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Singularities, Representation of Algebras, and Vector Bundles

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Editors

Gert-Martin Greuel
Günther Trautmann
Fachbereich Mathematik, Universität Kaiserslautern
Erwin-Schrödinger-Straße, 6750 Kaiserslautern, Federal Republic of Germany

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INTRODUCTION

These are the Proceedings of the Symposium "Singularities, Representation of Algebras, and Vectorbundles" held December 13-17, 1985 at the Pfalzakademie in Lambrecht/Pfalz, West Germany.

The purpose of the symposium was to discuss and promote recent developments of the interaction of singularity theory with representation of algebras and vector bundles. The colloquium talks given during the conference initiated intensive and stimulating discussions among the participants of the different areas who usually do not meet at conferences. These discussions led to new insights as well as to new questions concerning the relationship between the three topics of the conference. They partly condensed subsequently in research articles which are - besides the revised texts of oral lectures - presented in this volume. It is the editors' hope that these notes stimulate further development and interaction.
It is nowadays well known that there are close relations between classes of singularities and representation theory by means of the McKay correspondence and representation theory and vector bundles on projective space via the Bernstein-Gelfand-Gelfand construction. On the other hand, these relations can not be considered to be either completely understood or fully exploited.

It became clear during the conference that at least the following can be said about the principal relations between the three areas. The questions and methods of representation theory (as finite and tame representation type, almost split sequences, quivers) have applications to singularities and to vector bundles depending whether one considers modules over complete local rings or graded modules over graded rings. These led in particular to the characterization of the simple singularities in the sense of Arnold by maximal Cohen-

Macaulay modules generalizing the work of Mc Kay, Auslander, Artin and Verdier. Representation theory on the other hand, which had primarily developed its methods for Artinian algebras, starts to investigate algebras of higher dimensions partly because of these applications. There are not only interesting examples within the class of singularities or algebraic varieties, there might be also future research in representation theory, stimulated by the classification of singularities and the highly developed theory of moduli for vector bundles.

Of course, there is the general problem for specialists to understand well enough topics in fields other than their own. In order to overcome this difficulty at least partly during the conference there were three survey talks on the different topics stressing the relationship to the other two: H. Knörrer on "Cohen-Macaulay Modules on Hypersurface Singularities", M.S. Narasimhan on "Moduli of Vector Bundles on Curves", and I. Reiten on "Representation of Algebras and Relations to Singularities". The latter two are presented in this volume, Knörrer's article has already appeared in the proceedings "Representation of Algebras", Lond. Math. Soc., Lect. Notes, Series 116, 147-164, Cambridge Univ. Press. His article surveys the recent development of the interaction of singularity theory and representation theory and is warmly recommended. We are grateful to F.-O. Schreyer for having expanded his oral lecture in order to give a partial survey, together with new results and open problems and conjectures.

Not all the oral lectures are published in these proceedings because they had already been submitted to other journals. On the other hand we are pleased to include others which fit well into the subject of the conference, in particular those which have been initiated by the symposium itself.

With the exception of the survey talks all articles contain original research. The participants served as referees and we owe them much thanks.

ACKNOWLEDGEMENTS

The symposium was made possible by the generous financial support of the Stiftung Volkswagenwerk. Further support was provided by the Kultusministerium des Landes Rheinland-Pfalz and by the Fachbereich Mathematik der Universität Kaiserslautern. We like to thank these institutions for their help.

The pleasant atmosphere of the Pfalzakademie and the beautiful surroundings contributed to the success of the symposium.

G.-M. Greuel

G. Trautmann

LIST OF TALKS NOT PRESENTED IN THIS VOLUME

W. Barth
Degeneration of Horrocks-Mumford surfaces

D. Eisenbud
Plane sections of determinantal varieties

A. Holme
On duality for projective varieties

K. Hulek
Complete families of stable vector bundles

M. Maruyama
On the rationality of moduli spaces of vector bundles of rank 2 on $\mathbb{P}_2$

M. Schneider
Linear normality for subvarieties of projective spaces

C.S. Seshadri
Singularities of Schubert varieties

D. Siersma
A class of non-isolated hypersurface singularities

S.A. Strømme
Quot schemes on $\mathbb{P}_1$

LIST OF PARTICIPANTS

M. Auslander
Brandeis

W. Barth
Erlangen

J. Christophersen
Oslo

W. Decker
Kaiserslautern

E. Dieterich
Bielefeld

P. Dowbor
Paderborn

J.M. Drezet
Paris 7

D. Eisenbud
Brandeis

H. Flenner
Göttingen

O. Forster
München

W. Geigle
Paderborn

G.-M. Greuel
Kaiserslautern

D. Happel
Bielefeld

J. Herzog
Essen

A. Holme
Bergen

G. Horrocks
Newcastle

K. Hulek
Bayreuth

Th. de Jong
Leiden

C. Kahn
Hamburg

H. Knörrer
Bonn

H. Kröning
Kaiserslautern

H. Lenzing
Paderborn

M. Maruyama
Kyoto

M.S. Narasimhan
Bombay

J. Le Potier
Paris 7

C.J. Rego
Bombay

I. Reiten
Trondheim

O. Riemenschneider
Hamburg

C.M. Ringel
Bielefeld

K.W. Roggenkamp
Stuttgart

A. Schappert
Kaiserslautern

M. Schlichenmaier
Mannheim

G. Schneider
Kaiserslautern

M. Schneider
Bayreuth

F.-O. Schreyer
Kaiserslautern

C.S. Seshadri
Madras

D. Siersma
Utrecht

W. Singhof
Kaiserslautern

H. Spindler
Göttingen

J.H. Steenbrink
Leiden

D. van Straten
Leiden

St.A. Strømme
Bergen

G. Trautmann
Kaiserslautern

B. Uher
Erlangen

L. Unger
Bielefeld

H. Völlinger
Kaiserslautern

A. Wiedemann
Stuttgart

AUTHORS' ADDRESSES

M. Auslander	Department of Mathematics, Brandeis University, Waltham, Mass. 02257, USA
W. Böhmer	Fachbereich Mathematik der Universität Erwin-Schrödinger-Str., 6750 Kaiserslautern
R.O. Buchweitz	Institut für Mathematik der Universität Welfengarten 1, 3000 Hannover 1
J. Christophersen	Matematisk Institutt, Universitetet i Oslo, P.b. 1053 Blindern, Oslo 3, Norwegen
E. Dieterich	Fakultät für Mathematik der Universität Universitätsstr., 4800 Bielefeld 1
J.M. Drezet	Université de Paris VII, Mathématiques 2, Place Jussieu, 75 230 Paris Cedex 05
D. Eisenbud	Department of Mathematics, Brandeis University, Waltham, Mass. 02254, USA
C. Ellingsrud	Matematisk Institutt, Universitetet i Oslo P.b. 1053 Blindern, Oslo 3, Norwegen
O. Forster	Mathematisches Institut der Universität Theresienstrasse 39, 8000 München 2
W. Geigle	Fachbereich 17 - Mathematik - Informatik der Universität, Warburger Str. 100 4790 Paderborn
D. Happel	Fakultät für Mathematik der Universität Universitätsstr. 4800 Bielefeld 1
J. Herzog	FB Mathematik der Gesamthochschule Universitätsstr. 2, 4300 Essen
H. Knörrer	Mathematisches Institut der Universität Universitätsstr. 1, 4000 Düsseldorf 1
H. Lenzing	Fachbereich 17 - Mathematik - Informatik der Universität, Warburger Str. 100 4790 Paderborn

M.S. Narasimhan	Tata Institute of Fundamental Research Homi Bhabha Road, Bombay 400 005, Indien
C.J. Rego	19, 'Prasanna', Nesbit Road, Bombay 400 010, Indien
I. Reiten	Department of Mathematics, University of Trondheim, Trondheim, Norwegen
K.W. Roggenkamp	Fakultät für Mathematik und Informatik Universität, Pfaffenwaldring 57, 7000 Stuttgart 80
A. Schappert	Fachbereich Mathematik der Universität Erwin-Schrödinger-Str., 6750 Kaiserslautern
F.-O. Schreyer	Fachbereich Mathematik der Universität Erwin-Schrödinger-Str., 6750 Kaiserslautern
C.S. Seshadri	The Institute of Mathematical Sciences Madras 600 113, Indien
J.H.M. Steenbrink	Subfakulteit der Wiskunde, Rijksuniversiteit te Leiden, Wassenaarseweg 80, 23 RA Leiden, Holland
D. van Straten	Subfaculteit der Wiskunde, Rijksuniversiteit te Leiden, Wassenaarseweg 80, 23 RA Leiden, Holland
St.A. Strømme	Matematisk Institutt, Universitetet i Bergen 5014 Bergen, Norwegen
K. Wolffhardt	Mathematisches Institut der Universität Theresienstr. 39, 8000 München 2
J. Wunram	Mathematisches Seminar der Universität Bundesstr. 55, 2000 Hamburg 13
St. Zucker	Department of Mathematics, Johns Hopkins University, Baltimore MD 21218, USA

TABLE OF CONTENTS

Survey-Talks

Research-Articles

Singularities

Representation of Algebras

Vector Bundles

SURVEY OF VECTOR BUNDLES ON CURVES

M.S. Narasimhan
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005
INDIA

1. Moduli problem for vector bundles on curves.

Let X be a compact Riemann surface, or what is the same, a smooth projective irreducible algebraic curve over $\mathbb{C}$. It is well known that the set of isomorphism classes of holomorphic (or algebraic) line bundles of degree d has a natural structure of a smooth projective variety; if $d = 0$ we obtain an abelian variety, the Jacobian of X. Moreover any line bundle of degree 0 on X is associated to a (unitary) character of the fundamental group of X [9].

The corresponding 'moduli problem' for (algebraic) vector bundles of higher rank on X was first envisaged by A. Weil in 1938 in a famous paper [30]. Naively formulated, the question is whether there is a natural structure of a variety on the set of isomorphism classes of vector bundles of a given rank and degree on X. However it is easy to see that even 'locally' around certain bundles of rank ≥ 2 one can not have a structure of a variety, for example due to the jump-phenomenon [see 17, p. 126]. This suggests that one can expect a structure of variety only on a suitable subset of the isomorphism classes of vector bundles.

In what follows we shall assume that the genus g of X is ≥ 2. It is well known that any vector bundle on a curve of genus 0 is a direct sum of line bundles. Vector bundles on curves of genus 1 were investigated in detail by M.F. Atiyah [1].

2. Flat bundles and a theorem of A. Weil.

We shall consider in this section a class of vector bundles on X, namely flat bundles, which play an important role in the moduli problem for vector bundles on curves.

Let $\widetilde{X}$ be a universal covering of X and let the fundamental group $\pi = \pi_1(X)$ act on $\widetilde{X}$ on the right. If ρ is a homomorphism of π into the full linear group $GL(r,\mathbb{C})$, we can construct a holomorphic vector bundle $W_\rho = \widetilde{X} \times_\pi \mathbb{C}^r$ on X; the bundle W_ρ is the quotient of $\widetilde{X} \times \mathbb{C}^r$ under the action of π given by :

$$(\widetilde{x},v)\gamma = (\widetilde{x}\gamma,\ \rho(\gamma)^{-1}v),\ \widetilde{x} \in \widetilde{X},\ v \in \mathbb{C}^r,\ \gamma \in \pi .$$

We say that a holomorphic vector bundle V on X arises from a representation (resp. unitary representation) of π, if V is isomorphic to W_ρ where ρ is a homomorphism of π into $GL(r,\mathbb{C})$ (resp. into to the unitary group $U(r)$).

Among other results A. Weil proved in [30]

Theorem 2.1. A vector bundle on X arises from a representation of the fundamental group of X if and only if each of its indecomposable components is of degree zero.

A. Weil also expected that bundles which arise from unitary representations would play an important role.

3. Stable and semi-stable bundles.

The crucial step in the progress of moduli problem for vector bundles on curves was the introduction by David Mumford of the all important notion of stable vector bundles. This concept was motivated by the geometric invariant theory [6].

Definition 3.1. A vector bundle V on X is said to be stable (resp. semi-stable) if for every proper subbundle W of V we have

$$\frac{\deg W}{\operatorname{rank} W} < \frac{\deg V}{\operatorname{rank} V} \text{ (resp. } \frac{\deg W}{\operatorname{rank} W} \leq \frac{\deg V}{\operatorname{rank} V}) ,$$

where $\deg (V) = C_1(V)[X]$, $C_1(V)$ denoting the first chern class of V.

Observe that a semi-stable bundle is automatically stable, if its rank and degree are coprime.

D. Mumford proved [7]

Theorem 3.1. The set of isomorphism classes of stable vector bundles on X of rank r and degree d has a natural structure of a smooth quasi-projective variety of dimension $r^2(g-1)+1$.

4. Stable bundles and unitary bundles.

The following basic theorem was proved by M.S. Narasimhan and C.S. Seshadri [15,16].

Theorem 4.1. A vector bundle on X of degree 0 is stable if and only if it arises from an irreducible unitary representation of the fundamental group of X.

As a consequence one sees that a vector bundle arises from a unitary representation of $\pi_1(X)$ if and only if each of its indecomposable components is of degree 0 and stable. Moreover it is easy to show that two vector bundles arising from unitary representations are isomorphic if and only if the representations are equivalent.

In the same paper a characterisation similar to Theorem 4.1 was also given for stable bundles of arbitrary degree in terms of irreducible unitary representations of a certain Fuchsian group. This result implies that, if $(r,d) = 1$, the space of isomorphism classes of stable bundles of rank r and degree d is compact and is hence a smooth projective variety.

5. The moduli space of semi-stable bundles.

The results stated in § 4 suggest a natural compactification of the space of stable bundles of degree 0, namely the space of equivalence classes of unitary representations (not necessarily irreducible) of a given rank. C.S. Seshadri proved that this compactification is a projective variety [23]. Before stating his result precisely we will introduce an equivalence relation among semi-stable bundles.

Let V be a semi-stable vector bundle on X. Then V has a strictly decreasing filtration by subbundles

$$V = V_0 \supset V_1 \supset \cdots \supset V_k = (0)$$

such that for $1 \leq i \leq k$ the bundle $W_i = V_i/V_{i-1}$ is stable and satisfies $\frac{\deg W_i}{\operatorname{rank} W_i} = \frac{\deg V}{\operatorname{rank} V}$. Moreover the bundle $Gr(V) = \bigoplus_{i=1}^{k} (V_i/V_{i-1})$ is uniquely determined by V upto isomorphism (Jordan-Hölder theorem). We say that two semi-stable bundles V_1 and V_2 are S-equivalent if $Gr(V_1)$ is isomorphic to $Gr(V_2)$. Observe that two stable bundles are S-equivalent if and only if they are isomorphic. It is clear, using Theorem 4.1, that the set of equivalence classes of unitary representations is in canonical bijective correspondence with the set of S-equivalence classes of semi-stable vector bundles of degree 0.

C.S. Seshadri, using geometric invariant theory, proved [23]

Theorem 5.1. There is a unique structure of a normal projective (irreducible) variety $U(r,d)$ on the set of S-equivalence classes of semi-stable vector bundles on X of rank r and degree d such that the following property holds : if $\{V_t\}_{t \in T}$ is an algebraic family of semi-stable bundles on X of rank r and degree d parametrised by an algebraic variety T, then the map $T \longrightarrow U(r,d)$, sending $t \in T$ to the S-equivalence class of V_t, is a morphism.

We shall call the variety given by Theorem 5.1 the moduli space of (semi-stable) vector bundles of rank r and degree and denote it by $U(r,d)$.

Theorem 5.1 is also valid for a curve X over an algebraically closed field of arbitrary characteristic [25].

6. Singularities of $U(r,d)$.

The set of singular points of $U(r,d)$ has been determined by M.S. Narasimhan and S. Ramanan [11].

Theorem 6.1. The set of non-singular points of $U(r,d)$ is precisely the set of stable points in $U(r,d)$ except when $g = 2$, $r = 2$ and d even. In this exceptional case $U(r,d)$ is smooth.

Explicit desingularisations of $U(2,0)$ have been given by M.S.Narasimhan-S. Ramanan [14] and by C.S. Seshadri [24].

7. Poincaré bundles.

Let $U_S(r,d)$ denote the open set of stable points in $U(r,d)$.

Definition 7.1. Let Ω be a non-empty open subset of $U_S(r,d)$. A Poincaré family of vector bundles parametrised by Ω is an algebraic vector bundle P on $\Omega \times X$ such that for any $\omega \in \Omega$ the bundle on X obtained by restricting P to $\omega \times X$ is in the isomorphism class ω.

It is not hard to see that when $(r,d) = 1$ there is a Poincaré bundle on $U(r,d) \times X$ [8]. S. Ramanan proved [22]

Theorem 7.1. If r and d are not coprime there is no Poincaré family on X parametrised by any non-empty open subset of $U_S(r,d)$.

The special case of this theorem where $g = 2$, $r = 2$, d even, was proved earlier in [10].

8. The variety $S(r,d)$.

In order to study the varieties $U(r,d)$ let us fix a line bundle L of degree d and consider the subvariety of $U(r,d)$ corresponding to semi-stable bundles V with $\Lambda^r V \simeq L$. We will denote this variety by $S_X^L(r,d)$ or simply by $S(r,d)$. The varieties $S(r,d)$ have been studied intensively by G. Harder, M.S. Narasimhan, P.E. Newstead, S. Ramanan and A. Tjurin, especially in the case $(r,d) = 1$. The results obtained pertain to the computation of numerical invariants like the Betti numbers, questions concerning the rationality of these varieties, the relation between the moduli of curves and the moduli of the varieties $S(r,d)$ and the explicit determination of these varieties in low genus or rank.

9. Betti numbers of $S(r,d)$.

The Betti numbers of $S(r,d)$ were first calculated by P.E. Newstead in the case $r = 2$, $d = 1$, by topological methods using the results of § 4 [18]. Based on these results G. Harder verified the Weil conjecture for $S(2,1)$ in the case of a curve defined over a finite field [4], at a time when the Weil conjecture was not proved in general. Harder observed in turn that the Betti numbers of $S(2,1)$ can be computed by arithmetical methods on the basis of the Weil conjecture.

Harder's method was generalised by him and M.S. Narasimhan to bundles of arbitrary rank [5]. It was shown, in the case $(r,d) = 1$, that the ζ-function of $S(r,d)$ can be calculated from the ζ-function of X. This result, together with Weil conjecture proved by P. Deligne, gives a method for computing the Betti numbers of $S(r,d)$ when $(r,d) = 1$.