AMERICAN MATHEMATICAL SOCIETY COLLOQUIUM PUBLICATIONS

Volume XXXVI

AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS
VOLUME XXXVI

TOPOLOGICAL DYNAMICS

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PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
80 WATERMAN STREET, PROVIDENCE, R. I.
1955

PREFACE

By *topological dynamics* we mean the study of transformation groups with respect to those topological properties whose prototype occurred in classical dynamics. Thus the word "topological" in the phrase "topological dynamics" has reference to mathematical content and the word "dynamics" in the phrase has primary reference to historical origin.

Topological dynamics owes its origin to the classic work of Henri Poincaré and G. D. Birkhoff. It was Poincaré who first formulated and solved problems of dynamics as problems in topology. Birkhoff contributed fundamental concepts to topological dynamics and was the first to undertake its systematic development.

In the classic sense, a dynamical system is a system of ordinary differential equations with at least sufficient conditions imposed to insure continuity and uniqueness of the solutions. As such, a dynamical system defines a (one-parameter or continuous) flow in a space. A large body of results for flows which are of interest for classical dynamics has been developed, since the time of Poincaré, without reference to the fact that the flows arise from differential equations. The extension of these results from flows to transformation groups has been the work of recent years. These extensions and the concomitant developments are set forth in this book.

Part One contains the general theory. Part Two contains notable examples of flows which have contributed to the general theory of topological dynamics and which in turn have been illuminated by the general theory of topological dynamics.

In addition to the present Colloquium volume, the only books which contain extensive related developments are G. D. Birkhoff [2, Chapter 7], Niemytzki and Stepanoff [1, Chapter 4 of the 1st edition, Chapter 5 of the 2nd edition] and G. T. Whyburn [1, Chapter 12]. The contents of this volume meet but do not significantly overlap a forthcoming book by Montgomery and Zippin.

The authors wish to express their appreciation to the American Mathematical Society for the opportunity to publish this work. They also extend thanks to Yale University and the Institute for Advanced Study for financial aid in the preparation of the manuscript. The second named author extends to the American Mathematical Society his thanks for the invitation to give the Colloquium Lectures in which some aspects of the subject were discussed. Some of his work has been supported by the United States Air Force through the Office of Scientific Research of the Air Research and Development Command.

PHILADELPHIA, PENNSYLVANIA
NEW HAVEN, CONNECTICUT
July, 1954

CONVENTIONS AND NOTATIONS

Each of the two parts of the book is divided into sections and each section into paragraphs. Cross references are to paragraphs. 4.6 is the sixth paragraph of section 4. In general, a paragraph is either a definition, lemma, theorem or remark. A "remark" is a statement, the proof of which is left to the reader. These proofs are not always trivial, however.

References to the literature are, in general, given in the last paragraph of each section. Numbers in brackets following an author's name refer to the bibliography at the end of the book. Where there is joint authorship, the number given refers to the article or book as listed under the first named author.

An elementary knowledge of set theory, topology, uniform spaces and topological groups is assumed. Such can be gained by reading the appropriate sections of Bourbaki [1, 2, 3]. With a few exceptions to be noted, the notations used are standard and a separate listing seemed unnecessary.

Unless the contrary is specifically indicated, groups are taken to be multiplicative. Topological groups are not assumed to be necessarily separated (Hausdorff). The additive group of integers will be denoted by $\mathcal{I}$ and the additive group of reals by $\mathcal{R}$.

Contrary to customary usage, the function or transformation sign is usually placed on the right. That is, if X and Y are sets, f denotes a transformation of X into Y and $x \in X$, then xf denotes the unique element of Y determined by x and f.

In connection with uniform spaces, the term *index* is used to denote an element of the filter defining the uniform structure, thus replacing the term *entourage* as used by Bourbaki [2]. In keeping with the notation for the value of a function, if X is a uniform space, α is an index of X and $x \in X$, then $x\alpha$ denotes the set of all $y \in X$ such that $(x, y) \in \alpha$. Unless the contrary is stated, a uniform space is not necessarily separated.

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PART I. THE THEORY

1. TRANSFORMATION GROUPS

1.01. Definition. A *topological transformation group*, or more briefly, a *transformation group*, is defined to be an ordered triple (X, T, π) consisting of a topological space X, a topological group T and a mapping $\pi : X \times T \to X$ such that:

(1) (Identity axiom) $(x, e)\pi = x$ $(x \in X)$ where e is the identity element of T.

(2) (Homomorphism axiom) $((x, t)\pi, s)\pi = (x, ts)\pi$ $(x \in X; t, s \in T)$.

(3) (Continuity axiom) π is continuous.

If (X, T, π) is a transformation group, then $\{X\}\{T\}\{\pi\}$ is called the *phase* $\{$*space*$\}\{$*group*$\}\{$*projection*$\}$ *of* (X, T, π).

1.02. Definition. Let X, Y be $\{$topological$\}\{$uniform$\}$ spaces and let (X, T, π), (Y, S, ρ) be transformation groups.

A $\{$*topological*$\}\{$*uniform*$\}$ *isomorphism of* (X, T, π) *onto* (Y, S, ρ) is defined to be a couple (h, φ) consisting of a $\{$homeomorphism$\}\{$unimorphism$\}$ h of X onto Y and a homeomorphic group-isomorphism φ of T onto S such that $(xh, t\varphi)\rho = (x, t)\pi h$ $(x \in X, t \in T)$.

The transformation groups (X, T, π) and (Y, S, ρ) are said to be $\{$*topologically*$\}\{$*uniformly*$\}$ *isomorphic* (each *to* or *with* the other) provided there exists a $\{$topological$\}\{$uniform$\}$ isomorphism of (X, T, π) onto (Y, S, ρ).

1.03. Definition. Let X be a $\{$topological$\}\{$uniform$\}$ space and let (X, T, π) be a transformation group. An *intrinsic* $\{$*topological*$\}\{$*uniform*$\}$ property of (X, T, π) is a property of (X, T, π) definable solely in terms of the $\{$topological$\}$ $\{$uniform$\}$ structure of X, the topological structure of T, the group structure of T, and the mapping π.

1.04. Remark. We propose in this monograph to study certain intrinsic properties of transformation groups. It is clear that intrinsic $\{$topological$\}$ $\{$uniform$\}$ properties of transformation groups are invariant under $\{$topological$\}$ $\{$uniform$\}$ isomorphisms.

1.05. Notation. Let (X, T, π) be a transformation group. If $x \in X$ and if $t \in T$, then $(x, t)\pi$ is denoted more concisely by xt when there is no chance for ambiguity. Then the identity and homomorphism axioms may be restated as follows:

(1) $xe = x$ $(x \in X)$.

(2) $(xt)s = x(ts)$ $(x \in X; t, s \in T)$.

1.06. Terminology. The statement "(X, T, π) is a transformation group" may be paraphrased as "T $\{$is$\}\{$acts as$\}$ a transformation group $\{$of$\}\{$on$\}$ X

with respect to π". By virtue of 1.05 it often happens that a symbol for the phase projection does not occur in a discussion of a transformation group. In such an event we may speak simply of (X, T) as the transformation group where X is the phase space, T is the phase group and the phase projection is understood. The statement "(X, T) is a transformation group" may be paraphrased as "T {is}{acts as} a transformation group {of}{on} X". Thus the transformation group (X, T, π) may be denoted by (X, T) or even by T provided no ambiguity can occur.

Generally speaking, the statement that the transformation group (X, T, π) has a certain property may be paraphrased as either T has the property *on* X or X has the property *under* T. If $x \in X$, then the statement that (X, T, π) has a certain property at x may be paraphrased as either T has the property *at* x or x has the property *under* T.

1.07. Standing notation. Throughout the remainder of this section (X, T, π) denotes a transformation group.

1.08. Definition. If $t \in T$, then the *t-transition of* (X, T, π), denoted π^t, is the mapping $\pi^t : X \to X$ such that $x\pi^t = (x, t)\pi = xt$ $(x \in X)$. The *transition group of* (X, T, π) is the set $G = [\pi^t \mid t \in T]$. The *transition projection of* (X, T, π) is the mapping $\lambda : T \to G$ such that $t\lambda = \pi^t$ $(t \in T)$.

If $x \in X$, then the *x-motion of* (X, T, π), denoted π_x, is the mapping $\pi_x : T \to X$ such that $t\pi_x = (x, t)\pi = xt$ $(t \in T)$. The *motion space of* (X, T, π) is the set $M = [\pi_x \mid x \in X]$. The *motion projection of* (X, T, π) is the mapping $\mu : X \to M$ such that $x\mu = \pi_x$ $(x \in X)$.

1.09. Definition. The transformation group (X, T) is said to be *effective* provided that if $t \in T$ with $t \neq e$, then $xt \neq x$ for some $x \in X$.

1.10. Remark. Let $\{G\}\{\lambda\}\{M\}\{\mu\}$ be the {transition group}{transition projection}{motion space}{motion projection} of (X, T, π). Then

(1) π^e is the identity mapping of X.
(2) If $t, s \in T$, then $\pi^t\pi^s = \pi^{ts}$.
(3) If $t \in T$, then π^t is a one-to-one mapping of X onto X and $(\pi^t)^{-1} = \pi^{t^{-1}}$.
(4) If $t \in T$, then π^t is a homeomorphism of X onto X.
(5) G is a group of homeomorphisms of X onto X.
(6) λ is a group-homomorphism of T onto G. This justifies the name "homomorphism axiom" of 1.01 (2).
(7) λ is one-to-one if and only if (X, T, π) is effective.
(8) If $x \in X$, then π_x is a continuous mapping of T into X.
(9) μ is a one-to-one mapping of X onto M.

1.11. Remark. Let $t \in T$ and let $\varphi_t : T \to T$ be defined by $\tau\varphi_t = t^{-1}\tau t$ $(\tau \in T)$. Then (π^t, φ_t) is a topological isomorphism of (X, T, π) onto (X, T, π).

1.12. Definition. Let X be a topological space. A *topological homeomorphism group of* X is a topologized group Φ of homeomorphisms of X onto X

such that Φ is a topological group and $\rho : X \times \Phi \to X$ is continuous where ρ is defined by $(x, \varphi)\rho = x\varphi$ $(x \in X, \varphi \in \Phi)$.

1.13. Remark. The effective topological transformation groups and the topological homeomorphism groups are essentially identical in the following sense:

(1) If (X, T, π) is an effective topological transformation group, then the transition group G of (X, T, π), topologized so that the transition projection of (X, T, π) becomes a group-isomorphic homeomorphism of T onto G, is a topological homeomorphism group of X.

(2) If Φ is a topological homeomorphism group of a topological space X, then (X, Φ, ρ) is an effective topological transformation group where $\rho : X \times \Phi \to X$ is defined by $(x, \varphi)\rho = x\varphi$ $(x \in X, \varphi \in \Phi)$.

In particular, a notion defined for topological transformation groups is automatically defined for topological homeomorphism groups.

1.14. Definition. A *discrete transformation group* is a topological transformation group whose phase group is discrete. A *discrete homeomorphism group* is a topological homeomorphism group provided with its discrete topology. A *homeomorphism group* is a group of homeomorphisms. The *total homeomorphism group* of a topological space X is the group of all homeomorphisms of X onto X.

1.15. Remark. It is clear from 1.13 that the effective discrete transformation groups and the discrete homeomorphism groups are to be considered as identical. Since a homeomorphism group may be considered as a discrete homeomorphism group, a notion defined for transformation groups is automatically defined for homeomorphism groups.

1.16. Notation. If $A \subset X$ and if $B \subset T$, then $(A \times B)\pi = [xt \mid x \in A \& t \in B]$ is denoted more concisely by AB when there is no chance for ambiguity. In particular, we write At in place of $A[t]$ where $A \subset X$ and $t \in T$; and we write xB in place of $[x]B$ where $x \in X$ and $B \subset T$. By the homomorphism axiom, xts is unambiguously defined, where $x \in X$ and $t, s \in T$; likewise ABC where $A \subset X$ and $B, C \subset T$; etc.

1.17. Lemma. *Let X, Y, Z be topological spaces and let $\varphi : X \times Y \to Z$ be continuous. If A, B are compact subsets of X, Y and if W is a neighborhood of $(A \times B)\varphi$, then there exist neighborhoods U, V of A, B such that $(U \times V)\varphi \subset W$.*

Proof. We write $(x, y)\varphi = xy$ $(x \in X, y \in Y)$.

Let $x \in A$. We show there exist open neighborhoods U, V of x, B such that $UV \subset W$. For each $y \in B$ there exist open neighborhoods U_y, V_y of x, y such that $U_yV_y \subset W$. Choose a finite subset F of B for which $B \subset \bigcup_{y\in F} V_y$. Define $U = \bigcap_{y\in F} U_y$ and $V = \bigcup_{y\in F} V_y$.

For each $x \in A$ there exist open neighborhoods U_x, V_x of x, B such that $U_xV_x \subset W$. Choose a finite subset E of A for which $A \subset \bigcup_{x\in E} U_x$. Define $U = \bigcup_{x\in E} U_x$ and $V = \bigcap_{x\in E} V_x$. Then U, V are neighborhoods of A, B such that $UV \subset W$.

1.18. LEMMA. *The following statements are valid:*

(1) *If* $A \subset X$ *and if* $t \in T$, *then* $\overline{At} = \overline{A}t$.

(2) *If* $A \subset X$ *and if* $B \subset T$, *then* $\overline{A}\,\overline{B} \subset \overline{AB}$ *and* $\overline{\overline{A}\,\overline{B}} = \overline{A\overline{B}} = \overline{AB}$.

(3) *If* A, B *are compact subsets of* X, T, *then* AB *is a compact subset of* X.

(4) *If* A, B *are compact subsets of* X, T *and if* W *is a neighborhood of* AB, *then there exist neighborhoods* U, V, *of* A, B *such that* $UV \subset W$.

(5) *If* A *is a closed subset of* X *and if* B *is a compact subset of* T, *then* AB *is a closed subset of* X.

(6) *If* $A \subset X$ *and if* B *is a compact subset of* T, *then* $\overline{AB} = \overline{A}B$.

PROOF. (1) Since $\pi^t : X \to X$ is a homeomorphism onto, $\overline{At} = \overline{A\pi^t} = \overline{A}\pi^t = \overline{A}t$.

(2) Since $\pi : X \times T \to X$ is continuous, $\overline{A}\,\overline{B} = (\overline{A} \times \overline{B})\pi = (\overline{A \times B})\pi \subset \overline{(A \times B)\pi} = \overline{AB}$. The last conclusion follows from $AB \subset \overline{A}\,\overline{B} \subset \overline{AB}$ and $AB \subset A\overline{B} \subset \overline{AB}$.

(3) $AB = (A \times B)\pi$ is a continuous image of the compact set $A \times B$.

(4) Use 1.17.

(5) Let $x \in X - AB$. Then $xB^{-1} \cap A = \emptyset$. By (4) there exists a neighborhood U of x such that $UB^{-1} \cap A = \emptyset$ whence $U \cap AB = \emptyset$ and $U \subset X - AB$.

(6) By (2) and (5), $\overline{AB} = \overline{\overline{A}B} = \overline{A}B$.

1.19. LEMMA. *Let* X, Y *be uniform spaces, let* $\varphi : X \to Y$ *be continuous and let* A *be a compact subset of* X. *If* β *is an index of* Y, *then there exists an index* α *of* X *such that* $x \in A$ *implies* $x\alpha\varphi \subset x\varphi\beta$.

PROOF. Let γ be a symmetric index of Y such that $\gamma^2 \subset \beta$. For each $x \in A$ there exists a symmetric index α_x of X such that $x\alpha_x^2\varphi \subset x\varphi\gamma$. Choose a finite subset E of A for which $A \subset \bigcup_{x \in E} x\alpha_x$. Define $\alpha = \bigcap_{x \in E} \alpha_x$. Let $x \in A$. There exists $z \in E$ such that $x \in z\alpha_z$. Since $x\varphi \in z\alpha_z^2\varphi \subset z\varphi\gamma$, it follows that $x\alpha\varphi \subset z\alpha_z\alpha\varphi \subset z\alpha_z^2\varphi \subset z\varphi\gamma \subset x\varphi\gamma^2 \subset x\varphi\beta$. The proof is completed.

1.20. LEMMA. *Let* X *be a uniform space, let* A, B *be compact subsets of* X, T *and let* α *be an index of* X. *Then:*

(1) *There exists an index* β *of* X *and a neighborhood* V *of* e *such that* $x \in A$ *and* $t \in B$ *implies* $x\beta tV \subset xt\alpha$ *and* $x\beta Vt \subset xt\alpha$.

(2) *There exists an index* β *of* X *such that* $x \in A$ *and* $t \in B$ *implies* $x\beta t \subset xt\alpha$ *and* $xt\beta \subset x\alpha t$.

(3) *There exists an index* β *of* X *such that* $x \in A$ *implies* $x\beta B \subset xB\alpha$ *and* $xB\beta \subset x\alpha B$.

(4) *There exists an index* β *of* X *such that* $t \in B$ *implies* $A\beta t \subset At\alpha$ *and* $At\beta \subset A\alpha t$.

PROOF. Since $\pi : X \times T \to X$ is continuous and $A \times B$ is a compact subset of $X \times T$, (1) follows from 1.19. The first part of (2) follows immediately from (1). Since AB and B^{-1} are compact, there exists an index β of X such that $x \in AB$ and $t \in B^{-1}$ implies $x\beta t \subset xt\alpha$. Hence, $x \in A$ and $t \in B$ implies $xt\beta t^{-1} \subset xtt^{-1}\alpha$

and $xl\beta \subset x\alpha t$. This proves the second part of (2). Finally (3) and (4) are easy consequences of (2).

1.21. LEMMA. *Let X be a compact uniform space, let α be an index of X and let K be a compact subset of T. Then there exists an index β of X such that:*
(1) *$x \in X$ and $k \in K$ implies $x\beta k \subset xk\alpha$.*
(2) *$x \in X$ and $k \in K$ implies $xk\beta \subset x\alpha k$.*
(3) *$(x, y) \in \beta$ and $k \in K$ implies $(xk, yk) \in \alpha$.*
(4) *$(x, y) \in \alpha'$ and $k \in K$ implies $(xk, yk) \in \beta'$.*

PROOF. Use 1.20 (2).

1.22. DEFINITION. Let $A \subset X$ and let $S \subset T$. The set A is said to be *invariant under* S or *S-invariant* provided that $AS \subset A$. When $S = T$, the qualifying phrase "under T" and the prefix "T-" may be omitted.

1.23. REMARK. The following statements are valid:
(1) If $A \subset X$, then the following statements are pairwise equivalent: A is T-invariant, that is, $AT \subset A$; $AT = A$; $t \in T$ implies $At \subset A$; $t \in T$ implies $At = A$; $t \in T$ implies $At \supset A$.
(2) X and $\emptyset$ are T-invariant.
(3) If A is a T-invariant subset of X, then $A' = X - A$, $\bar{A}$, int A are T-invariant.
(4) If A and B are T-invariant subsets of X, then $A - B$ is T-invariant.
(5) If $\mathfrak{A}$ is a class of T-invariant subsets of X, then $\bigcap\mathfrak{A}$ and $\bigcup\mathfrak{A}$ are T-invariant.
(6) If $A \subset X$ and if $S \subset T$, then A is S-invariant if and only if A' is S^{-1}-invariant.

1.24. REMARK. Let $Y \subset X$, let S be a subgroup of T, let Y be S-invariant and let $\rho = \pi \mid Y \times S$. Then (Y, S, ρ) is a transformation group. In particular, T acts as a transformation group on every T-invariant subset of X, and every subgroup of T acts as a transformation group on X.

1.25. DEFINITION. Let Y be a T-invariant subset of X. The transformation group (X, T) is said to have a certain property *on* Y provided that the transformation group (Y, T) has this property.

1.26. DEFINITION. Let $x \in X$ and let $S \subset T$. The *orbit of x under S* or the *S-orbit of x* is defined to be the subset xS of X. The *orbit-closure of x under S* or the *S-orbit-closure of x* is defined to be the subset $\overline{xS}$ of X. An {*orbit*}{*orbit-closure*} *under* S or an {*S-orbit*}{*S-orbit-closure*} is defined to be a subset A of X such that A is the {S-orbit}{S-orbit-closure} of some point of X. When $S = T$, the phrase "under T" and the prefix "T-" may be omitted.

1.27. DEFINITION. Let X be a set. A *partition of X* is defined to be a disjoint class $\mathfrak{A}$ of nonvacuous subsets of X such that $X = \bigcup\mathfrak{A}$.

1.28. Remark. The following statements are valid:

(1) If $x \in X$, then the orbit of x under T is the least T-invariant subset of X which contains the point x.

(2) If $x \in X$ and if $y \in xT$, then $yT = xT$.

(3) The class of all orbits under T is a partition of X.

(4) If $x \in X$, then the orbit-closure of x under T is the least closed T-invariant subset of X which contains the point x.

(5) If $x \in X$ and if $y \in \overline{xT}$, then $\overline{yT} \subset \overline{xT}$.

(6) The class of all orbit-closures under T is a covering of X.

1.29. Remark. The following definitions describe various methods of constructing transformation groups.

1.30. Definition. Let n be a positive integer. An *n-parameter* {*discrete*} {*continuous*} *flow* is defined to be a transformation group whose phase group is $\{\mathscr{I}^n\}\{\mathscr{R}^n\}$. The phrase "one-parameter {discrete}{continuous} flow" is shortened to "{discrete}{continuous} flow".

1.31. Remark. Let n be a positive integer. An n-parameter discrete flow $(X, \mathscr{I}^n, \pi)$ is characterized in an obvious manner by n pairwise commuting homeomorphisms of X onto X, namely $\pi^{(1,0,\cdots,0)}, \ldots, \pi^{(0,\cdots,0,1)}$ which are said to *generate* $(X, \mathscr{I}^n, \pi)$. In particular, a discrete flow $(X, \mathscr{I}, \pi)$ is characterized by a single homeomorphism of X onto X, namely π^1, which is said to *generate* $(X, \mathscr{I}, \pi)$. The properties of a discrete flow $(X, \mathscr{I}, \pi)$ are often attributed to its generating homeomorphism π^1.

1.32. Definition. Let S be a subgroup of T and define $\rho = \pi \mid X \times S$. The transformation group (X, S, ρ) is called the *S-restriction of* (X, T, π) or a *subgroup-restriction of* (X, T, π).

Let Y be a subset of X such that $(Y \times T)\pi = Y$ and define $\rho = \pi \mid Y \times T$. The transformation group (Y, T, ρ) is called the *Y-restriction of* (X, T, π) or a *subspace-restriction of* (X, T, π).

Let S be a subgroup of T, let Y be a subset of X such that $(Y \times S)\pi = Y$ and define $\rho = \pi \mid Y \times S$. The transformation group (Y, S, ρ) is called the *(Y, S)-restriction of* (X, T, π) or a *transformation subgroup of* (X, T, π).

Let S be a topological group, let $\varphi : S \to T$ be a continuous homomorphism into and let $\rho : X \times S \to X$ be defined by $(x, s)\rho = (x, s\varphi)\pi$ $(x \in X, s \in S)$. The transformation group (X, S, ρ) is called the *(S, φ)-restriction of* (X, T, π).

Let S be a topological group, let $\varphi : S \to T$ be a continuous homomorphism into, let Y be a subset of X such that $(Y \times S\varphi)\pi = Y$ and let $\rho : Y \times S \to Y$ be defined by $(y, s)\rho = (y, s\varphi)\pi$ $(y \in Y, s \in S)$. The transformation group (Y, S, ρ) is called the *(Y, S, φ)-restriction of* (X, T, π).

1.33. Remark. We consider every partition $\mathscr{A}$ of a topological space X to be itself a topological space provided with its partition topology, namely, the greatest topology which makes the projection of X onto $\mathscr{A}$ continuous.

1.34. DEFINITION. Let X be a set, let $\mathfrak{A}$ be a partition of X and let $E \subset X$. The *star of E in $\mathfrak{A}$* or the *$\mathfrak{A}$-star of E* or the *saturation of E in $\mathfrak{A}$* or the *$\mathfrak{A}$-saturation of E*, denoted $E\mathfrak{A}$, is the subset $\bigcup[A \mid A \in \mathfrak{A}, A \cap E \neq \emptyset]$ of X. The set E is *saturated in $\mathfrak{A}$* or *$\mathfrak{A}$-saturated* in case $E = E\mathfrak{A}$.

1.35. DEFINITION. Let X be a topological space and let $\mathfrak{A}$ be a partition of X. The partition $\mathfrak{A}$ is said to be {*star-open*}{*star-closed*} provided that the $\mathfrak{A}$-star of every {open}{closed} subset of X is {open}{closed} in X.

1.36. REMARK. Let X be a topological space and let $\mathfrak{A}$ be a partition of X. Then the following statements are pairwise equivalent

(1) $\mathfrak{A}$ is {star-open}{star-closed}.

(2) If $x \in X$ and if U is a neighborhood of $\{x\}\{x\mathfrak{A}\}$, then there exists a neighborhood V of $\{x\mathfrak{A}\}\{x$ and therefore $x\mathfrak{A}\}$ such that $\{V \subset U\mathfrak{A}\}\{V\mathfrak{A} \subset U\}$.

(3) The projection of X onto $\mathfrak{A}$ is {open}{closed}.

1.37. DEFINITION. Let X be a topological space. A *decomposition of X* is a partition $\mathfrak{A}$ of X such that every member of $\mathfrak{A}$ is compact.

1.38. REMARK. Let X be a compact metrizable space and let $\mathfrak{A}$ be a decomposition of X. Then $\mathfrak{A}$ is {star-open}{star-closed} if and only if $x_0, x_1, x_2, \cdots \in X$ with $\lim_{n\to\infty} x_n = x_0$ implies

$$\{x_0\mathfrak{A} \subset \liminf_{n\to\infty} x_n\mathfrak{A}\}\{\limsup_{n\to\infty} x_n\mathfrak{A} \subset x_0\mathfrak{A}\}.$$

1.39. DEFINITION. Let $\mathfrak{A}$ be a {star-open partition}{star-closed decomposition} of X, let $A\pi^t \in \mathfrak{A}$ $(A \in \mathfrak{A}, t \in T)$ and let $\rho : \mathfrak{A} \times T \to \mathfrak{A}$ be defined by $(A\ T)\rho = A\pi^t$ $(A \in \mathfrak{A}, t \in T)$. The transformation group $(\mathfrak{A}, T, \rho)$ is called the *partition transformation group of $\mathfrak{A}$ induced by (X, T, π)*.

1.40. DEFINITION. Let Φ be a group of homeomorphisms of X onto X such that $\varphi\pi^t = \pi^t\varphi$ $(\varphi \in \Phi, t \in T)$ and let $\mathfrak{A} = [x\Phi \mid x \in X]$ whence $\mathfrak{A}$ is a star-open partition of X such that $A\pi^t \in \mathfrak{A}$ $(A \in \mathfrak{A}, t \in T)$. The partition transformation group of $\mathfrak{A}$ induced by (X, T, π) is called the *Φ-orbit partition transformation group induced by* (X, T, π).

1.41. DEFINITION. Let Φ be a group of homeomorphisms of X onto X such that $\varphi\pi^t = \pi^t\varphi$ $(\varphi \in \Phi, t \in T)$ and let $\mathfrak{A} = [\overline{x\Phi} \mid x \in X]$ be a partition of X whence $\mathfrak{A}$ is a star-open partition of X such that $A\pi^t \in \mathfrak{A}$ $(A \in \mathfrak{A}, t \in T)$. The partition transformation group of $\mathfrak{A}$ induced by (X, T, π) is called the *Φ-orbit-closure partition transformation group induced by* (X, T, π).

1.42. DEFINITION. {Let φ be a continuous-open mapping of X onto X} {Let X be a compact T_2-space, let φ be a continuous mapping of X onto X} such that $\varphi\pi^t = \pi^t\varphi$ $(t \in T)$ and let $\mathfrak{A} = [x\varphi^{-1} \mid x \in X]$ whence $\mathfrak{A}$ is a {star-open partition}{star-closed decomposition} of X such that $A\pi^t \in \mathfrak{A}$ $(A \in \mathfrak{A},$

$t \in T$). The partition transformation group of $\mathfrak{A}$ induced by (X, T, π) is called the *φ-inverse partition transformation group induced by* (X, T, π).

1.43. Definition. Let S be a topological group, let T be a topological subgroup of S, for $x \in X$ and $\sigma \in S$ define $\{A(x, \sigma) = [(x\pi^\tau, \sigma\tau) \mid \tau \in T]\}$ $\{A(x, \sigma) = [(x\pi^\tau, \tau^{-1}\sigma) \mid \tau \in T]\}$, define the star-open partition $\mathfrak{A} = [A(x, \sigma) \mid x \in X, \sigma \in S]$ of $X \times S$ and let $\rho : \mathfrak{A} \times S \to \mathfrak{A}$ be defined by $\{(A(x, \sigma), s)\rho = A(x, s^{-1}\sigma)\}\{(A(x, \sigma), s)\rho = A(x, \sigma s)\}$ $(x \in X; \sigma, s \in S)$. The transformation group $(\mathfrak{A}, S, \rho)$ is called the $\{$*left*$\}\{$*right*$\}$ *S-extension of* (X, T, π).

1.44. Remark. We adopt the notation of 1.43. Consider the transformation group $(X \times S, S, \eta)$ where $\eta : (X \times S) \times S \to X \times S$ is defined by $\{((x, \sigma), s)\eta = (x, s^{-1}\sigma)\}\{((x, \sigma), s)\eta = (x, \sigma s)\}$ $(x \in X; \sigma, s \in S)$. The partition transformation group of $\mathfrak{A}$ induced by $(X \times S, S, \eta)$ coincides with the {left}{right} S-extension of (X, T, π).

1.45. Remark. Let S be a topological group and let T be a discrete topological subgroup of S. Then (X, T, π) is isomorphic to a transformation subgroup of the {left}{right} S-extension of (X, T, π).

1.46. Notation. The cartesian product of a family $(X_\iota \mid \iota \in I)$ of sets is denoted $\times_{\iota \in I} X_\iota$. The direct product of a family $(G_\iota \mid \iota \in I)$ of groups is denoted ${}_D\times_{\iota \in I} G_\iota$.

1.47. Remark. We consider the cartesian product of every family of {topological}{uniform} spaces to be itself a {topological}{uniform} space provided with its product {topology}{uniformity}, namely, the least {topology}{uniformity} which makes all the projections onto the factor spaces {continuous}{uniformly continuous}.

1.48. Definition. Let $((X_\iota, T_\iota, \pi_\iota) \mid \iota \in I)$ be a family of transformation groups. The $\{$*cartesian*$\}\{$*direct*$\}$ *product of* $((X_\iota, T_\iota, \pi_\iota) \mid \iota \in I)$, denoted $\{\times_{\iota \in I} (X_\iota, T_\iota, \pi_\iota)\}$ $\{{}_D\times_{\iota \in I} (X_\iota, T_\iota, \pi_\iota)\}$, is the transformation group (X, T, π) where $X = \times_{\iota \in I} X_\iota$, $\{T = \times_{\iota \in I} T_\iota\}\{T = {}_D\times_{\iota \in I} T_\iota\}$ and $\pi : X \times T \to X$ is defined by $(x, t)\pi = (x_\iota \pi_\iota^{t_\iota} \mid \iota \in I)$ $(x = (x_\iota \mid \iota \in I) \in X, t = (t_\iota \mid \iota \in I) \in T)$.

1.49. Definition. Let $((X_\iota, T, \pi_\iota) \mid \iota \in I)$ be a family of transformation groups. The *space product of* $((X_\iota, T, \pi_\iota) \mid \iota \in I)$, denoted ${}_S\times_{\iota \in I} (X_\iota, T, \pi_\iota)$, is the transformation group (X, T, π) where $X = \times_{\iota \in I} X_\iota$ and $\pi : X \times T \to X$ is defined by $(x, t)\pi = (x_\iota \pi_\iota^t \mid \iota \in I)$ $(x = (x_\iota \mid \iota \in I) \in X, t \in T)$.

1.50. Remark. Both the direct and space products of a family of transformation groups are subgroup-restrictions of the cartesian product of the family.

1.51. Definition. Let T be a topological group.

The *left transformation group of* T is defined to be the transformation group (T, T, λ) where $\lambda : T \times T \to T$ is defined by $(\tau, t)\lambda = t^{-1}\tau$ $(\tau, t \in T)$.

The *right transformation group of* T is defined to be the transformation group (T, T, μ) where $\mu : T \times T \to T$ is defined by $(\tau, t)\mu = \tau t$ $(\tau, t \in T)$.

The *bilateral transformation group of T* is defined to be the transformation group $(T, T \times T, \xi)$ where $\xi : T \times (T \times T) \to T$ is defined by $(\tau, (t, s))\xi = t^{-1}\tau s$ $(\tau, t, s \in T)$.

1.52. DEFINITION. Let X be a uniform space and let $x \in X$. The transformation group (X, T, π) is said to be {*equicontinuous at x*}{*equicontinuous*}{*uniformly equicontinuous*} provided that the transition group $[\pi^t \mid t \in T]$ is {equicontinuous at x}{equicontinuous}{uniformly equicontinuous}.

1.53. REMARK. Let X be a uniform space. The following statements are pairwise equivalent:

(1) (X, T, π) is uniformly equicontinuous.

(2–5) If α is an index of X, then there exists an index β of X such that

$$\{x \in X \text{ and } t \in T \text{ implies } x\beta t \subset xt\alpha\}$$
$$\{x \in X \text{ and } t \in T \text{ implies } xt\beta \subset x\alpha t\}$$
$$\{(x, y) \in \beta \text{ and } t \in T \text{ implies } (xt, yt) \in \alpha\}$$
$$\{(x, y) \in \alpha' \text{ and } t \in T \text{ implies } (xt, yt) \in \beta'\}.$$

1.54. REMARK. The {left}{right} transformation group of a topological group is uniformly equicontinuous relative to the {left}{right} uniformity of the phase space.

1.55. NOTATION. Let H be a subgroup of a group G. The left quotient space $[xH \mid x \in G]$ of G by H is denoted G/H. The right quotient space $[Hx \mid x \in G]$ of G by H is denoted $G\backslash H$.

1.56. DEFINITION. Let S be a subgroup of a topological group T.

The *left transformation group of T/S induced by T* is defined to be the transformation group $(T/S, T, \lambda)$ where $\lambda : T/S \times T \to T/S$ is defined by $(A, t)\lambda = t^{-1}A$ $(A \in T/S, t \in T)$.

The *right transformation group of T\S induced by T* is defined to be the transformation group $(T\backslash S, T, \mu)$ where $\mu : T\backslash S \times T \to T\backslash S$ is defined by $(A, t)\mu = At$ $(A \in T\backslash S, t \in T)$.

1.57. REMARK. Let S be a subgroup of a topological group T and let (T, T, η) be the {left}{right} transformation group of T. Then the {left}{right} transformation group of $\{T/S\}\{T\backslash S\}$ induced by T coincides with the partition transformation group of $\{T/S\}\{T\backslash S\}$ induced by (T, T, η).

1.58. DEFINITION. Let φ be a continuous homomorphism of a topological group T into a topological group S and let $\rho : S \times T \to S$ be defined by $\{(s, t)\rho = (t^{-1}\varphi)s\}\{(s, t)\rho = s(t\varphi)\}$ $(s \in S, t \in T)$. The transformation group (S, T, ρ) is called the {*left*}{*right*} *transformation group of S induced by T under* φ.

1.59. REMARK. Let φ be a continuous homomorphism of the topological group T into the topological group S. Then the {left}{right} transformation group of S induced by T under φ coincides with the (T, φ)-restriction of the {left}{right} transformation group of S.

1.60. Remark. Let φ be a continuous homomorphism of a topological group T into a topological group S. Then the {left}{right} transformation group of S induced by T under φ is uniformly equicontinuous relative to the {left}{right} uniformity of S.

1.61. Remark. Let φ be a homomorphism of $\mathscr{I}$ into a topological group T, let $t = 1\varphi$ and let θ be the {left}{right} translation of T induced by $\{t^{-1}\}\{t\}$. Then the {left}{right} transformation group of T induced by $\mathscr{I}$ under φ coincides with the discrete flow on T generated by θ.

1.62. Definition. Let T be a topological group, let Y be a uniform space, let Φ be the class of all {right}{left} uniformly continuous functions on T to Y, let Φ be provided with its space-index uniformity and let $\rho : \Phi \times T \to \Phi$ be defined by $\{(\varphi, t)\rho = (t\tau\varphi \mid \tau \in T)\}\{(\varphi, t)\rho = (\tau t^{-1}\varphi \mid \tau \in T)\}$ $(\varphi \in \Phi, t \in T)$. The uniformly equicontinuous transformation group (Φ, T, ρ) is called the *{left}{right} uniform functional transformation group over T to Y.*

1.63. Definition. Let T be a locally compact topological group, let Y be a uniform space, let Φ be the class of all continuous functions on T to Y, let Φ be provided with its compact-index uniformity and let $\rho : \Phi \times T \to \Phi$ be defined by $\{(\varphi, t)\rho = (t\tau\varphi \mid \tau \in T)\}\{(\varphi, t)\rho = (\tau t^{-1}\varphi \mid \tau \in T)\}$ $(\varphi \in \Phi, t \in T)$. The transformation group (Φ, T, ρ) is called the *{left}{right} functional transformation group over T to Y.*

1.64. Remark. A particular case of 1.63 arises when T is discrete. In such an event a different notation may be used, as indicated by the following statements:

(1) $\Phi = Y^T = \times_{\tau \in T} Y_\tau$ where $Y_\tau = Y$ $(\tau \in T)$.

(2) The point-index (= compact-index) uniformity of Φ coincides with the product uniformity of $\times_{\tau \in T} Y_\tau$.

(3) If $y = (y_\tau \mid \tau \in T) \in \times_{\tau \in T} Y_\tau$ and if $t \in T$, then $\{(y, t)\rho = (y_{t\tau} \mid \tau \in T)\}$ $\{(y, t)\rho = (y_{\tau t^{-1}} \mid \tau \in T)\}$.

1.65. Lemma. *Let T be a locally compact topological group, let Y be a uniform space, let (Φ, T, ρ) be the {left}{right} functional transformation group over T to Y, let $\varphi \in \Phi$ and let $\Psi \subset \Phi$. Then:*

(1) *The orbit φT of φ is totally bounded if and only if φ is {left}{right} uniformly continuous and bounded.*

(2) *If Y is complete, then the orbit-closure $\overline{\varphi T}$ of φ is compact if and only if φ is {left}{right} uniformly continuous and bounded.*

(3) *ΨT is totally bounded if and only if Ψ is {left}{right} uniformly equicontinuous and bounded.*

(4) *If Y is complete, then $\overline{\Psi T}$ is compact if and only if Ψ is {left}{right} uniformly equicontinuous and bounded.*

Proof. Use 11.31 and 11.32.

1.66. DEFINITION. Let X be a uniform space, let each transition $\pi^t : X \to X$ $(t \in T)$ be uniformly continuous, let the motion space $[\pi_x : T \to X \mid x \in X]$ be equicontinuous, let Y be a uniform space, let Φ be the class of all uniformly continuous functions on X to Y, let Φ be provided with its space-index uniformity and let $\rho : \Phi \times T \to \Phi$ be defined by $(\varphi, t)\rho = \pi^{t^{-1}}\varphi$ $(\varphi \in \Phi, t \in T)$. The uniformly equicontinuous transformation group (Φ, T, ρ) is called the *uniform functional transformation group over* (X, T, π) *to* Y.

1.67. REMARK. Let T be a topological group, let (T, T, η) be the {left} {right} transformation group of T and let Y be a uniform space. Then the uniform functional transformation group over (T, T, η) to Y coincides with the {left}{right} uniform functional transformation group over T to Y.

1.68. DEFINITION. Let T be locally compact, let Y be a uniform space, let Φ be the class of all continuous functions on X to Y, let Φ be provided with its compact-index uniformity and let $\rho : \Phi \times T \to \Phi$ be defined by $(\varphi, t)\rho = \pi^{t^{-1}}\varphi$ $(\varphi \in \Phi, t \in T)$. The transformation group (Φ, T, ρ) is called the *functional transformation group over* (X, T, π) *to* Y.

1.69. REMARK. Let T be a locally compact topological group, let (T, T, η) be the {left}{right} transformation group of T and let Y be a uniform space. Then the functional transformation group over (T, T, η) to Y coincides with the {left}{right} functional transformation group over T to Y.

1.70. NOTES AND REFERENCES.

(1.01) The concept of a transformation group for which the topology of the group plays a role appears to have originated in the latter part of the nineteenth century (cf., e.g., Lie and Engel [1]). A system of n differential equations of the first order defines, under suitable conditions, a transformation group (X, T, π) for which X is an n-dimensional manifold and T is the additive group of reals. Thus a classical dynamical system with n degrees of freedom defines a transformation group for which the *phase space* is the $2n$-dimensional manifold customarily associated with the term. See also Zippin [1].

(1.35) For a decomposition of a compact metric space, the equivalence of {star-open}{star-closed} with {lower semi-continuous}{upper semi-continuous} is readily verified (cf. Whyburn [1], Ch. VII).

(1.40) Φ-orbit partition transformation groups arise naturally in the study of geodesic flows on manifolds (cf. §13).

2. ORBIT-CLOSURE PARTITIONS

2.01. **Standing notation.** Throughout this section T denotes a topological group.

2.02. **Definition.** A subset A of T is said to be *{left} {right} syndetic in T* provided that $\{T = AK\} \{T = KA\}$ for some compact subset K of T.

2.03. **Remark.** The following statements are valid.

(1) If $A \subset T$, then A is {left} {right} syndetic in T if and only if there exists a compact subset K of T such that every {left} {right} translate of K intersects A.

(2) If $A \subset B \subset T$ and if A is {left} {right} syndetic in T, then so also is B.

(3) If $A \subset T$, then A is {left} {right} syndetic in T if and only if A^{-1} is {right} {left} syndetic in T.

(4) If $A \subset T$ and if A is symmetric or invariant (in particular, if A is a subgroup of T or if T is abelian), then A is left syndetic in T if and only if A is right syndetic in T. In such an event, the equivalent phrases "left syndetic", "right syndetic" are contracted to "syndetic".

(5) If A is a syndetic subgroup of T, then the left, right quotient spaces T/A, $T\backslash A$ are compact.

(6) If T is locally compact, if A is a subgroup of T and if some one of the left, right quotient spaces T/A, $T\backslash A$ is compact, then A is syndetic in T.

(7) If T is discrete and if A is a subgroup of T, then A is syndetic in T if and only if A is of finite index in T.

(8) If A is a {left} {right} syndetic subset of T and if U is a compact neighborhood of e, then $\{AU\} \{UA\}$ is {left} {right} syndetic relative to the discrete topology of T.

2.04. **Example.** Let T be the discrete free group on 2 generators a, b and let $\{A\} \{B\}$ be the set of all words of T which in reduced form do not {end} {begin} with $\{a^1\} \{b^1\}$. Then:

(1) A is left syndetic in T but A is not right syndetic in T.

(2) B is right syndetic in T but B is not left syndetic in T.

(3) $A \cup B$ is both left and right syndetic in T but there is no compact (= finite) subset K of T such that every bilateral translate of K intersects $A \cup B$.

2.05. **Definition.** Let G be a group. A *semigroup in G* is defined to be a subset H of G such that $HH \subset H$.

2.06. **Lemma.** *Let S be a left or right syndetic closed semigroup in T. Then S is a subgroup of T.*

Proof. We assume without loss that S is left syndetic. Let $s \in S$ and let