ANALYSIS AND LINEAR ALGEBRA

HENRY HELSON

Pilot Edition

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Foreword

Over the past decades calculus texts have evolved into a refined, smooth product. Instructors joke about how similar the latest one is to the preceding six or seven. The books have grown to monstrous size so that no potential user will be disappointed by the omission of some topic considered essential for physics or engineering students. Chapters are divided into sections of equal and measured difficulty, each followed by a very large number of trivial exercises and some problems pretending to have relevance to physical science.

Experience proves that these books are very successful in imparting a little knowledge about calculus to a great number of students who would rather not have any. Their aversion is presently justified, for this homogenized mathematics is not the lovely collection of ideas that made their instructors want to become mathematicians. Is this mistreatment of young minds really necessary in order to teach our large, heterogeneous classes?

This book is an attempt to provide an alternative. It is offered with humility, and the sincere hope that it can soon be improved upon. It is written on hypotheses directly opposed to the prevailing ones, so that at any rate its faults are not those of other texts just described.

First of all mathematics is by definition abstraction from the real world. The first goal of a mathematics course must be to increase the student's powers of abstraction. Learning of any particular techniques is of secondary importance. Pretending that mathematics is really physics or engineering or economics only misleads the student.

It is hard to apply mathematics, harder than learning mathematics itself. Fabricated and unconvincing "applications" are useful as illustrative material, but they can give the student wrong ideas about genuine applied mathematics, and they divert attention from what is important. They should stay in the background.

It is not profitable for a student to grind out an immense number of repetitive exercises each week (while watching TV). Once a point has been grasped, and solidified by a few easy calculations, he should be asked to think about the ramifications of the idea just presented, not continue with make-work problems.

Students are not alike: they learn in different ways as well as at different speeds. Building mathematics like an office tower, one unit on top of another, serves no student well. Rather each lecture, each section of text should to some extent be about all of mathematics. Of course mathematics consists of detail. But the detail will fall into place if the pattern is clear.

Such are the hypotheses underlying this writing. They apply equally to students who are gifted in mathematics, and to those who are not. The book has grown out of a course offered for many years to social scientists who had avoided mathematics until their own departments obliged them to take some. Their innate ability has ranged from very little to quite a lot, speaking about mathematics alone. The mathematically talented among them have gone on after one year to junior-level mathematics courses; the least gifted have often accomplished more than they thought possible.

The topics are conventional. After a review of high school algebra (appropriate for students who have not taken mathematics for some time) the material of one-variable and many-variable calculus follows in order. Generally a topic is introduced at a concrete, computational level and subsequently is treated more rigorously and abstractly. An instructor can give a more elementary course by omitting later parts of several chapters. The relatively difficult theorems based on the Mean Value Theorem have been gathered into one chapter.

However Chapter 8 is radically different from the usual treatment of the trigonometric and exponential functions based on contemplation of triangles. The most important part of elementary analysis is the collection of results about complex numbers, power series, the complex exponential function, and finally the trigonometric functions. These topics fit together beautifully and they are all fundamental. Why are they kept a secret for so long? They are presented here in a mathematically correct way without concealing any difficulties. If necessary the proofs of theorems about power series can be deferred. That will not diminish the profit in seeing how the trigonometric and exponential functions are related in the complex domain.

The chapters on linear algebra constitute an independent text on that subject. The presentation has been made as simple as possible, but all results are proved. Sections on factor analysis and regression present these genuine applications of non-trivial mathematics to statistics.

The final chapter on differential equations is a modest introduction to that subject. Linear difference equations are mentioned as well, because they are useful to social scientists.

The book contains enough material for three semesters at least, even with omissions. It is not intended that one class will start at the beginning and work to the end; the difference in level is too great. But different classes can use the book in different ways. An honors class can start with Chapter 2 or beyond and proceed linearly. A class of older students who are not science majors (such as the students I have taught) should spend considerable time on Chapter 1, but then could touch all the main topics in a year. A normal first-year class in calculus can start in Chapter 2, but go on rapidly to Chapter 4, and then omit parts of Chapters 5, 7 and 8.

This book embodies my ideas on how mathematics should and can be taught to most students. It does not follow that every elementary mathematics course can be taught successfully from it. More than with conventional texts the student must be persuaded that mathematics is important, and must be willing to turn off the TV (literally and figuratively) in order to concentrate on it. The instructor must judge how much his class can accomplish, and then not try to get it too fast. Where these prerequisites are not met, a text that presents mathematics as a cut-and-dried subject will undoubtedly be more satisfactory.

Contents

Chapter 1

Algebra

1. Manipulation of symbols

A symbol is a mark on paper that stands in our thought for a substantial object. We can perform operations on symbols more easily than we can think about the objects they represent. The rules for manipulating symbols depend, of course, on the properties of the objects. The manipulation of symbols according to stated rules is called algebra. In school, algebra was the manipulation of symbols standing for numbers or measurements of objects according to the rules of arithmetic. Algebra is a major branch of mathematics and means much more than that, but this chapter is a review of elementary algebra as learned in school.

A problem in algebra gives some information and asks us to find an unknown quantity that is determined by that information. The given information is written down, say in the form of an equation, using a symbol (by tradition the letter x) to denote the unknown quantity. This equation is really a sentence about x. Now x is a number that can be combined with other numbers by the operations of arithmetic, even though we do not know its value. So arithmetic operations on the equation give other sentences about x. If we can get to a sentence "x = (some particular number)" the problem is solved.

This description is too simple, but it can serve to introduce the mathematical process. Here is an example for illustration.

A son was born when his father was 26 years old. Now their ages have sum equal to 100. How old are they?

Denote the son's age by x. Then the father's age is $x + 26$, an unknown quantity too but expressed now in terms of the son's age. Their ages add up to 100:

$$x + (x + 26) = 100. \tag{1.1}$$

By arithmetic we get successively $2x + 26 = 100$, $2x = 74$, and finally $x = 37$. That is, the son is 37 years old and the father must be $37 + 26 = 63$ years old.

This problem and even others that are more involved can be solved without using symbols or writing down an equation. But the solution would take thought, and there is a limit to the complication we can keep track of. The symbolic method enables us to solve even very complicated problems in a mechanical way.

Problems.

1. Solve these equations for x. (a) $2x - 3 = 7$ (b) $x - 3 = 2x - 3$ (c) $2x + 4 = 16$ (d) $x - 1 = x + 2$. [If you have difficulty solving (d) try to say something sensible about it.]

2. A small country has a million inhabitants, of whom 100,000 are foreigners. The citizens consist of two ethnic groups, one twice as numerous as the other. How many are there in each group?

3. The gross national product of that country was 10% larger in 1978 than in 1977; the labor force was, on average, 4% larger. How many times as much did the average worker produce in 1978 as in 1977?

4. A number is multiplied by 2, the result is squared, eight subtracted, the cube root taken, and this subtracted from 10. If we get 8, what was the number?

2. Simultaneous equations

A company shipped 500 transformers of type A and 300 of type B during one month, and billed its customers \$550,000. The next month it shipped 400 of each type, and billed \$600,000. What was the price of each type?

The two billings constitute independent pieces of information, and it is not easy to incorporate all the information in one equation. So we write down two equations, each containing two unknowns: x is the price of type A, and y of type B:

(1.2) $$500x + 300y = 550{,}000 \text{ and } 400x + 400y = 600{,}000$$

We seek numbers x and y such that both equations are true about x and y. In usual terms, we want to solve the equation simultaneously.

We assume that some numbers do satisfy both equations, even though that might not be true. Then correct manipulations of these equations will lead to new equations that are true statements about those numbers. Finally we expect to derive the equations "x = (some number), y = (some number)". This means, logically, that if any numbers satisfy (1.2), then these must be the numbers. In order to justify the assumption we substitute the numbers we have found into (1.2) and verify directly that they furnish a simultaneous solution of the equations.

That is the logic of a solution; here are the manipulations. First divide all terms by 100 to simplify writing. Now multiply the terms of the first equation by 4, and the terms of the second equation by 5:

(1.3) $$20x + 12y = 22{,}000 \text{ and } 20x + 20y = 30{,}000.$$

The reason for doing that was to make the coefficients of x the same in the two equations. Now subtracting the first equation from the second gives

$$(1.4) \qquad 8y = 8000, \quad y = 1000.$$

If we put 1000 in place of y in the first equation of (1.2) (or in the second one for that matter) we get $20x = 10,000$, $x = 500$.

Thus $x = 500$, $y = 1000$ are the only numbers that could satisfy (1.2). When they are substituted into (1.2) they do indeed give true statements, and so the transformers cost \$500 for type A and \$1000 for type B.

Problems.

1. Solve these pairs of equations simultaneously, if possible. (a) $3x - 2y = 0$, $x + y = 1$ (b) $x - y = 2$, $x = 1$ (c) $x + 4y = 5$, $3x - y = 6$ (d) $x - 3y = 16$, $2x - 6y = 4$.

2. Two psychological traits can be measured by means of two tests, A and B. The scores S_A and S_B that a subject makes on the tests both depend on both traits, but not to the same degree. The authors of the tests have computed that

$$S_A = 2x + 3y, \qquad S_B = 6x + y,$$

where x is the strength of the first trait in the subject, and y the strength of the second trait. One subject made scores $S_A = 11$, $S_B = 9$. Find x and y for this subject.

3. A bartender uses Bourbon costing \$4 and vermouth costing \$2 per bottle. A Manhattan costs him \$.50 in ingredients. If he should go over to Bourbon costing \$5 and vermouth costing \$3, the same mixture would cost him \$.65. What is his recipe?

4. Suppose, in the last problem, that the bottles are not the same size: Bourbon bottles hold a quart and vermouth bottls 4/5 quart. Find the recipe in this case.

3. Quadratic equations

A park of rectangular shape is 4 miles longer than wide. Its area is 32 square miles. What are its dimensions?

Let x be the width of the park; its length then is x+4, and its area is x(x+4). Thus x must satisfy

$$x(x + 4) = 32 \quad \text{or} \quad x^2 + 4x - 32 = 0. \tag{1.5}$$

This is a quadratic equation.

The simplest way to solve (1.5) is to write it in the form

$$x^2 + 4x - 32 = (x - 4)(x + 8) = 0. \tag{1.6}$$

We have factored the quadratic expression as a product of linear ones. Now (1.6) says the product of two numbers x-4 and x+8 equals 0. This is the case if and only if one of the factors, at least, is 0. That is, (1.6) has two solutions: $x = 4$ and $x = -8$.

We have now solved (1.5) but not our problem. The number 4 is an acceptable answer: if the park has width 4 miles, and length 4+4 = 8 miles, then indeed its area is 32 square miles. But what of the meaningless solution $x = -8$ of (1.5)?

The equation (1.5) is a statement about the width of the park, but it is not the only true statement about this quantity. For example, this width must be positive, but (1.5) does not contain that information. So when we solve (1.5) we prove that no numbers except 4 and -8 can be the width in question, but not that either number actually is an answer. We have to check each one. Obviously we accept 4 and discard -8.

This example shows that you may lose information in condensing a real problem into formulas. Mathematical analysis, when successful, leads to numbers that are candidates for answers. These numbers always have to be checked against the original statement of the problem for a decision as to whether they really are a solution.

There is a second, opposite danger. A mathematical formulation may contain too much information instead of too little, and the information may be internally inconsistent. A banal example will illustrate the point, although in practice the trouble may not be so obvious. If we measure the sum of two quantities x and y twice we shall naturally get slightly different results, say

$$x + y = 100, \qquad x + y = 101. \tag{1.7}$$

This is a pair of equations in x and y and we might be tempted to solve them simultaneously. Of course we cannot. Now (1.7) is a legitimate account of two experiments, and the fact that the equations are inconsistent is no criticism; however we cannot blindly use them together.

It is useful to be able to spot equations without solution, or inconsistent simultaneous equations. There are two examples below.

Problems.

1. Solve these equations by factoring. (a) $x^2 + x = 6$ (b) $x^2 + 3 = 2$ (c) $x^2 - x = 0$ (d) $x^3 - 2x^2 + x = 0$ (e) $x^2 = 21 - 4x$ (f) $6x^2 + 5x - 1 = 0$

2. Solve these pairs of simultaneous equations. In each case find all the solutions, and write the answers down carefully. (a) $x^2 + y^2 = 1$, $x + y = 0$ (b) $x^2 + y^2 = 2$, $xy = 1$ (c) $xy = 1$, $x + y = 0$.

3. An electric utility company charges industrial customers $.005 + .5\sqrt{x}$ dollars for x units of electricity per month. One month a customer was billed exactly 100 dollars. How much electricity had he used?

4. More quadratic equations

The factoring method is the easiest way to solve a quadratic equation if the factoring can be found without too much trouble, but there is a systematic procedure that always succeeds, and actually gives a formula for the solutions. It is called completing the square. We use it to solve the equation

$$(1.8)\qquad x^2 + bx + c = 0$$

where b and c are any numbers.

Add $b^2/4 - c$ to both sides of (1.8):

$$(1.9)\qquad x^2 + bx + b^2/4 = b^2/4 - c = \tfrac{1}{4}(b^2 - 4c).$$

Notice that the left side now equals $(x + b/2)^2$. (Multiply the square out in order to check this.) Therefore (1.8) has exactly the same solutions, if any, as

$$(1.10)\qquad (x + b/2)^2 = \tfrac{1}{4}(b^2 - 4c).$$

The square of any real number is positive, except that the square of 0 is 0. Thus the left side of (1.10) is positive or 0, whatever value x has. If $b^2 - 4c$ is negative (1.10) has to be false for every x; that is, (1.8) has no solution in this case. If $b^2 - 4c = 0$, then $x + b/2$ is a number whose square is 0, so $x + b/2 = 0$ or $x = -b/2$. If $b^2 - 4c$ is positive, it is the square of two numbers, namely $\tfrac{1}{2}\sqrt{b^2 - 4c}$ and $-\tfrac{1}{2}\sqrt{b^2 - 4c}$, and (1.10) is true if $x + b/2$ is either of these numbers. That is, the two numbers

$$(1.11)\qquad -b/2 + \tfrac{1}{2}\sqrt{b^2 - 4c}, \qquad -b/2 - \tfrac{1}{2}\sqrt{b^2 - 4c}$$

are both solutions of (1.10) and therefore of (1.8). If $b^2 - 4c = 0$ these solutions both collapse to $-b/2$ as we should hope.

We see that depending on the coefficients b and c, (1.8) may have two solutions, or one solution, or no solution, according as $b^2 - 4c$ is positive, or zero, or negative.

The general quadratic equation is

$$ax^2 + bx + c = 0, \tag{1.12}$$

where a, b, c are numbers. If $a = 0$ the equation is actually linear and easy to solve. We exclude that case, because the following analysis would be meaningless. In order to solve (1.12) with $a \neq 0$ divide the equation by a; we get

$$x^2 + Bx + C = 0, \tag{1.13}$$

where $B = b/a$, $C = c/a$. This equation is like (1.8), and from (1.11) we see that its solutions are $-B/2 + \frac{1}{2}\sqrt{B^2 - 4C}$, $-B/2 - \frac{1}{2}\sqrt{B^2 - 4C}$ provided $B^2 - 4C$ is positive or zero. If we replace B and C by b/a and c/a respectively and simplify the expressions that result, we find that (1.13) has solutions

$$\frac{1}{2a}(-b + \sqrt{b^2 - 4ac}), \quad \frac{1}{2a}(-b - \sqrt{b^2 - 4ac}). \tag{1.14}$$

Furthermore $B^2 - 4C = (b^2 - 4ac)/a^2$, and a^2 is a positive number, so $B^2 - 4C$ is positive, negative or zero at the same time as $b^2 - 4ac$.

Thus we have proved this important theorem: (1.12) _has two solutions given by_ (1.14) _if_ $b^2 - 4ac$ _is positive; it has one solution_ $-b/2a$ _if the quantity is_ 0; _and it has no solution if the quantity is negative_. The number $b^2 - 4ac$ is called the _discriminant_ of the quadratic expression on the left side of (1.12).

The two numbers in (1.14) are conventionally written together for convenience: $\frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$. This is merely an abbreviation for (1.14), and no matter of principle is involved.