda\sigma}$. Consequently $s_\lambda r_\sigma = 0$ for any $\sigma \neq e$, so we have that $s r_\sigma = 0$ for any $s \in R$. In particular for $s = 1$ we obtain that $r_\sigma = 0$ for any $\sigma \neq e$. Hence $1 = r_e \in R_e$.

2. Assume that $r \in U(R) \cap R_\lambda$. If $r^{-1} = \sum_{\sigma \in G} (r^{-1})_\sigma$ with $(r^{-1})_\sigma \in R_\sigma$, then $1 = rr^{-1} = \sum_{\sigma \in G} r(r^{-1})_\sigma$. Since $1 \in R_e$ and $r(r^{-1})_\sigma \in R_{\lambda\sigma}$, we have that $r(r^{-1})_\sigma = 0$ for any $\sigma \neq \lambda^{-1}$. Since $r \in U(R)$ we get that $(r^{-1})_\sigma \neq 0$ for $\sigma \neq \lambda^{-1}$, therefore $r^{-1} = (r^{-1})_{\lambda^{-1}} \in R_{\lambda^{-1}}$.

3. Suppose that $1 \in R_\sigma R_{\sigma^{-1}}$ for any $\sigma \in G$. Then for $\sigma, \tau \in G$ we have:

$$R_{\sigma\tau} = R_e R_{\sigma\tau} = (R_\sigma R_{\sigma^{-1}}) R_{\sigma\tau} = R_\sigma (R_{\sigma^{-1}} R_{\sigma\tau}) \subseteq R_\sigma R_\tau$$

therefore $R_{\sigma\tau} = R_\sigma R_\tau$, which means that R is strongly graded. The converse is clear. □

1.1.2 Remark

The previous proposition shows that $R_e R_\sigma = R_\sigma R_e = R_\sigma$, proving that R_σ is an R_e-bimodule.

If R is a G-graded ring, we denote by $U^{\mathrm{gr}}(R) = \cup_{\sigma \in G} (U(R) \cap R_\sigma)$ the set of the invertible homogeneous elements. It follows from Proposition 1.1.1 that $U^{\mathrm{gr}}(R)$ is a subgroup of $U(R)$. Clearly the degree map $\deg : U^{\mathrm{gr}}(R) \to G$ is a group morphism with $\mathrm{Ker}(\deg) = U(R_e)$.

A G-graded ring R is called a crossed product if $U(R) \cap R_\sigma \neq \emptyset$ for any $\sigma \in G$, which is equivalent to the map *deg* being surjective. Note that a G-crossed product $R = \oplus_{\sigma \in G} R_\sigma$ is a strongly graded ring. Indeed, if $u_\sigma \in U(R) \cap R_\sigma$, then $u_\sigma^{-1} \in R_{\sigma^{-1}}$ (by Proposition 1.1.1), and $1 = u_\sigma u_\sigma^{-1} \in R_\sigma R_{\sigma^{-1}}$.

1.2 The Category of Graded Rings

The category of all rings is denoted by RING. If G is a group, the category of G-graded rings, denoted by G-RING, is obtained by taking the G-graded rings for the objects and for the morphisms between G-graded rings R and S we take the ring morphisms $\varphi : R \to S$ such that $\varphi(R_\sigma) \subseteq S_\sigma$ for any $\sigma \in G$.

Note that for $G = \{1\}$ we have G-RING=RING. If R is a G-graded ring, and X is a non-empty subset of G, we denote $R_X = \oplus_{x \in X} R_x$. In particular, if $H \leq G$ is a subgroup, $R_H = \oplus_{h \in H} R_h$ is a subring of R. In fact R_H is an H-graded ring. If $H = \{e\}$, then $R_H = R_e$. Clearly the correspondence $R \mapsto R_H$ defines a functor $(-)_H : G - \text{RING} \to H - \text{RING}$.

1.2.1 Proposition

The functor $(-)_H$ has a left adjoint.

Proof Let $S \in H - \text{RING}$, $S = \oplus_{h \in H} S_h$. We define a G-graded ring $\overline{S}$ as follows: $\overline{S} = S$ as rings, and $\overline{S}_\sigma = S_\sigma$ if $\sigma \in H$, and $\overline{S}_\sigma = 0$ elsewhere. Then the correspondence $S \mapsto \overline{S}$ defines a functor which is a left adjoint of $(-)_H$.

□

We note that if $S \in \text{RING} = H - \text{RING}$ for $H = \{1\}$, then the G-graded ring $\overline{S}$ is said to have the **trivial G-grading**. Let $H \trianglelefteq G$ be a normal subgroup. Then we can consider the factor group G/H. If $R \in G - \text{RING}$, then for any class $C \in G/H$ let us consider the set $R_C = \oplus_{x \in C} R_x$. Clearly $R = \oplus_{C \in G/H} R_C$, and $R_C R_{C'} \subseteq R_{CC'}$ for any $C, C' \in G/H$. Therefore R has a natural G/H-grading, and we can define a functor $U_{G/H} : G - \text{RING} \to G/H - \text{RING}$, associating to the G-graded ring R the same ring with the G/H-grading described above. If $H = G$, then $G/G - \text{RING} = \text{RING}$, and the functor $U_{G/G}$ is exactly the forgetful functor $U : G - \text{RING} \to \text{RING}$, which associates to the G-graded ring R the underlying ring R.

1.2.2 Proposition

The functor $U_{G/H} : G - \text{RING} \to G/H - \text{RING}$ has a right adjoint.

Proof Let $S \in G/H - \text{RING}$. We consider the group ring $S[G]$, which is a G-graded ring with the natural grading $S[G]_g = Sg$ for any $g \in G$. Since $S = \oplus_{C \in G/H} S_C$, we define the subset A of $S[G]$ by $A = \oplus_{C \in G/H} S_C[C]$. If $g \in G$, there exists a unique $C \in G/H$ such that $g \in C$; define $A_g = S_C g$. Clearly the A_g's define a G-grading on A, in such a way that A becomes a G-graded subring of $S[G]$. We have defined a functor $F : G/H - \text{RING} \to G - \text{RING}$, associating to S the G-graded ring A. This functor is a right adjoint of the functor $U_{G/H}$. Indeed, if $R \in G - \text{RING}$ and $S \in G/H - \text{RING}$, we define a map

$$\varphi : \text{Hom}_{G/H-\text{RING}}(U_{G/H}(R), S) \to \text{Hom}_{G-\text{RING}}(R, F(S))$$

in the following way: if $u \in \text{Hom}_{G/H-\text{RING}}(U_{G/H}(R), S)$, then $\varphi(u)(r_g) = u(r_g)g$ for any $r_g \in R_g$. Then φ is a natural bijection; its inverse is defined by $\varphi^{-1}(v) = \varepsilon \circ i \circ v$ for any $v \in \text{Hom}_{G-\text{RING}}(R, A)$, where $i : A \to S[G]$ is the inclusion map, and $\varepsilon : S[G] \to S$ is the augmentation map, i.e. $\varepsilon(\sum_{g\in G} s_g g) = \sum_{g\in G} s_g$. In case S is a strongly graded ring (resp. a crossed product, then the ring A, constructed in the foregoing proof, is also strongly graded (resp. a crossed product). □

Clearly if $H \leq G$ and R is a G-strongly graded ring (respectively a crossed product), then R_H is an H-strongly graded ring (respectively a crossed product). Moreover, if $H \trianglelefteq G$ is a normal subgroup, then $U_{G/H}(R)$ is a G/H-strongly graded ring (respectively a crossed product). □

1.2.3 Remark

The category G-RING has arbitrary direct products. Indeed, if $(R_i)_{i\in I}$ is a family of G-graded rings, then $R = \oplus_{\sigma\in G}(\prod_i (R_i)_\sigma)$ is a G-graded ring, which is the product of the family $(R_i)_{i\in I}$ in the category G-RING. Note that R is a subring of $\prod_{i\in I} R_i$, the product of the family in the category RING. The ring R is denoted by $\prod_{i\in I}^{\text{gr}} R_i$. If G is finite or I is a finite set, we have $\prod_{i\in I}^{\text{gr}} R_i = \prod_{i\in I} R_i$.

1.2.4 Remark

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a G-graded ring. We denote by R^o the opposite ring of R, i.e. R^o has the same underlying additive group as R, and the multiplication defined by $r \circ r' = r'r$ for $r, r' \in R$. The assignment $(R^o)_\sigma = R_{\sigma^{-1}}$ makes R into a G-graded ring. The association $R \mapsto R^o$ defines an isomorphism between the categories $G - \text{RING}$ and $G - \text{RING}$.

1.3 Examples

1.3.1 Example *The polynomial ring*

If A is a ring, then the polynomial ring $R = A[X]$ is a $\mathbb{Z}$-graded ring with the standard grading $R_n = AX^n$ for $0 \leq n$, and $R_n = 0$ for $n < 0$. Clearly R is not strongly graded.

1.3.2 Example *The Laurent polynomial ring*

If A is a ring, let $R = A[X, X^{-1}]$ be the ring of Laurent polynomials with the indeterminate X. An element of R is of the form $\sum_{i\geq m} a_i X^i$ with $m \in \mathbb{Z}$

and finitely many non-zero a_i's. Then R has the standard $\mathbb{Z}$-grading $R_n = AX^n$, $n \in \mathbb{Z}$. Clearly R is a crossed product.

1.3.3 Example *Semitrivial extension*

Let A be a ring and ${}_AM_A$ a bimodule. Assume that $\varphi = [-,-] : M \otimes_A M \to A$ is an $A-A$-bilinear map such that $[m_1, m_2]m_3 = m_1[m_2, m_3]$ for any $m_1, m_2, m_3 \in M$. Then we can define a multiplication on the abelian group $A \times M$ by

$$(a, m)(a', m') = (aa' + [m, m'], am' + ma')$$

which makes $A \times M$ a ring called the semi-trivial extension of A by M and φ, and is denoted by $A \times_\varphi M$. The ring $R = A \times_\varphi M$ can be regarded as a $\mathbb{Z}_2$-graded ring with $R_0 = A \times \{0\}$ and $R_1 = \{0\} \times M$. We have that $R_1R_1 = Im\varphi \times \{0\}$, so if φ is surjective then R is a $\mathbb{Z}_2$-strongly graded ring.

1.3.4 Example *The "Morita Ring"*

Let $(A, B, {}_AM_B, {}_BN_A, \varphi, \psi)$ be a Morita context, where $\varphi : M \otimes_B N \to A$ is an $A-A$-bimodule morphism, and $\psi : N \otimes_A M \to B$ is a $B-B$-bimodule morphism such that $\varphi(m \otimes n)m' = m\psi(n \otimes m')$ and $\psi(n \otimes m)n' = n\varphi(m \otimes n')$ for all $m, m' \in M, n, n' \in N$. With this set-up we can form the Morita ring

$$R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

where the multiplication is defined by means of φ and ψ. Moreover, R is a $\mathbb{Z}$-graded ring with the grading given by:

$$R_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad R_1 = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \quad R_{-1} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}$$

and $R_i = 0$ for $i \neq -1, 0, 1$.

Since $R_1R_{-1} = \begin{pmatrix} Im\,\varphi & 0 \\ 0 & 0 \end{pmatrix}$ and $R_{-1}R_1 = \begin{pmatrix} 0 & 0 \\ 0 & Im\,\psi \end{pmatrix}$, then R is not strongly graded.

1.3.5 Example *The matrix rings*

Let A be a ring, and $R = M_n(A)$ the matrix ring. Let $\{e_{ij} | 1 \le i, j \le n\}$ be the set of matrix units, i.e. e_{ij} is the matrix having 1 on the (i, j)-position and 0 elsewhere. We have that $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for any i, j, k, l, where δ_{jk} is Kronecker's symbol. For $t \in \mathbb{Z}$ set $R_t = 0$ if $|t| \ge n$, $R_t = \sum_{i=1,n-t} Re_{i,i+t}$ if $0 \le t < n$, and $R_t = \sum_{i=-t+1,n} Re_{i,i+t}$ if $-n < t < 0$. Clearly $R = \oplus_{t \in \mathbb{Z}} R_t$, and this defines a $\mathbb{Z}$-grading on R.

On the other hand we can define various gradings on the matrix ring. We mention an example of a $\mathbb{Z}_2$-grading on $R = M_3(A)$, defined by :

$$R_0 = \begin{pmatrix} A & A & 0 \\ A & A & 0 \\ 0 & 0 & A \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & A \\ A & A & 0 \end{pmatrix}$$

Since $R_1 R_1 = R_0$, R is a strongly $\mathbb{Z}_2$-graded ring; however R is not a crossed product, since there is no invertible element in R_1. It is possible to define such "block-gradings" on every $M_n(A)$.

1.3.6 Example *The $G \times G$-matrix ring*

Let G be a finite group and let A be an arbitrary ring. We denote by $R = M_G(A)$ the set of all $G \times G$-matrices with entries in A. We view such a matrix as a map $\alpha : G \times G \to A$. Then R is a ring with the multiplication defined by:

$$(\alpha\beta)(x, y) = \sum_{z \in G} \alpha(x, z)\beta(z, y)$$

for $\alpha, \beta \in R$, $x, y \in G$. If

$$R_g = \{\alpha \in M_G(R) | \alpha(x, y) = 0 \quad \text{for every} \quad x, y \in G \text{ with } x^{-1}y \neq g\}$$

for $g \in G$, then R is a G-graded ring with g-homogeneous component R_g. Indeed, let $\alpha \in R_g, \beta \in R_{g'}$. Then for every $x, y \in G$ such that $x^{-1}y \neq gg'$, and any $z \in G$, we have either $x^{-1}z \neq g$ or $z^{-1}y \neq g'$, therefore $(\alpha\beta)(x, y) = \sum_{z \in G} \alpha(x, z)\beta(z, y) = 0$, which means that $\alpha\beta \in R_{gg'}$. If for $x, y \in G$ we consider $e_{x,y}$ the matrix having 1 on the (x, y)-position, and 0 elsewhere, then $e_{x,y}e_{u,v} = \delta_{y,u}e_{x,v}$. Clearly $e_{x,xg} \in R_g$, $e_{yg,y} \in R_{g^{-1}}$, and $(\sum_{x \in G} e_{x,xg})(\sum_{y \in G} e_{yg,y}) = 1$, hence R is a crossed product.

1.3.7 Example *Extensions of fields*

Let $K \subseteq E$ be a field extension, and suppose that $E = K(\alpha)$, where α is algebraic over K, and has minimal polynomial of the form $X^n - a$, $a \in K$ (this means that E is a radical extension of K). Then the elements $1, \alpha, \alpha^2, ..., \alpha^{n-1}$ form a basis of E over K. Hence $E = \oplus_{i=0,n-1} K\alpha^i$, and this yields a $\mathbb{Z}_n$-grading of E, with $E_0 = K$. Moreover E is a crossed product with this grading.

As particular examples of the above example we obtain two very interesting ones:

1.3.8 Example

Let $k(X)$ be the field of rational fractions with the indeterminate X over the field k. Then the conditions in the previous example are satisfied by the