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Differential Equations

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Translated from the Russian
by A. B. Sossinskij

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Foreword

The proposed book is one of a series called "A Course of Higher Mathematics and Mathematical Physics" edited by A. N. Tikhonov, V. A. Ilyin and A. G. Sveshnikov.

The book is based on a lecture course which, for a number of years now has been taught at the Physics Department and the Department of Computational Mathematics and Cybernetics of Moscow State University. The exposition reflects the present state of the theory of differential equations, as far as it is required by future specialists in physics and applied mathematics, and is at the same time elementary enough.

An important part of the book is devoted to approximation methods for the solution and study of differential equations, e.g. numerical and asymptotic methods, which at the present time play an essential role in the study of mathematical models of physical phenomena. Less attention is paid to the integration of differential equations in elementary functions than to the study of algorithms on which numerical solution methods of differential equations for computers are based.

The reader will become acquainted with various methods for the numerical solution of initial values as well as boundary value problems, and with such fundamental notions of the theory of numerical methods as the convergence of difference schemes, approximation and stability. The chapter concerned with asymptotic methods contains, in particular, information on the so-called method of singular perturbations (the averaging method), the method of boundary functions, the WKB method; these methods have rapidly developed in the last decade in connection with the requirements of such branches of physics and technology as the theory of automatic control, hydrodynamics, quantum mechanics, kinetics, the theory of non-linear oscillations, etc.

The English translation of the book includes important changes from the first Russian edition. The most important ones concern the existence and uniqueness theorem for the solution of initial value problems. The new proofs are based on the method of differential inequalities. These same ideas are applied in the study of the dependence on parameters of solutions of differential equation systems. The use of the method of differential inequalities considerably simplifies the proofs, makes them more uniform and allows us to state the results in more general form.

The manuscript of the book was read through by E. A. Grebennikov and L. D. Kudryavtsev, who made a number of important remarks. Inestimable assistance in the preparation of the manuscript for publication was rendered by B. I. Volkov. To all of them the authors express their sincere gratitude.

The authors

Contents

Chapter I
Introduction

§1. The Concept of a Differential Equation

The present book is concerned with differential equations, i. e. relations between an unknown function, its derivatives and independent variables. Equations containing derivatives with respect to several independent variables are called *partial differential equations*. Equations containing derivatives with respect to only one of the independent variables are called *ordinary differential equations*. This book mainly deals with the properties and solution methods of ordinary differential equations; only the last chapter is devoted to certain special classes of partial differential equations.

The independent variable with respect to which the derivatives in an ordinary differential equation are taken is usually denoted by the letter x (or the letter t, since time often plays the role of the independent variable). The unknown function is denoted by $y(x)$.

An ordinary differential equation may be written as a relation of the form

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right)=0. \tag{1.1}$$

Besides the unknown function, its derivatives with respect to the independent variable x and the independent variable x itself, equation (1.1) may involve additional variables $\mu_1, \ldots, \mu_k$. In this case we say that the unknown function depends on the variables $\mu_1, \ldots, \mu_k$ as parameters.

The order of the highest order derivative contained in equation (1.1) is known as the *order of the equation*. A first order equation is of the form

$$F\left(x, y, \frac{dy}{dx}\right)=0 \tag{1.2}$$

and is a relation between three expressions – the unknown function, its derivative and the independent variable. It is often possible to write this relation in the form

$$\frac{dy}{dx}=f(x, y). \tag{1.3}$$

Equation (1.3) is called a first order equation *resolved* with respect to the derivative. We shall begin our study of the theory of ordinary differential equations with equation (1.3).

Along with differential equations (1.1)–(1.3) with one unknown function, the theory of ordinary differential equations deals with systems of equations. A system of first order equations resolved with respect to the derivatives

$$\frac{dy_i}{dx}=f_i(x, y_1, \ldots, y_n), \qquad (i=1, \ldots, n), \tag{1.4}$$

is called a *normal system*. Introducing vector functions

$$\boldsymbol{y}=(y_1, \ldots, y_n), \quad \boldsymbol{f}=(f_1, \ldots, f_n),$$

we may write the system (1.4) in vector form

$$\frac{d\boldsymbol{y}}{dx}=\boldsymbol{f}(x, \boldsymbol{y}). \tag{1.5}$$

It is easy to see that the n-th order equation (1.1), resolved with respect to the highest order derivative

$$\frac{d^n y}{dx^n}=f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right), \tag{1.6}$$

may be reduced to a normal system. Indeed, introducing the notation

$$y(x)=y_1(x), \frac{dy}{dx}=\frac{dy_1}{dx}=y_2(x), \ldots, \frac{d^{n-1}y}{dx^{n-1}}=\frac{dy_{n-1}}{dx}=y_n(x). \tag{1.7}$$

and using the obvious equality

$$\frac{d^n y}{dx^n}=\frac{dy_n}{dx},$$

we obtain the normal system

$$\begin{aligned} \frac{dy_1}{dx}&=y_2, \\ &\cdots\cdots\cdots \\ \frac{dy_{n-1}}{dx}&=y_n, \\ \frac{dy_n}{dx}&=f(x, y_1, \ldots, y_n). \end{aligned} \tag{1.8}$$

corresponding to equation (1.6).

In equations (1.1)–(1.5), the independent variable will be assumed real. The unknown functions may be real-valued as well as complex-valued functions of a real variable.

Obviously, if

$$y(x) = \bar{y}(x) + i\bar{\bar{y}}(x),$$

where $\bar{y}(x)$ and $\bar{\bar{y}}(x)$ are the real and imaginary parts of the function $y(x)$ respectively, then equation (1.3) is equivalent to the following system of ordinary differential equations for real-valued functions

$$\frac{d\bar{y}}{dx} = \text{Re } f(x, \bar{y}, \bar{\bar{y}}), \qquad \frac{d\bar{\bar{y}}}{dx} = \text{Im } f(x, \bar{y}, \bar{\bar{y}}).$$

The solution of the system of differential equations (1.4) is, by definition, any family of functions which satisfy the equations identically. As a rule, and this will be clear from examples given below (see §2), if a differential equation is soluble, then it has an infinite set of solutions. The procedure of finding the solutions is known as *integration of differential equations.*

Any solution $y_i(x)$ $(i = 1, \ldots, n)$ of the system (1.4) may be interpreted geometrically as a curve in the $(n+1)$-dimensional space of the variables $x, y_1, \ldots, y_n$; this curve is called the *integral curve*. The subspace of the variables $y_1, \ldots, y_n$ is called the *phase space*, while the projection of the integral curve on the phase space is the *phase trajectory*.

Equation (1.4) determines a direction given by the vector $\tau = (1, f_1, \ldots, f_n)$ at every point of the domain D. Such a domain in space, with a direction given at each point, is said to be a *direction field.* The integration of the system of equations (1.4) may be interpreted geometrically as finding curves whose tangents at each point coincide with the direction τ determined by the given field of directions at that point.

As we pointed out above, differential equations have an infinite set of solutions in general. Therefore, when we integrate the system (1.4), we will find an infinite set of integral curves contained in the domain where the right-hand sides of the system (1.4) are defined. In order to distinguish an individual integral curve in the set of all solutions, thus specifying a so-called *particular solution* of the system (1.4), we must impose additional conditions. In many cases such additional conditions are the *initial conditions*

$$y_i(x_0) = y_i^0 \qquad (i = 1, \ldots, n), \tag{1.9}$$

which determine a point of $(n+1)$-dimensional space of the variables $x, y_1, \ldots, y_n$ through which the required integral curve passes.

The problem of integrating system (1.4) with initial conditions (1.9) is known as the *initial value problem* or *Cauchy problem.*

In the simplest case of one equation

$$\frac{dy}{dx} = f(x, y), \tag{1.10}$$

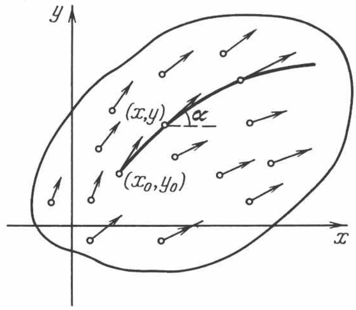

Fig. 1

the function $f(x,y)$ determines a direction field in the domain D (of the (x,y)-plane) where the right-hand side of (1.10) is defined. This direction field is given at each point of the domain D by the vector $\tau(x,y)$ with slope $f(x,y)$ ($\tan\alpha = f(x,y)$) (Fig. 1).

In order to solve the initial value problem with the condition $y(x_0)=y_0$ in this case, we must construct, in the domain D, the integral curve $y=y(x)$ which starts at the initial point (x_0, y_0) and is tangent to the vector τ of slope $f(x,y)$ at every one of its points (x,y).

Theorem 1.1 (*Chaplygin's theorem on differential inequalities*). *If for* $x\in[x_0, X]$ *there exists a solution of the initial value problem*

$$\frac{dy}{dx}=f(x,y),\ y(x_0)=y_0, \tag{1.11}$$

and, if $z(x)$ *is a continuous and continuously differentiable function on* $[x_0,X]$ *satisfying*

$$\begin{gathered}\frac{dz}{dx}<f(x,z),\ x\in[x_0,X],\\ z(x_0)<y_0,\end{gathered} \tag{1.12}$$

then we have the inequality

$$z(x)<y(x),\ x\in(x_0,X]. \tag{1.13}$$

Indeed, by the assumptions of the theorem, inequality (1.13) is satisfied at the point x_0. Therefore, by the continuity of $y(x)$ and $z(x)$, it is satisfied also in some neighbourhood to the right of the point x_0. Assume that $x_1\in[x_0, X]$ is the nearest point to x_0 in which the inequality (1.13) fails to hold, i. e. $z(x_1)=y(x_1)$. Geometrically, this means that the curves $z(x)$ and $y(x)$ intersect or are tangent when $x=x_1$. But then we must have $\frac{dz}{dx}(x_1)\geq f(x_1, y(x_1))$, which contradicts (1.12). The theorem is proved.

Now for some remarks – which will be important later – about the theorem on differential inequalities just proved above.

Remarks. 1. We assumed that $z(x_0)<y_0$, but the theorem remains valid if $z(x_0)=y_0$. In this case, the existence of a neighbourhood (to the right of the point x_0) in which the inequality (1.13) holds follows from the fact that

$$\frac{dz}{dx}(x_0)<f(x_0,z(x_0))=f(x_0,y_0)=\frac{dy}{dz}(x_0)$$

The rest of the argument is exactely the same as in the case $z(x_0)<y_0$

2. If the function $z(x)$ satisfies the inequalities

$$\frac{dz}{dx}\geq f(x,z),\ x\in[x_0,X]$$

$$z(x_0)=y_0,$$

then the sign of the inequality in (1.13) should also be changed to the opposite one.

3. The theorem remains valid in the case when $z(x)$ is piecewise differentiable on $[x_0,X]$ and the inequality (1.12) is satisfied for the limiting values of the derivative $\frac{dz}{dx}$ at the points of discontinuity.

In order to answer a number of questions, it is convenient to reduce certain problems concerning differential equations to corresponding problems about integral equations.

Lemma 1.1. *Suppose $f(x,y)$ is a continuous function of the point (x,y) in some rectangle $D=\{|x-x_0|\leq a,\ |y-y_0|\leq b\}$. Then the initial value problem*

$$\frac{dy}{dx}=f(x,y),\ y(x_0)=y_0 \tag{1.14}$$

is equivalent to the integral equation

$$y(x)=y_0+\int_{x_0}^{x}f(\xi,y(\xi))\,d\xi. \tag{1.15}$$

Proof. Suppose that there exists a solution $y(x)$ of the initial value problem on the segment $|x-x_0|\leq a$ and we have $y_0-b\leq y(x)\leq y_0+b$ (these inequalities mean that for $|x-x_0|\leq a$ the integral curve is located within the domain D where $f(x,y)$ is continuous). Substituting $y(x)$ into equation (1.14), we obtain an identity. Integrating this identity from x_0 to $x\in[x_0-a,x_0+a]$ and using the initial condition $y(x_0)=y_0$, we obtain (1.15). Therefore, the solution of the initial value problem (1.14) satisfies the integral equation (1.15). On the other hand, if there exists a continuous solution of the integral equation (1.15) – the function $y(x)$, where $y_0 - b \leq y(x) \leq y_0 + b$, then, by the continuity of $f(x, y)$ and the continuity of the function $f(\xi, y(\xi))$ as a function of ξ which follows, the integral in the right-hand side of (1.15) is a continuously differentiable function of ξ. Therefore, the left-hand side of (1.15), i.e. the function $y(x)$, possesses a continuous derivative and this derivative equals $f(x, y(x))$, so that $y(x)$ is a solution of equation (1.14). The fact that the initial condition is satisfied can be checked directly. The lemma is proved.

Remark. A similar theorem on equivalence holds also for systems of differential equations, i.e. for the problem (1.4), (1.9).

One usually considers systems (1.4) whose right-hand sides are continuous in some domain D where the unknown functions y_i and the independent variable x vary. Obviously, in this case the solution will be a continuously differentiable function. However, in applications, one often meets with equations whose right-hand sides have discontinuities in the variable x (for example, in the description of instantaneously applied forces or concentrated forces), therefore these solutions will also possess discontinuous derivatives. It is then natural to consider, as solutions of (1.4), continuous functions $y_i(x)$ with piecewise continuous derivatives. In substituting them into the equation, they are to be differentiated everywhere except at points of discontinuity and points where derivatives do not exist. It is natural to call such solutions generalised solutions.

Lemma 1.1 remains valid in the case when the function $f(x, y)$ is a piecewise continuous function of the variable x. Then the integral equation (1.15) has a continuous solution $y(x)$ which is a piecewise differentiable function of x. This solution satisfies equation (1.14) in those intervals where the function $f(x, y)$ is continuous.

There are other ways of specifying supplementary conditions which determine a particular solution of the system (1.4). Among them let us note: the so-called boundary value problems, in which a particular solution is determined by certain supplementary conditions at some points of its domain of definition; the eigenvalue problem, which involves determining certain parameters appearing in the equation so that particular solutions exist and satisfy some supplementary requirements; and the problem of finding periodic solutions and a number of other specifications uniquely determining the required particular solution of the equation.

§ 2. Physical Problems Leading to Differential Equations

In this section we shall present some typical problems in physics and mechanics whose study, by means of mathematical models, leads to the investigation of differential equations.

1. Radioactive disintegration. The physical law which describes the process of radioactive disintegration states that the rate of disintegration is negative and proportional to the amount of non-disintegrated matter at the given moment of time. The coefficient of proportionality α, which is a constant characterising the given type of matter, does not depend on time and is known as the disintegration coefficient. The mathematical expression of the law of radioactive disintegration is

$$\frac{dm}{dt} = -\alpha m(t), \tag{1.16}$$

where $m(t)$ is the amount of non-disintegrated matter at the moment of time t. This relation is a first order differential equation resolved with respect to the derivative.

It is easy to check (by direct substitution) that there is solution of (1.16) of the form

$$m(t) = Ce^{-\alpha t}, \tag{1.17}$$

where C is an arbitrary constant, which may be determined from some supplementary condition, for example, from the initial condition $m(t_0) = m_0$ specifying the amount of matter at the initial moment t_0. A particular solution of the corresponding initial value problem is

$$m(t) = m_0 e^{-\alpha(t-t_0)}. \tag{1.18}$$

One of the important physical characteristics of the process of radioactive disintegration is the half-life, the time T needed for the amount of non-disintegrated matter to decrease by half. It follows from (1.18) that

$$\frac{m_0}{2} = m_0 e^{-\alpha T},$$

so that we get

$$T = \frac{1}{\alpha} \log 2 \tag{1.19}$$

Note that equation (1.16) is the mathematical model not only of the process of radioactive disintegration, but also of many other processes of splitting or multiplication characterized by the fact that the rate of splitting (multiplication) is proportional to the amount of matter at the given moment of time, the coefficient of proportionality being a certain constant for the given process. As we shall see, a typical way of setting the problem for this class of equations is the initial value problem (the Cauchy problem).

2. The motion of a system of particles. The mathematical models of motion for a system of particles of mass $m_i (i = 1, \ldots, N)$, usually accepted in theoretical mechanics, are the equations of motion which follow from Newton's second law:

$$m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} = \boldsymbol{F}_i\left(t, \boldsymbol{r}_j, \frac{d\boldsymbol{r}_j}{dt}\right) \qquad (i,j=1,\dots,N). \tag{1.20}$$

Here m_i are the particle masses, which do not change in time, $\boldsymbol{r}_i$ are the radius-vectors of the particles, and $\boldsymbol{F}_i$ is the force vector acting on the i-th particle and depending, in general, on time, on the coordinates of the i-th particle and the position of the particles of the system, as well as their velocities. The system (1.20) is a system of N vector equations of the second order. If the masses of the particles do not change in the process of motion, then, by denoting the Cartesian coordinates of the radius-vector $\boldsymbol{r}_i$ by x_i, y_i, z_i and introducing new variables $v_{ix} = \frac{dx_i}{dt}$, $v_{iy} = \frac{dy_i}{dt}$, $v_{iz} = \frac{dz_i}{dt}$ (the components of the velocity vector of the i-th particle), we can write (1.20) in the form of a normal system of first order equations

$$\frac{dx_i}{dt} = v_{ix}, \quad \frac{dy_i}{dt} = v_{iy}, \quad \frac{dz_i}{dt} = v_{iz},$$
$$\frac{dv_{ix}}{dt} = \frac{1}{m_i} F_{ix}, \quad \frac{dv_{iy}}{dt} = \frac{1}{m_i} F_{iy}, \quad \frac{dv_{iz}}{dt} = \frac{1}{m_i} F_{iz}. \tag{1.21}$$

The difficulty in integrating system (1.21) is mainly determined by the form of the right-hand sides, i. e. the functional dependence of the components of the force vectors on the variables $t, x_i, y_i, z_i, v_{ix}, v_{iy}, v_{iz}$. In many cases it is possible to obtain the value of a particular solution of the system with a given degree of precision only by means of numerical methods, using computers. A typical problem for the system (1.21) is the initial value problem, which consists in determining the trajectory of the particles when their position and velocity at the initial moment of time t_0 are given,

$$\boldsymbol{r}_i(t_0) = \boldsymbol{r}_i^0, \quad \boldsymbol{v}_i(t_0) = \boldsymbol{v}_i^0, \tag{1.22}$$

the right-hand sides being given functions (given external forces acting on the system and interaction forces between the particles themselves). Another typical problem for the system (1.21) is the boundary value problem which consists in determining the trajectory passing through the given initial and terminal points in the phase space. This is the problem that must be solved when we calculate the trajectory of a spacecraft leaving the Earth for the Moon or for some planet.

In a number of cases, other means of specifying a particular solution of the system (1.21) are considered.

An important special case of the system (1.20) is the oscillation equation of the physical pendulum. Usually, by a physical pendulum one means an absolutely rigid body, which can rotate, under the action of the force of gravity,

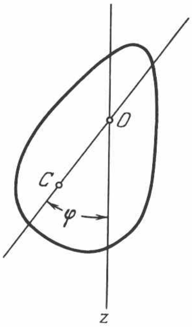

Fig. 2

about a motionless horizontal axis not passing through the centre of mass C (Fig. 2).

Consider a section of the solid by a plane perpendicular to the rotation axis and passing through the centre of mass. Denote the intersection point of the axis and the plane by O. Obviously, the position of the physical pendulum at any moment of time may be characterized by the angle φ made by the line OC with the vertical axis z passing through the point O. To deduce the equation of motion, let us use Newton's second law, as applied to rotational motion (the angular acceleration is proportional to the principal moment of the exterior forces). Then, ignoring friction forces, we obtain

$$I\frac{d^2\varphi}{dt^2} = -mgd \sin \varphi, \tag{1.23}$$

where I is the moment of inertia of the solid with respect to the axis of rotation and d is the distance from the point O to the centre of mass C.

The general equation (1.23) of oscillation of the physical pendulum is non-linear. In the case of small oscillations, restricting ourselves to the first term of the expansion of the function $\sin\varphi$, we get

$$\frac{d^2\varphi}{dt^2} + \omega^2\varphi = 0, \tag{1.24}$$

where ω^2 denotes the quotient $\omega^2 = mgd/I$. Obviously, from the point of view of dimension, $[\omega] = \sec^{-1}$, which justifies this notation. Note that in the case of equation (1.24) the returning force is proportional to the displacement from the position of equilibrium.

It is easy to check (by direct substitution) that equation (1.24) possesses periodic solutions of frequency ω

$$\varphi(t) = A \cos \omega t + B \sin \omega t, \tag{1.25}$$

where A and B are arbitrary constants determining the amplitude of periodic oscillations.

If we take into account friction forces proportional to angular velocity, equation (1.24) will become an equation of the form

$$\frac{d^2\varphi}{dt^2}+\alpha\frac{d\varphi}{dt}+\omega^2\varphi=0. \tag{1.26}$$

As will be shown later (see Chapter 3) equation (1.26) determines damped oscillations.

3. The transfer equations. Suppose that air flows along a pipe of constant perpendicular section, whose axis coincides with the x axis, the velocity along the axis of the pipe at the point x at time t being a given function $v(x, t)$. Suppose the air carries a certain amount of matter whose linear density in the section of the pipe with coordinate x at time t will be denoted by $u(x, t)$. In the transfer process, some matter settles on the inner walls of the pipe. We shall assume that the density of distribution of the matter which settles is given by the expression $f(x,t)\ u(x,t)$ ($f(x,t)$ is a given function), i.e. is proportional to the concentration of matter; this may be viewed as a linear approximation (to a more complicated law) valid for sufficiently small u. This means that the amount of matter settling on the part of the pipe located between the sections x and $x+\Delta x$ during time $[t, t+\Delta t]$ is given by the expression

$$\int_x^{x+\Delta x}\int_t^{t+\Delta t} f(\xi,\tau)\,u(\xi,\tau)\,d\xi d\tau.$$

To obtain a differential equation with respect to u, consider the balance of matter in the domain between the sections x and $x+\Delta x$. The process of diffusion will not be taken into consideration, which is natural if the velocity v is sufficiently large.

During the period of time Δt, the change in the amount of matter in the domain under consideration equals

$$\int_x^{x+\Delta x} [u(\xi, t+\Delta t)-u(\xi, t)]\,d\xi.$$

This change is determined, firstly, by the difference of flows of matter: the amount which flows in through the section x and equals

$$\int_t^{t+\Delta t} v(x,\tau)\ u(x,\tau)\ d\tau$$

and the amount which flows out through the section $x+\Delta x$ and equals

$$\int_t^{t+\Delta t} v(x+\Delta x, \tau)\, u(x+\Delta x, \tau)\, d\tau,$$

and, secondly, by the decrease of the amount of matter due to settling on the walls of the pipe, which equals

$$-\int_x^{x+\Delta x}\int_t^{t+\Delta t} f(\xi, \tau)\, u(\xi, \tau)\, d\xi d\tau.$$

Thus the law of conservation of matter gives

$$\int_x^{x+\Delta x} [u(\xi, t+\Delta t)-u(\xi, t)]\, d\xi = \int_t^{t+\Delta t} [v(x, \tau)\, u(x, \tau)-v(x+\Delta x, \tau)u(x+\Delta x, \tau)] \cdot d\tau - \int_x^{x+\Delta x}\int_t^{t+\Delta t} f(\xi, \tau)\, u(\xi, \tau)\, d\xi d\tau. \quad (1.27)$$

Using the mean-value theorem of the differential calculus for the expressions under the integral signs, assuming that continuous partial derivatives of the given functions exist, and calculating the integrals by the mean value theorem of the integral calculus, we obtain

$$\frac{\partial u}{\partial t}(x^*, t)\Big|_{t=t^*} \Delta x\Delta t = -\frac{\partial}{\partial x}(v(x, t^{**})u(x, t^{**})\Big|_{x=x^{**}} \Delta x\Delta t - f(x^{***}, t^{***})\, u(x^{***}, t^{***})\, \Delta x\Delta t, \quad (1.28)$$

where $x^*, x^{**}, x^{***}, t^*, t^{**}, t^{***}$ are certain points on the segments $[x, x+\Delta x]$, $[t, t+\Delta t]$ respectively. Dividing the relation (1.28) by $\Delta x\Delta t$ and assuming that Δx and Δt tend to zero, in view of the continuity of all terms of relation (1.28), we finally obtain the equation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(uv)+fu=0, \quad (1.29)$$

or

$$\frac{\partial u}{\partial t}+v(x, t)\,\frac{\partial u}{\partial x}+c(x, t)\, u=0, \quad (1.30)$$

where

$$c(x, t)=\frac{\partial v}{\partial x}(x, t)+f(x, t). \quad (1.31)$$

Equation (1.30) is a first order partial differential equation. The following problem, for example, may be considered for this equation. Suppose we know