ADVANCED GENERAL RELATIVITY

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This book is a self-contained introduction to key topics in advanced general relativity. The opening chapter reviews the subject, with strong emphasis on the geometric structures underlying the theory. The second chapter discusses 2-component spinor theory, its usefulness for describing zero-mass fields and its practical application via Newman-Penrose formalism, together with examples and applications. There follows an account of the asymptotic theory far from a strong gravitational source, describing the mathematical theory by which measurements of the far-field and gravitational radiation emanating from a source can be used to describe the source itself. Finally, the characteristic initial value problem is described, first in general terms, and then with particular reference to relativity, concluding with its relation to Arnol'd's singularity theory. Exercises are included throughout.

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PREFACE

General relativity is the flagship of applied mathematics. Although from its inception this has been regarded as an extraordinarily difficult theory, it is in fact the simplest theory to consummate the union of special relativity and Newtonian gravity. Einstein's 'popular articles' set a high standard which is now emulated by many in the range of introductory textbooks. Having mastered one of these the new reader is recommended to move next to one of the more specialized monographs, e.g. Chandrasekhar, 1983, Kramers et al., 1980, before considering review anthologies such as Einstein (centenary), Hawking and Israel, 1979, Held, 1980 and Newton (tercentary), Hawking and Israel, 1987. As plausible gravitational wave detectors come on line in the next decade (or two) interest will focus on gravitational radiation from isolated sources, e.g., a collapsing star or a binary system including one, and I have therefore chosen to concentrate in this book on the theoretical background to this topic.

The material for the first three chapters is based on my lecture courses for graduate students. The first chapter of this book presents an account of local differential geometry for the benefit of the beginner and as a reminder of notation for more experienced readers. Chapter 2 is devoted to two-component spinors which give a representation of the Lorentz group appropriate for the description of gravitational radiation. (The relationship to the more common Dirac four-component spinors is discussed in an appendix.) Far from an isolated gravitating object one might expect spacetime to become asymptotically Minkowskian, so that the description of the gravitational field would be especially simple. This concept, *asymptopia* (asymptotic Utopia) is discussed in chapter

3, commencing with an account of the asymptotics of Minkowski spacetime, and going on to the definitions of asymptotic flatness and radiating spacetimes. (For a more detailed development of the material in chapters 2 and 3, the reader is referred to Penrose and Rindler, 1984, 1986.) The book concludes with a self-contained discussion of the characteristic initial value problem, caustics and their relation to the singularity theory of Arnol'd.

Exercises form an integral part of each chapter giving the reader a chance to check his understanding of intricate material or offering straightforward extensions of the mainstream discussions. *Problems* are even more important, for they are not only more challenging exercises, but can frequently be combined to produce significant results, encouraging the student to develop his understanding by deriving much material which is not explicitly spelt out. The brevity of this book is deceptive.

Finally I acknowledge the considerable benefit of discussions with many of my colleagues, especially Jürgen Ehlers, Bernd Schmidt and Martin Walker. In particular I owe especial thanks to Helmut Friedrich for teaching me (almost) all I know about the characteristic initial value problem.

John Stewart

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1

DIFFERENTIAL GEOMETRY

The natural arena for physics is spacetime. As **we shall see later spacetime is** curved. It is necessary therefore to introduce **a fair amount** of mathematics in order to understand the physics. **Fortunately** we shall only need a local theory, so that problems from **differential** topology will not occur.

1.1 Differentiable manifolds

The simplest example of a curved space is the surface of a sphere S^2, such as the surface of the earth. One can set up local coordinates, e.g., latitude and longitude, which map S^2 onto a plane piece of paper R^2, known to sailors as a chart. Collections of charts are called atlases. Perusal of any atlas will reveal that there is no 1-1 map of S^2 into R^2; we need several charts to cover S^2. Let us state this more formally.

(1.1.1) DEFINITION

Given a (topological) space M, a ***chart on M*** *is a 1-1 map ϕ from an open subset $U \subset M$ to an open subset $\phi(U) \subset R^n$, i.e., a map $\phi : M \to R^n$. A chart is often called a* ***coordinate system.***

Now suppose the domains U_1, U_2 of two charts ϕ_1, ϕ_2 overlap in $U_1 \cap U_2$. Choose a point x_1 in $\phi_1(U_1 \cap U_2)$. It corresponds to a point p in $U_1 \cap U_2$, where $p = \phi_1{}^{-1}(x_1)$. Since p is in U_2 we can map it to a point x_2 in $\phi_2(U_2)$. We shall require the map $x_1 \mapsto x_2$ to be smooth, see figure 1.1.1 so that in the next section the definitions of smooth

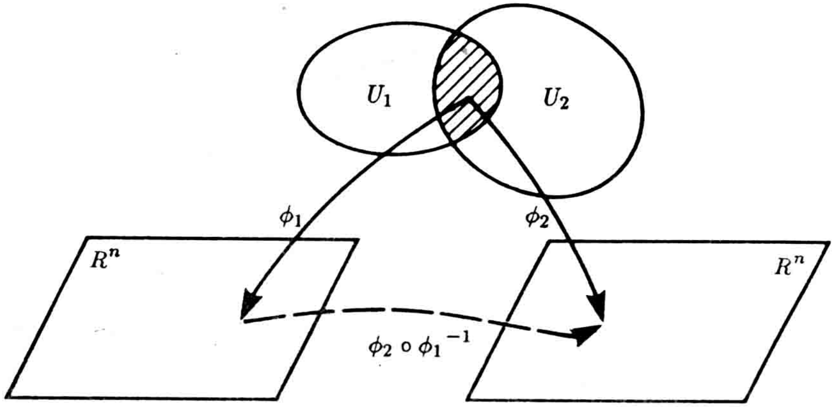

Fig. 1.1.1 When the domains of two charts ϕ_1 and ϕ_2 overlap they are required to mesh smoothly, i.e., $\phi_2 \circ \phi_1^{-1}$ must be smooth.

curves and tangent vectors will be coordinate-independent. More precisely:

(1.1.2) DEFINITION

*Two charts ϕ_1, ϕ_2 are **C^∞-related** if both the map*

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2),$$

*and its inverse are C^∞. A collection of C^∞-related charts such that every point of M lies in the domain of at least one chart forms an **atlas**. The collection of all such C^∞-related charts forms a **maximal atlas**. If M is a space and A its maximal atlas, the set (M, A) is a **(C^∞-) differentiable manifold**. If for each ϕ in the atlas the map $\phi : U \to R^n$ has the same n, then the manifold has **dimension** n.*

When problems of differentiablity arise we can similarly define C^k-related charts and C^k-manifolds.

The reader for whom these ideas are new is strongly recommended to peruse a geographical atlas and identify the features described above. Further examples worth examining include the plane R^2, the circle S^1, the sphere S^2 and a Möbius band.

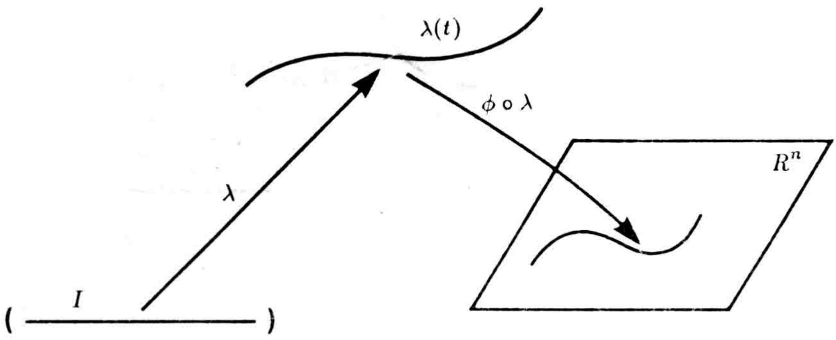

Fig. 1.2.1 A curve $\lambda : I \to M$, $t \mapsto \lambda(t)$ is smooth iff its image under a chart is a smooth curve in R^n.

1.2 Tangent vectors and tangent spaces

Most concepts in physics involve the concept of differentiability, and, as we shall see, an essential ingredient is the generalization of the idea of a vector. Simple naïve definitions of vectors do not work in general manifolds. The vector London → Paris may be parallel to the vector London → Dublin in one chart and perpendicular to it in another. Some experiments will show that there are severe problems in establishing chart-independence for the usual properties of vectors in any non-local definition. The following approach may not be an obvious one but it will capture the intuitive concepts. We start by defining a curve within a manifold.

(1.2.1) DEFINITION

A C^∞***-curve*** *in a manifold* M *is a map* λ *of the open interval* $I = (a, b) \in R \to M$ *such that for any chart* ϕ, $\phi \circ \lambda : I \to R^n$ *is a* C^∞ *map.*

There are a number of points to note about this definition. Firstly C^k-curves are defined in the obvious way. Either or both of a, b can be infinite. By considering half-closed or closed intervals I one or both endpoints can be included. Finally the definition

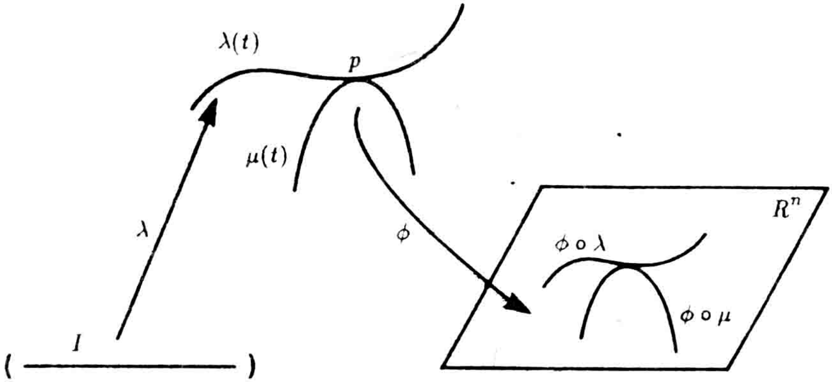

Fig. 1.2.2 Two curves $\lambda(t)$, $\mu(s)$ are tangent at p iff their images under a chart ϕ are tangent at $\phi(p)$ in R^n.

implies that the curve is parametrized. As an abbreviation one often speaks of the ***curve*** $\lambda(t)$, with $t \in (a,b)$.

Let $f : M \to R$ be a smooth function on M. Consider the map $f \circ \lambda : I \to R$, $t \mapsto f(\lambda(t))$. This has a well-defined derivative, the rate of change of f along the curve. Now suppose that two curves $\lambda(t), \mu(s)$ meet at a point p where $t = t_o$, $s = s_o$. Suppose that

$$\frac{d}{dt}(f \circ \lambda) = \frac{d}{ds}(f \circ \mu) \qquad \text{at} \quad t = t_o, \; s = s_o, \tag{1.2.1}$$

for all functions f. This is a precise way of stating that "λ, μ pass through p with the same velocity". To see this we consider:

(1.2.2) LEMMA

Suppose that ϕ is any chart whose domain of dependence includes p. Let ϕ map $q \in M$ to $x^i(q)$ where $x^i(q)$ are the coordinates of q. Then (1.2.1) holds if and only if for all charts and each i

$$\left[\frac{d}{dt}(x^i \circ \lambda)\right]_{t=t_o} = \left[\frac{d}{ds}(x^i \circ \mu)\right]_{s=s_o}. \tag{1.2.2}$$

Proof: It is trivial to show that (1.2.1) implies (1.2.2). To proceed in the other direction write $f \circ \lambda = (f \circ \phi^{-1}) \circ (\phi \circ \lambda)$. Now $f \circ \phi^{-1}$

is a map $R^n \to R$, $x^i \mapsto f(x^i) = f(\phi^{-1}(x^i))$. Also $\phi \circ \lambda$ is a map $I \to R^n$, $t \mapsto x^i(\lambda(t))$. Using the chain rule for differentiation

$$\frac{d}{dt}(f \circ \lambda) = \sum_1^n \left[\frac{\partial}{\partial x^i}(f(x^i))\right] \frac{d}{dx^i}(\lambda(t)). \qquad (1.2.3)$$

A similar expression holds for $f \circ \mu$ which then proves the result.

Thus given a curve $\lambda(t)$ and a function f we can obtain a new number $[d(f \circ \lambda)/dt]_{t=t_o}$, the rate of change of f along the curve $\lambda(t)$ at $t = t_o$. We now use this result to remove the non-locality from the idea of a vector.

(1.2.3) DEFINITION

The ***tangent vector*** $\dot{\lambda}_p = (d\lambda/dt)_p$ ***to a curve*** $\lambda(t)$ ***at a point p on it*** *is the map from the set of real functions f defined in a neighbourhood of p to R, defined by*

$$\dot{\lambda}_p : f \mapsto \left[\frac{d}{dt}(f \circ \lambda)\right]_p = (f \circ \lambda)^{\cdot}_p = \dot{\lambda}_p(f). \qquad (1.2.4)$$

Given a chart ϕ with coordinates x^i, ***the components of*** $\dot{\lambda}_p$ ***with respect to the chart*** *are*

$$(x^i \circ \lambda)^{\cdot}_p = \left[\frac{d}{dt} x^i(\lambda(t))\right]_p .$$

The set of tangent vectors at p is the ***tangent space*** $T_p(M)$ ***at*** *p.*

This accords with the usual algebraists' idea of vectors, as we see from the following result.

(1.2.4) THEOREM

If the dimension of M is n then $T_p(M)$ is a vector space of dimension n.

Proof: We show first that $T_p(M)$ is a vector space, i.e., if $X_p, Y_p \in T_p(M)$ and a is a real number then

$$X_p + Y_p, \quad aX_p \qquad \in T_p(M).$$

In other words we have first to show that there is a curve $\nu(t)$ through p, $t = t_o$ such that

$$\dot{\nu}_p(f) = (f \circ \nu)^{\cdot}_p = X_p f + Y_p f. \qquad (1.2.5)$$

Let λ, μ be curves through p with $\lambda(t_o) = \mu(t_o) = p$, and $\dot{\lambda}_p = X_p$, $\dot{\mu}_p = Y_p$. Then $\tilde{\nu} : t \mapsto \phi \circ \lambda + \phi \circ \mu - \phi(p)$ is a curve in R^n (where $+,-$ have their usual meaning) and $\nu : t \mapsto \phi^{-1} \circ \tilde{\nu}$ is a curve in M satisfying (1.2.5). The second part of this proof is left as an exercise for the reader.

Finally we have to show that a basis of $T_p(M)$ contains n vectors. We first establish a useful result. Let ϕ be a chart with coordinates x^i. Consider n curves $\lambda_k(t)$ defined as follows

$$\phi\,(\lambda_k(t)) = (x^1(p), \ldots, x^{k-1}(p), x^k(p) + t, x^{k+1}(p), \ldots, x^n(p))\,,$$

i.e., only the k'th coordinate varies. We denote the tangent vector at p, $t = 0$ by

$$\dot{\lambda}_k(0) = \left(\frac{\partial}{\partial x^k}\right)_p . \qquad (1.2.6)$$

Note the simple result $(x^i \circ \lambda_k)^{\cdot} = \delta^i{}_k$. Then using the chain rule (1.2.3) we have

$$\begin{aligned}
\left(\frac{\partial}{\partial x^k}\right)_p f &= \frac{d}{dt}\,(f \circ \lambda_k)_p \\
&= \frac{d}{dt}\,\left[(f \circ \phi^{-1}) \circ (\phi \circ \lambda_k)\right]_p \\
&= \sum_1^n \frac{\partial}{\partial x^m}(f \circ \phi^{-1})\,\frac{d}{dt}\,(x^m \circ \lambda_k) \\
&= \sum_1^n \frac{\partial}{\partial x^m}\,(f \circ \phi^{-1})\,\delta_k{}^m = \left[\frac{\partial}{\partial x^k}(f \circ \phi^{-1})\right]_p . \qquad (1.2.7)
\end{aligned}$$

Next we show that each vector at p is a linear combination of the $(\partial/\partial x^k)_p$. To see this let X_p be a tangent vector to the curve $\lambda(t)$ with $\lambda(0) = p$. Then

$$\begin{aligned} X_p f &= (f \circ \lambda)^{\cdot}(0) = [f \circ \phi^{-1} \circ \phi \circ \lambda]^{\cdot}(0) \\ &= \sum_k \frac{\partial}{\partial x^k}(f \circ \phi^{-1})\,(x^k \circ \lambda)^{\cdot}(0) \\ &= \sum_k \left(\frac{\partial}{\partial x^k}\right)_p f \;\; (X_p x^k), \end{aligned} \tag{1.2.8}$$

where (1.2.7) has been used in the last step. Thus

$$X_p = \sum_k (X_p x^k) \left(\frac{\partial}{\partial x^k}\right)_p, \tag{1.2.9}$$

and so the $(\partial/\partial x^k)_p$ span $T_p(M)$. Finally to show that they are linearly independent suppose that $\sum A^k \, (\partial/\partial x^k)_p = 0$. Then

$$0 = \sum_k A^k \left(\frac{\partial}{\partial x^k}\right)_p x^i = \sum_k A^k \delta_k{}^i = A^i.$$

Thus each $A^i = 0$, i.e., the $(\partial/\partial x^k)_p$ form a basis. From (1.2.9) we see that $X_p x^i$ are the components of X_p with respect to the given basis.

We shall subsequently use the ***Einstein summation convention***: in an expression where the same index occurs twice, once up once down, it is to be summed. (In a fraction up in a denominator counts as down in the numerator and vice versa.) Thus

$$\sum_k A^k \left(\frac{\partial}{\partial x^k}\right)_p \rightarrow A^k \left(\frac{\partial}{\partial x^k}\right)_p$$

WARNING: Do not confuse the differential operator acting on functions in R^n, $\partial/\partial x^k$ with the vector $(\partial/\partial x^k)_p$ in $T_p(M)$.

The final result needed to capture the concept of a tangent vector as a derivative is left as an exercise.