DYNAMIC SYSTEMS CONTROL

LINEAR SYSTEMS ANALYSIS AND SYNTHESIS

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To
Judy, David, Buzz,
Leigh, and Jeff,
who paid the full price
of the text.

Preface

This text evolved from lecture notes for a two-year course sequence at Purdue and is intended for both self-study and for a dual-level graduate–undergraduate classroom use. The features of this book are briefly mentioned here to help the student and instructor plan their reading. The homework problems appear as exercises at appropriate places in the text as opposed to a collection of problems at the end of each chapter. This, I believe, will better involve the reader in the real time development of the ideas so that the reader may determine his or her level of understanding of each concept before introducing another. In fact, the whole book is arranged in order of increasing sophistication of the engineering concepts. As a result, the topics do not appear in order of sophistication of mathematics. For example, the linear algebra in Chapter 2 involves the simplest engineering concepts but not the simplest mathematics. The building of engineering concepts is our focus, and the mathematical tools are required to fit where needed and are summarized in theorem form for easy reference.

The reader is presumed to have some background in linear algebra and an undergraduate course in the Laplace transform and the single-input/output methods of Evans (root locus), Nyquist, and Bode.

The chapters are arranged as follows. Chapter 2 reviews the needed background in linear algebra and related mathematics. This is a study of linear systems of algebraic equations and is the precursor to our study of linear dynamic systems in Chapters 3 and 4. Chapter 2 should be used as a reference of handy facts and may be skipped or read lightly on first reading. The instructor may choose selected lectures from this chapter to set the stage for notation, but it may be boring for the engineering student to digest *all* the topics of linear algebraic systems before receiving some of the motivations from linear dynamic systems (Chapters 3 and 4). Nonetheless, these are the most fundamental concepts of the book and are presented first to be consistent with the goal of presentation in order of increasing sophistication of the engineering concepts. Education is an *iterative* process, and the student will truly learn Chapter 2 and the power of its applications only by repeated referrals back to Chapter 2 during the study of each of the subsequent chapters of the book. In this way the mathematics is learned in the context of its engineering use. The student may therefore expect that the difficult Chapter 2 will come alive at the end of the book rather than at the beginning.

Chapter 3 develops the equations of motion for some simple dynamic systems by using the language of Chapter 2. This chapter is simply a collection of systems to be used as examples throughout the text as we develop each new concept. There is no methodology taught in this chapter.

The first discussion of the properties of linear dynamic systems begins in Chapter 4, and it is possible to begin instruction here if time is short and if the reader has a good background in linear algebra. After the treatment of the time-varying systems, this chapter discusses the relationships between state variable and transfer function descriptions of dynamic systems. This chapter introduces the deterministic concept of "time correlation" between two vectors, which ties together all remaining chapters.

Chapter 5 focuses on the fundamental controllability and observability properties of linear systems.

Chapter 6 provides the concepts of equivalent systems and develops three different types of equivalence: equivalence with respect to transfer functions, with respect to output correlations, and with respect to a quadratic performance metric called the "cost function." The relationships between these descriptions of a dynamic system are described. Even though this book is entirely deterministic, the introduction of a deterministic version of "time correlation" (a simple integral of two variables) allows the development of mathematical machinery and results that are strikingly similar to the covariance analysis of stochastic systems. This is one advantage of this type of treatment of deterministic systems, even though the student will not realize this advantage until a later course on stochastic processes.

Chapter 7 focuses primarily on two types of stability definitions: Liapunov stability and bounded-input, bounded-output stability. Connections with Chapter 6 will be obvious by the equivalence of stability properties between the original system and its simpler cost-equivalent realization.

Chapter 8 uses the least squares theory of Chapter 2 to solve the simplest of all optimal control problems: quadratic performance measures and linear dynamics. The method chosen for the derivation of these results is based upon the matrix calculus and trace identities of Chapter 2, which allow many different control problems to be viewed within a *common* framework: state feedback control, output feedback control, and dynamic controllers. The instructor will find some new results in Chapter 8, including a completely deterministic theory of optimal dynamic controllers, which eliminates the prior ad hoc practice of putting observers together with optimal state feedback control laws.

Chapter 9 introduces the concept of state estimation from measurement data.

Chapter 10 is a most important chapter because it cautions the reader against misuse of the other nine chapters. Until the tenth chapter the mathematical models of the underlying dynamic system are presumed accurate. Control in the presence of inaccurate models is the necessity of every practical application of control theory. In fact, the limitation in performance of every control design is eventually due to the effect of modeling errors. Hence, great care must be used in the application of

Chapters 1 through 9. Chapter 10 charts the care to be taken, and without Chapter 10 the book would fall short of its goal to provide practical and yet theoretically sound control design techniques. Chapter 10 is an attempt to formally introduce the very fundamental notion that *the modeling problem and the control problem are not independent*. This seemingly innocent concept leads one into a virtual minefield of potential pitfalls in the use of linear systems theory (Chapters 2 through 9). Hence, an introduction to linear systems is incomplete (and perhaps even deceitful) without some elementary introduction to model error concepts and consequences.

The general linear dynamic system is first treated in vector first-order (state) form. However, there are many engineering applications in circuits, acoustics, and mechanics in which the dynamics naturally take on a vector second-order form, $\mathscr{M}\ddot{q} + \mathscr{D}\dot{q} + \mathscr{K}q = f$. Utilizing the special structure of these systems allows many results to be greatly simplified, and these results deserve textbook presentation, in addition to the standard vector first-order theory. For courses that focus only on the basic theory of state space (vector first-order form), the following sections on vector second-order systems can be omitted without loss of continuity: Chapter 3, Sections 5.2.1.4, 5.4.4, 8.7, and 10.2.2.

If this introductory book is used in a transitional course where the students expect to continue their graduate education afterward, the topics will require two semesters and a slower more thoughtful pace in which all theorems are proved. If this book is used in a "terminal" course where students do not expect to continue their controls education, then the instructor may choose to focus on a presentation of the facts without developing the proofs so that the topics can be finished in one semester.

Many Purdue students in A & AE 564 solved numerous example problems. Their feedback influenced the final text in many helpful ways, and without the interaction with them and our joint desire to make that interaction successful, this book would not have been written. It is my pleasure to acknowledge helpful reviews and suggestions from P. Likins, A. Frazho, M. Corless, T. Dwyer, P. Kabamba, and S. Meerkov.

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CHAPTER 1

Introduction

There is only an ill-defined boundary between that body of knowledge called "control theory" and that called "systems analysis." Systems analysis explains *why* a system response behaves the way it does. Control theory deals with modifications to the system which will alter the response in a desirable manner. It is no surprise then that one's degree of success in the latter task (control) depends critically upon the relative completeness of an understanding of the first task (systems analysis). The system dynamics may be modified (to improve the response) in two fundamentally different ways: (i) by modifying parameters of the system or by (ii) modifying the forcing functions in the system differential equations. The second approach (ii) is commonly understood to be the purpose of *control* theory, but in the discussion of systems with equivalent behavior (from *systems analysis*) it may be possible to obtain the same response by technique (i) or by a simpler combination of (i) and (ii). A suitable introduction to systems analysis is therefore desirable prior to the introduction of control methods. Chapters 2–7 focus on system analysis and Chapters 7–10 introduce control design.

To illustrate these points consider the rocket depicted in Fig. 1.1 whose linear dynamics are described by these differential equations derived in Chapter 3:

$$
\begin{aligned}
J\ddot{\alpha} - \rho L V^2 \alpha + \rho L V \dot{r}_x &= F D \theta, \\
m\ddot{r}_x + \rho V \dot{r}_x - F\alpha &= -F\theta, \\
m\ddot{r}_y &= F - \rho V^2 - mG,
\end{aligned}
\tag{1.1}
$$

$\dot{r}_x$	Horizontal speed
$\dot{r}_y$	Vertical speed (total speed $V = \sqrt{\dot{r}_x^2 + \dot{r}_y^2}$)
α	Attitude of vehicle with respect to inertial space
θ	Gimbal angle of rocket engine
F	Magnitude of thrust (assumed constant)
D	Distance from mass center to engine gimbal

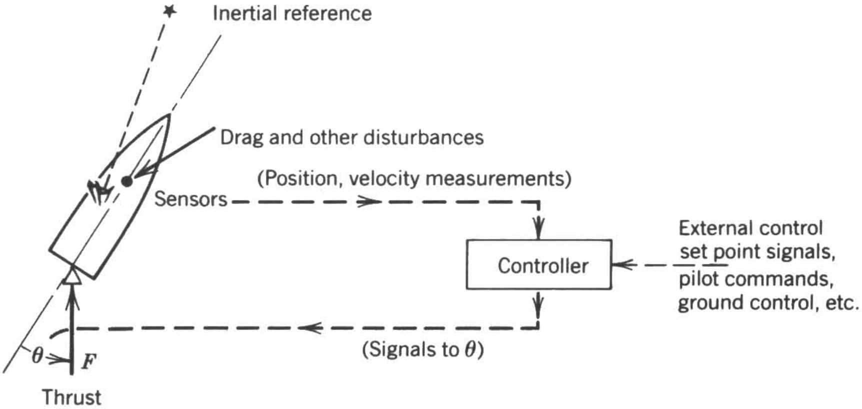

Figure 1.1 Feedback Control Concepts

L	Distance from mass center to aerodynamic pressure center
G	Gravity constant
ρV^2	Magnitude of drag force
m	Vehicle mass
J	Moment of inertia

The typical control problem is find a forcing function $\theta(t)$ so that the responses $\alpha(t)$, $r_x(t)$, $r_y(t)$ are acceptable. If $\theta(t)$ is specified as a function of time, then $\theta(t)$ is called an *open loop* control policy. If θ is specified as a function of the system responses, then $\theta(\alpha, \dot{\alpha}, r_x, \dot{r}_x, r_y, \dot{r}_y)$ is called a feedback, or a *closed loop* control policy. The physical and mathematical device that computes the desired θ, given the responses $(\alpha, \dot{\alpha}, r_x, \dot{r}_x, r_y, \dot{r}_y)$, is called the *controller* in Fig. 1.1.

A study of the physical sciences (electricity, mechanics, thermodynamics, chemistry, etc.) prepares the student to apply known physical laws in the derivation of mathematical models of the physical phenomenon such as shown in Fig. 1.1. This is the substance of the undergraduate engineering experience: learning of *how to model* physical systems; whereas the study of control is concerned with *what to do with the model* after it is available. This popular view is much too narrow, of course. It presumes that the modeling problem and the control problem are separable. They are not. One cannot know what level of detail is required in the model prior to knowledge of the accuracy required of the controlled performance and knowledge of the nature of the forces (or control inputs) required to achieve this performance. For example, one cannot know whether the coupling between the translational dynamics (involving $\dot{r}_x$) and the rotational dynamics (involving α) in equations (1.1) is important prior to knowledge of the precision required of the controlled performance and prior to knowledge of the forcing function $\theta(t)$ in (1.1). In fact, for the same reasons one cannot even know *a priori* whether the rigid body in (1.1) and Fig. 1.1 should be modeled as an elastic structure. This depends upon the relative magnitude (and frequency content) of the applied forces. Yet in feedback control the forces applied from the control policy for regulating θ *depend* upon the model chosen. The control problem is to find an appropriate θ, *given* the model (1.1). Hence, the model that is most appropriate for the control design and the control design itself must evolve in an *iterative* fashion. (Do not fall in love with your model!) Control theory and practice still have not, and cannot, produce a fool-proof procedure for accomplishing this [1.1], but the value of seeking a sound theoretical base for the engineer is to help accelerate the convergence of these iterations, with a blend of theory (in the text) and the engineer's judgment and insight (developed on the job). In the first nine chapters of this book we too shall make the traditional assumptions of absolute correctness of our model and the separability of model development from the control design problem. The flaw in this unwritten but commonly evoked *separation* principle will not be corrected until Chapter 10. Yet it is important to learn the traditional wisdom of control theory (Chapters 2 through 9) in full light of its premises, which we here state.

Suppose the horizontal speed of the rocket in (1.1) were truly negligible, then the rotation is governed by

$$J\ddot{\alpha} - \rho L V^2 \alpha = FD\theta \tag{1.2}$$

and the Laplace transform of this equation leads to the transfer function $H(s)$:

$$\alpha(s) = \left[\frac{FD}{Js^2 - \rho L V^2}\right]\theta(s) = H(s)\theta(s), \tag{1.3}$$

where J, ρ, F, and D are positive quantities. The two methods of altering system response mentioned above were (i) modifying parameters of the system and (ii) choosing the forcing function, $\theta(s)$ in this case. Method (i) may be illustrated by changing L, which is positive if the center of pressure is *forward* of the center of mass and negative if the center of pressure is *aft* of the center of mass. The parameter L may be reduced or made negative by making the nose pointed and the tail section larger in diameter, or by adding fins on the tail of the rocket. The center of mass can be made even more forward of the center of pressure (making L more negative) by moving heavy objects (payload) as far forward as possible. Thus, the native Indian's design of an arrow [with rock (heaviest item) on the nose and feathers (large area) as fins on the tail] makes L as negative as possible. This native wisdom is verified by noting that this changes the poles of the transfer function $H(s)$ from the right half-plane ($\lambda = \pm\sqrt{\rho L V^2/J}$ when $L > 0$) to the imaginary axis ($\lambda = \pm j\sqrt{\rho(-L)V^2/J}$ when $L < 0$). This is an improvement since it changes the response from unstable to stable. In fact, aerodynamic damping effects ignored here in (1.1) would actually place the poles slightly inside the left half-plane yielding asymptotic stability of the open-loop system [$\theta(t) \to 0$]. That is, α returns to zero from some nonzero initial condition. However, the price paid in approach (i) is possibly to increase the weight of the rocket, which in turn reduces payload capability. Hence, one possible advantage of feedback control [method (ii) above] is the modification of system behavior without adding the weight or other undesirable features that a structural parameter change might require. Clearly, a trade-off is evident between increasing structural design modifications (parameter changes), which beef up the structure weight, and increasing control sophistication, which would be required as the basic structure degenerates to "cheapest" to build, lightest in weight. The arguments taken to the limit of this latter extreme would yield an absurd design. The control system would fly all the payload, engine, and instrument components in formation with hardly any structure holding them together at all, all of the required interaction forces to hold things together being provided by the addition of multiple control forces (besides the control of θ). Hence, even though this example points to an absurd extreme, it is nonetheless true that as the control requirements become more stringent in modern systems, it is usually necessary to add more "actuators" or control variables since those detailed things we wish to control can be "uncontrollable" with a single actuator. The time domain methods readily accommodate multiple inputs and outputs, and this is the focus of this text, whereas transform methods, equation (1.3), more readily treat the single-input/output class of systems.

Having anticipated that multiple actuators and sensors will be needed in a research frontier pressing for better performance capabilities of a controlled system, we shall become aware of fundamental limitations and dangers in pressing too far. As the controller becomes more complex, it may become less reliable, both from the possibility of failures and from the reliance upon increasing detail of mathematical models (upon which the control policy is based). Chapter 10 sorts out some of these difficulties and guards against a misuse of the first theory we learn in Chapters 2 through 9 (where correctness of the model is presumed).

The classical control tools of Bode, Nyquist, and Evans [1.2–1.4], all developed prior to 1950, are basically graphical in nature, with the following underlying design strategy: "Design for stability, then check for performance." These tools function in the frequency domain. The time domain tools of state space optimal control developed rapidly two decades later, with an eye on the mean squared performance: "Design for performance, then check for stability." These techniques are numerical rather than graphical. Thus, the early time domain design philosophy and the frequency domain design philosophy were *opposite*, and complementary insights are available by studying both. Of course, this is an oversimplification of the methods and their power, but nonetheless it is important for a student in control to enthusiastically embrace both points of view. This text focuses primarily on time domain methods.

While time domain techniques began their rapid development in the 1960s with the vector first-order (state) form of differential equations, it was Sir William Rowan Hamilton in 1835 who introduced state form equations with the presentation of his theory of "generalized momenta" [1.5]. This replaced the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathscr{T}}{\partial \dot{q}_i} - \frac{\partial \mathscr{T}}{\partial q_i} = Q_i, \qquad i = 1, \ldots, N,\ T = \text{kinetic energy},\ Q_i = \text{generalized forces}, \tag{1.4}$$

which always resulted in systems of equations of a vector second-order form,

$$\mathcal{M}\ddot{\mathbf{q}} + \mathcal{D}\dot{\mathbf{q}} + \mathcal{K}\mathbf{q} = \mathbf{f}, \tag{1.5}$$

by a set of generalized momenta equations,

$$p_i \triangleq \frac{\partial \mathcal{T}}{\partial \dot{q}_i}, \tag{1.6}$$

which always resulted in systems of equations of a vector first-order form,

$$\dot{\mathbf{p}} = \mathbf{A}\mathbf{p} + \bar{\mathbf{f}}. \tag{1.7}$$

For a century, dynamicists largely shunned Hamilton's generalized momenta equations (1.7) until modern computers were available to provide practical computations of solutions. His first-order "Hamilton's canonical" equations,

$$\dot{p}_i \triangleq -\frac{\partial \mathcal{H}}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i},$$

where $\mathcal{H} \triangleq \mathbf{p}^T\mathbf{q} - \mathcal{L}$, $\mathcal{L} \triangleq \mathcal{T} - \mathcal{U} + \mathcal{F}$, $\mathcal{U}$ = potential energy, and $\mathcal{F}$ = work done by nonconservative forces, became popular [1.6] many years after their introduction. Thus, it is no coincidence that the modern and rapid development of state space techniques closely followed the development of computers and efficient numerical methods.

If equations (1.1) were placed in the vector second-order from (1.5), we would have

$$\begin{bmatrix} J & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{pmatrix} \ddot{\alpha} \\ \ddot{r}_x \\ \ddot{r}_y \end{pmatrix} + \begin{bmatrix} 0 & \rho LV & 0 \\ 0 & \rho V & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{\alpha} \\ \dot{r}_x \\ \dot{r}_y \end{pmatrix} + \begin{bmatrix} -\rho LV^2 & 0 & 0 \\ -F & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \alpha \\ r_x \\ r_y \end{pmatrix}$$

$$= \begin{bmatrix} FD \\ -F \\ 0 \end{bmatrix}\theta + \begin{bmatrix} 0 \\ 0 \\ F - \rho V^2 - mG \end{bmatrix}. \tag{1.8}$$

And if placed in the vector first-order form (state form), we would have, from the generalized momenta equations,

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 1/J & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 1/m \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \qquad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \triangleq \begin{bmatrix} \alpha \\ r_x \\ r_y \end{bmatrix}, \tag{1.9a}$$

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & -\rho LV/m & 0 \\ 0 & -\rho V/m & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} \rho LV^2 q_1 + FD\theta \\ Fq_1 - F\theta \\ F - \rho V^2 - mG \end{bmatrix}, \tag{1.9b}$$