A first course in

general relativity

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Professor, Applied Mathematics and Astronomy, University College, Cardiff

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Preface

This book has evolved from lecture notes for a full-year undergraduate course in general relativity which I taught from 1975 to 1980, an experience which firmly convinced me that general relativity is not significantly more difficult for undergraduates to learn than the standard undergraduate-level treatments of electromagnetism and quantum mechanics. The explosion of research interest in general relativity in the past 20 years, largely stimulated by astronomy, has not only led to a deeper and more complete understanding of the theory; it has also taught us simpler, more physical ways of understanding it. Relativity is now in the mainstream of physics and astronomy, so that no theoretical physicist can be regarded as broadly educated without some training in the subject. The formidable reputation relativity acquired in its early years (Interviewer: 'Professor Eddington, is it true that only three people in the world understand Einstein's theory?' Eddington: 'Who is the third?') is today perhaps the chief obstacle that prevents it being more widely taught to theoretical physicists. The aim of this textbook is to present general relativity at a level appropriate for undergraduates, so that the student will understand the basic physical concepts and their experimental implications, will be able to solve elementary problems, and will be well prepared for the more advanced texts on the subject.

In pursuing this aim, I have tried to satisfy two competing criteria: first, to assume a minimum of prerequisites; and second, to avoid watering down the subject matter. Unlike most introductory texts, this one does not assume that the student has already studied electromagnetism in its manifestly relativistic formulation, the theory of electromagnetic waves, or fluid dynamics. The necessary fluid dynamics is developed in the relevant chapters. The main consequence of not assuming a familiarity with electromagnetic waves is that gravitational waves have to be introduced slowly: the wave equation is studied from scratch. A full list of prerequisites appears below.

The second guiding principle, that of not watering down the treatment, is very subjective and rather more difficult to describe. I have tried to introduce differential geometry fully, not being content to rely only on analogies with curved surfaces, but I have left out subjects that are not essential to general relativity at this level, such as nonmetric manifold theory, Lie derivatives, and fiber bundles.[1] I have introduced the full nonlinear field equations, not just those of linearized theory, but I solve them only in the plane and spherical cases, quoting and examining, in addition, the Kerr solution. I study gravitational waves mainly in the linear approximation, but go slightly beyond it to derive the energy in the waves and the reaction effects in the wave emitter. I have tried in each topic to supply enough foundation for the student to be able to go to more advanced treatments without having to start over again at the beginning.

The first part of the book, up to Ch. 8, introduces the theory in a sequence which is typical of many treatments: a review of special relativity, development of tensor analysis and continuum physics in special relativity, study of tensor calculus in curvilinear coordinates in Euclidean and Minkowski spaces, geometry of curved manifolds, physics in a curved spacetime, and finally the field equations. The remaining four chapters study a few topics which I have chosen because of their importance in modern astrophysics. The chapter on gravitational radiation is more detailed than usual at this level because the observation of gravitational waves may be one of the most significant developments in astronomy in the next decade. The chapter on spherical stars includes, besides the usual material, a useful family of exact compressible solutions due to Buchdahl. A long chapter on black holes studies in some detail the physical nature of the horizon, going as far as the Kruskal coordinates,

1 The treatment here is therefore different in spirit from that in my book *Geometrical Methods of Mathematical Physics* (Cambridge University Press 1980*b*), which may be used to supplement this one.

then exploring the rotating (Kerr) black hole, and concluding with a simple discussion of the Hawking effect, the quantum mechanical emission of radiation by black holes. The concluding chapter on cosmology derives the homogeneous and isotropic metrics and briefly studies the physics of cosmological observation and evolution. There is an appendix summarizing the linear algebra needed in the text, and another appendix containing hints and solutions for selected exercises. One subject I have decided not to give as much prominence to as other texts traditionally have is experimental tests of general relativity and of alternative theories of gravity. Points of contact with experiment are treated as they arise, but systematic discussions of tests now require whole books (Will 1981). Physicists today have far more confidence in the validity of general relativity than they had a decade or two ago, and I believe that an extensive discussion of alternative theories is therefore almost as out of place in a modern elementary text on gravity as it would be in one on electromagnetism.

The student is assumed already to have studied: special relativity, including the Lorentz transformation and relativistic mechanics; Euclidean vector calculus; ordinary and simple partial differential equations; thermodynamics and hydrostatics; Newtonian gravity (simple stellar structure would be useful but not essential); and enough elementary quantum mechanics to know what a photon is.

The notation and conventions are essentially the same as in Misner *et al.*, *Gravitation* (W. H. Freeman 1973), which may be regarded as one possible follow-on text after this one. The physical point of view and development of the subject are also inevitably influenced by that book, partly because Thorne was my teacher and partly because *Gravitation* has become such an influential text. But because I have tried to make the subject accessible to a much wider audience, the style and pedagogical method of the present book are very different.

Regarding the use of the book, it is designed to be studied sequentially as a whole, in a one-year course, but it can be shortened to accommodate a half-year course. Half-year courses probably should aim at restricted goals. For example, it would be reasonable to aim to teach gravitational waves and black holes in half a year to students who have already studied electromagnetic waves, by carefully skipping some of Chs. 1–3 and most of Chs. 4, 7, and 10. Students with preparation in special relativity and fluid dynamics could learn stellar structure and cosmology in half a year, provided they could go quickly through the first four chapters and then skip Chs. 9 and 11. A graduate-level course can, of course, go much more quickly, and it should be possible to cover the whole text in half a year.

Each chapter is followed by a set of exercises, which range from trivial ones (filling in missing steps in the body of the text, manipulating newly introduced mathematics) to advanced problems that considerably extend the discussion in the text. Some problems require programmable calculators or computers. I cannot overstress the importance of doing a selection of problems. The easy and medium–hard ones in the early chapters give essential practice, without which the later chapters will be much less comprehensible. The medium–hard and hard problems of the later chapters are a test of the student's understanding. It is all too common in relativity for students to find the conceptual framework so interesting that they relegate problem solving to second place. Such a separation is false and dangerous: a student who can't solve problems of reasonable difficulty doesn't really understand the concepts of the theory either. There are generally more problems than one would expect a student to solve; several chapters have more than 30. The teacher will have to select them judiciously. Another rich source of problems is the *Problem Book in Relativity and Gravitation*, Lightman *et al.* (Princeton University Press 1975).

I am indebted to many people for their help, direct and indirect, with this book. I would like especially to thank my undergraduates at University College, Cardiff, whose enthusiasm for the subject and whose patience with the inadequacies of the early lecture notes encouraged me to turn them into a book. And I am certainly grateful to Suzanne Ball, Jane Owen, Margaret Vallender, Pranoat Priesmeyer and Shirley Kemp for their patient typing and retyping of the successive drafts.

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Contents

1 Special relativity

1.1 Fundamental principles of special relativity theory (SR)

The way in which special relativity is taught at an elementary undergraduate level – the level which the reader is assumed competent at – is usually close in spirit to the way it was first understood by physicists. This is an algebraic approach, based on the Lorentz transformation (§ 1.7 below). At this basic level, one learns how to use the Lorentz tra sformation to convert between one observer's measurements and another's, to verify and understand such remarkable phenomena as time dilation and Lorentz contraction, and to make elementary calculations of the conversion of mass into energy.

This purely algebraic point of view began to change, to widen, less than four years after Einstein proposed the theory.[1] Minkowski pointed out that it is very helpful to regard (t, x, y, z) as simply four coordinates in a four-dimensional space which we now call spacetime. This was the beginning of the geometrical point of view which led directly to general relativity in 1914–16. It is this geometrical point of view on special relativity which we must study before all else.

1 Einstein's original paper was published in 1905, while Minkowski's discussion of the geometry of spacetime was given in 1908. Einstein's and Minkowski's papers are reprinted (in English translation) in *The Principle of Relativity* by A. Einstein, H. A. Lorentz, H. Minkowski & H. Weyl (Dover).

As we shall see, special relativity can be deduced from two fundamental postulates:

(1) *Principle of relativity* (Galileo): No experiment can measure the absolute velocity of an observer; the results of any experiment performed by an observer do not depend on his speed relative to other observers who are not involved in the experiment.

(2) *Universality of the speed of light* (Einstein): The speed of light relative to any unaccelerated observer is $c = 3 \times 10^8$ m s^{-1}, regardless of the motion of the light's source relative to the observer. Let us be quite clear about this postulate's meaning: two different unaccelerated observers measuring the speed of the *same photon* will each find it to be moving at 3×10^8 m s^{-1} relative to themselves, regardless of their state of motion relative to each other.

As noted above, the principle of relativity is not at all a modern concept: it goes back all the way to Galileo's hypothesis that a body in a state of uniform motion remains in that state unless acted upon by some external agency. It is fully embodied in Newton's second law, which contains only accelerations, not velocities themselves. Newton's laws are, in fact, all invariant under the replacement

$$\boldsymbol{v}(t) \to \boldsymbol{v}'(t) = \boldsymbol{v}(t) - \boldsymbol{V},$$

where $\boldsymbol{V}$ is any *constant* velocity. This equation says that a velocity $\boldsymbol{v}(t)$ relative to one observer becomes $\boldsymbol{v}'(t)$ when measured by a second observer whose velocity relative to the first is $\boldsymbol{V}$. This is called the Galilean law of addition of velocities.

By saying that Newton's laws are *invariant* under the Galilean law of addition of velocities, we are making a statement of a sort which we will often make in our study of relativity, so it is well to start by making it very precise. Newton's first law, that a body moves at a constant velocity in the absence of external forces, is unaffected by the replacement above, since if $\boldsymbol{v}(t)$ is really a constant, say $\boldsymbol{v}_0$, then the new velocity $\boldsymbol{v}_0 - \boldsymbol{V}$ is also a constant. Newton's second law,

$$\boldsymbol{F} = m\boldsymbol{a} = m\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t,$$

is also unaffected, since

$$\boldsymbol{a}' = \mathrm{d}\boldsymbol{v}'/\mathrm{d}t = \mathrm{d}(\boldsymbol{v} - \boldsymbol{V})/\mathrm{d}t = \mathrm{d}\boldsymbol{v}/\mathrm{d}t = \boldsymbol{a}.$$

Therefore the second law will be valid according to the measurements of both observers, provided that we add to the Galilean transformation law the statement that $\boldsymbol{F}$ and m are themselves invariant, i.e. the same regardless of which of the two observers measures them. Newton's third law, that the force exerted by one body on another is equal and opposite to that exerted by the second on the first, is clearly unaffected by the change of observers, again because we assume the forces to be invariant.

So there is no absolute velocity. Is there an absolute acceleration? Newton argued that there was. Suppose, for example, that I am in a train on a perfectly smooth track,[2] eating a bowl of soup in the dining car. Then if the train moves at constant speed the soup remains level, thereby offering me no information about what my speed is. But if the train changes its speed then the soup climbs up one side of the bowl, and I can tell by looking at it how large and in what direction the acceleration is.[3]

Therefore, it is reasonable and useful to single out a class of preferred observers: those who are unaccelerated. They are called *inertial observers*, and each one has a constant velocity with respect to any other one. These inertial observers are fundamental in special relativity, and when we use the term 'observer' from now on we will mean an inertial observer.

The postulate of the universality of the speed of light was Einstein's great and radical contribution to relativity. It smashes the Galilean law of addition of velocities because it says that if $\boldsymbol{v}$ has magnitude c then so does $\boldsymbol{v}'$, regardless of $\boldsymbol{V}$. Einstein felt that the postulate was forced on him by, among other things, the Michelson–Morley experiment. The counter-intuitive predictions of special relativity all flow from this postulate, and they are amply confirmed by experiment. In fact it is probably fair to say that special relativity has a firmer experimental basis than any other of our laws of physics, since it is tested every day in all the giant particle accelerators, which send particles nearly to the speed of light.

Although the concept of relativity is old, it is customary to refer to Einstein's theory simply as 'relativity'. The adjective 'special' is applied in order to distinguish it from Einstein's theory of gravitation, which acquired the name 'general relativity' because it permits one to describe physics from the point of view of both accelerated and inertial observers and is in that respect a more general form of relativity. But the real physical distinction between these two theories is that special relativity (SR) is capable of describing physics only in the absence of gravitational fields, while general relativity (GR) extends SR to describe gravitation

2 Physicists frequently have to make such idealizations, which often are far removed from common experience!

3 For Newton's discussion of this point, see the excerpt from his *Principia* in Williams (1968).

itself.[4] One can only wish that an earlier generation of physicists had chosen more appropriate names for these theories!

1.2 Definition of an inertial observer in SR

It is important to realize that an 'observer' is in fact a huge information-gathering system, not simply one man with binoculars. In fact, we shall remove the human element entirely from our definition, and say that an inertial observer is simply a coordinate system for spacetime, which makes an observation simply by recording the location (x, y, z) and time (t) of any event. This coordinate system must satisfy the following three properties to be called *inertial*:

(1) The distance between point P_1 (coordinates x_1, y_1, z_1) and point P_2 (coordinates x_2, y_2, z_2) is independent of time.

(2) The clocks that sit at every point ticking off the time coordinate t are synchronized and all run at the same rate.

(3) The geometry of space at any constant time t is Euclidean.

Notice that this definition does not mention whether the observer accelerates or not. That will come later. It will turn out that only an unaccelerated observer can keep his clocks synchronized. But we prefer to start out with this geometrical definition of an inertial observer. It is a matter for experiment to decide whether such an observer can exist: it is not self-evident that any of these properties *must* be realizable, although we would probably expect a 'nice' universe to permit them! However, we will see later in the course that a gravitational field does in fact make it impossible to construct such a coordinate system, and this is why GR is required. But let us not get ahead of the story. At the moment we are assuming that we *can* construct such a coordinate system (that, if you like, the gravitational fields around us are so weak that they do not really matter). One can envision this coordinate system, rather fancifully, as a lattice of rigid rods filling space, with a clock at every intersection of the rods. We shall now define how we use this coordinate system to make observations.

An *observation* made by the inertial observer is the act of assigning to any event the coordinates x, y, z of the location of its occurrence, and

4 It is easy to see that gravitational fields cause problems for SR. If an astronaut in orbit about Earth holds a bowl of soup, does the soup climb up the side of the bowl in response to the gravitational 'force' which holds the spacecraft in orbit? Two astronauts in different orbits accelerate relative to one another, but neither *feels* an acceleration. Problems like this make gravity special, and we will have to wait until Ch. 5 to resolve them. Until then, the word 'force' will refer to a nongravitational force.

the time read by the clock at (x, y, z) when the event occurred. It is *not* the time t on the wrist watch worn by a scientist located at (0, 0, 0) when he first learns of the event. A *visual* observation is of this second type: the eye regards as simultaneous all events it *sees* at the same time; an inertial observer regards as simultaneous all events that *occur* at the same time as recorded by the clock nearest them when they occurred. This distinction is important and must be borne in mind. Sometimes we will say 'an observer sees ...' but this will only be shorthand for 'measures'. We will never mean a *visual* observation unless we say so explicitly.

An inertial observer is also called an *inertial reference frame*, which we will often abbreviate to 'reference frame' or simply 'frame'.

1.3 New units

Since the speed of light c is so fundamental, we shall from now on adopt a new system of units for measurements in which c simply has the value 1! It is perfectly okay for slow-moving creatures like engineers to be content with the SI units: m, s, kg. But it seems silly in SR to use units in which the fundamental constant c has the ridiculous value 3×10^8. The SI units evolved historically. Meters and seconds are not fundamental; they are simply convenient for human use. What we shall now do is adopt a new unit for time, the meter. One meter of time is the time it takes light to travel one meter. (You are probably more familiar with an alternative approach: a year of distance – called a 'light year' – is the distance light travels in one year.) The speed of light in these units is:

$$c = \frac{\text{distance light travels in any given time interval}}{\text{the given time interval}}$$

$$= \frac{1\ \text{m}}{\text{the time it takes light to travel one meter}}$$

$$= \frac{1\ \text{m}}{1\ \text{m}} = 1.$$

So if we consistently measure time in meters, then c is not merely 1, it is also dimensionless! In converting from SI units to these 'natural' units, you can use any of the following relations:

$$3\times10^8\ \text{m s}^{-1} = 1,$$

$$1\ \text{s} = 3\times10^8\ \text{m},$$

$$1\ \text{m} = \frac{1}{3\times10^8}\ \text{s}.$$

The SI units contain many 'derived' units, such as joules and newtons, which are defined in terms of the basic three: m, s, kg. By converting from s to m these units simplify considerably: energy and momentum are measured in kg, acceleration in m^{-1}, force in kg m^{-1}, etc. Do the exercises on this. With practice, these units will seem as natural to you as they do to most modern theoretical physicists.

1.4 Spacetime diagrams

A very important part of learning the geometrical approach to SR is mastering the spacetime diagram. In the rest of this chapter we will derive SR from its postulates by using spacetime diagrams, because they provide a very powerful guide for threading one's way among the many pitfalls SR presents to the beginner. Fig. 1.1 below shows a two-dimensional slice of spacetime, the t–x plane, in which are illustrated

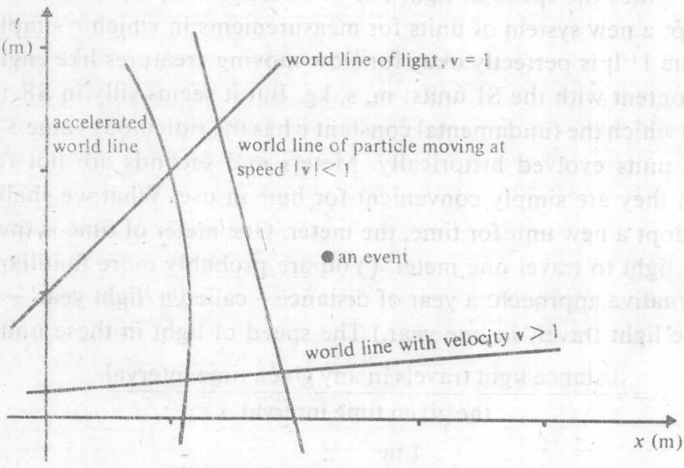

Fig. 1.1 A spacetime diagram in natural units.

the basic concepts. A single point in this space[5] is a point of fixed x *and* fixed t, and is called an *event*. A line in the space gives a relation $x = x(t)$, and so can represent the position of a particle at different times. This is called the particle's *world line*. Its slope is related to its velocity,

$$\text{slope} = dt/dx = 1/v.$$

Notice that a light ray (photon) always travels on a 45° line in this diagram.

5 We use the word 'space' in a more general way than you may be used to. We do not mean a Euclidean space in which Euclidean distances are necessarily physically meaningful. Rather, we mean just that it is a set of points which is continuous (rather than discrete, as a lattice is). This is the first example of what we will define in Ch. 5 to be a 'manifold'.