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Edited by A. Dold and B. Eckmann

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Recurrences and Discrete Dynamic Systems

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P R E F A C E

The idea to study recurrences took root in 1949 during an informal lecture of P. Montel. A discussion about possible types of fixed points in 1959 led to a lasting collaboration with C. Mira. The analytical insight increased more rapidly after an invariant curve germ, based on a Lattès series, was successfully continued by means of numerical computations. The material described in this monograph constitutes a synopsis of the slowly accumulated particular results. Frequent discussions with R. Thibault, R. Clerc, Ch. Hartmann, J. Couot, C. Gillot, O. Rössler and G. Targonski produced substantial improvments.

During the last few years recurrences appeared in several fields of applied science and they have become a major research topic of the interdisciplinary Dynamic Systems Research Group of Toulouse University. The creation of such a group would have been impossible without the continued support of J.C. Martin, President of Toulouse University.

The text was typed by Mrs. C. Grima and many of the figures were drawn by G. Roussel. The preparation of the typescript was encouraged by Ph. Leturcq. All contributions are gratefully acknowledged.

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CONTENTS

INTRODUCTION AND STATEMENT OF THE PROBLEM

The content of this monograph is intended to be easily accessible to readers from various disciplines, such as mathematics, physics, biology, biometry, medical biometry, ecology, etc., who face time-evolving phenomena, or as the latter are concisely called : dynamic systems. For this reason, the simplest possible terminology has been adopted, in order not to add artificial vocabulary-hurdles to the intrinsic difficulty of the subject. Ars-pro-artis generalisations, so popular in contemporary mathematics, have been deliberately sacrified in order not to obscure the dominating internal mechanisms of the evolution processes and thus preserve a phenomenological transparency. The emphasis on "what happens" instead of on symbols appears to be a valid motivation of all efforts to understand nature (as well as mathematics). The resulting formulation of problems may therefore appear primitive, sometimes even simplistic, and the terminology old-fashioned. Abstractly inclined readers will no doubt find occasions to feel irritated ; the authors accept in advance full blame for these distractions from the subject-matter in question. Experience in dynamic systems teaches, however, that apparently simple problems turn out to have complicated solutions, whose description happens to encroach on the "influence spheres" of several distinct mathematical disciplines, each having already a frozen professional vocabulary. One way out of this linguistic predicament appears to be the pursuit of maximal simplicity, and thus the search for a maximal interdisciplinary common part ; which is the exposition method adopted by the authors. The monograph is restricted to the study of discrete dynamic dystems, taken from a "natural" context, expressed in the form of one or two-dimensional real-valued point-mappings, or according to old French usage, recurrences.

Recurrences occur in many branches of mathematics, ranging from number theory to functional equations (c.f. [M 10]). They appear also independently as natural descriptions of evolution phenomena in physics, biology, etc. (c.f. [M 2]). Individual results on recurrences are therefore scattered in different types of publications, and are generally expressed in a widely varying vocalubary. The purpose of this monograph is to provide a systematic and unified treatment of presently known and physically, biologically, etc. relevant properties of first and second order real-valued recurrences. Several new results are published here for the first time. For conciseness, all constants, parameters, variables and functions are assumed to be real-valued, unless the contrary is explicitly stated.

The conventional specialized symbols used to express this property will be omitted. Functional spaces will be treated in the same manner ; in most cases only a verbal specification will be given. Readers in need of constantly repeated specialized symbols will experience no difficulty in supplying their own.

Autonomous first order recurrences of the form

(0-1) $$x_{n+1} = f(x_n, c), \qquad n = 0, \pm 1, \pm 2, \ldots$$

where c is a parameter and f(x, c) a single-valued smooth function of both x and c, constitute one of the oldest mathematical notions (examples : relation between two successive numbers of a sequence ; Hurwitz's definition of the logarithm in terms of the limit of x_n, $n \to \infty$ when $x_{n+1} = \sqrt{x_n}$, $x_o > 0$). A rather fundamental role is played by recurrences in the works of Poincaré, although at first sight it appears that they are introduced there merely as an artifice for the study of trajectories of dynamic systems, defined by a system of ordinary differential equations, where the independent variable represents time. The advantage obtained by means of this artifice consists in the reduction of the dimensionality of the problem by one unit, which is accomplished by the elimination of one dependent variable via a suitable "surface of section". As an illustration of the procedure consider a particular example [A 5] possessing explicit expressions in terms of elementary functions at every stage (this is completely untypical in a non linear context).

Let the dynamic system be of order two :

(0-2) $$\ddot{x} + 2b\dot{x} + x = 0,\ \dot{x} < 0\ ;\ = a,\ \dot{x} = 0\ ;\ = 1,\ \dot{x} > 0\ ;\ t \geqslant 0\ ,$$

where $0 < b \ll 1$, $0 < a < 1$, and $x = x(t)$, $\dot{x}(t) = y(t)$ are continuous functions. It is required to determine the nature of the solutions of (0-2) in the phase plane x, y ; and in particular to ascertain the possible existence of periodic solutions, which to quote Poincaré, constitute the main breach in the natural defences (i.e. in the intrinsic complexity) of non linear dynamic systems.

Since the dynamic system (0-2) is linear for $y < 0$ and $y > 0$, the corresponding component-solutions are :

(0-2') $$y < 0:\ x_-(t) = e^{-bt}(\bar{x} \cos \omega t + \bar{z} \sin \omega t),\ \bar{z} = \frac{1}{\omega}(\bar{y} + b\bar{x}),$$
$$y > 0:\ x_+(t) = 1 + x_-(t),\ \omega^2 = 1 - b^2,$$

where $\bar{x}$, $\bar{y}$ is either $x_-(0)$, $y_-(0)$ or $x_+(0)$, $y_+(0)$. The assumed continuity of x(t) and $\dot{x}(t)$ implies that all phase-plane trajectories G(x, y) = const. of (0-2), defined parametrically by (0-2'), are continuous, cross the x-axis, and are not tangent to the latter. Thus the x-axis, or a part of it, is an appropriate (one-dimensional) surface of section. Fix the initial point $(\bar{x}, \bar{y})$, $\bar{y} \neq 0$, and let $0 < x_o \neq a$ be the first x-axis crossing point of a trajectory of (0-2) passing through $(\bar{x}, \bar{y})$. It follows from (0-2') that the next positive x-axis crossing occurs at

$$x_1 = 1 + e^{-c} + x_o\, e^{-2c},\ c = \pi b/\omega\ ,$$

and, in general, two successive positive x-axis crossing points of G(x, y) = const. are related by the linear recurrence

(0-3) $$x_{n+1} = 1 + e^{-c} + x_n\, e^{-2c},\ x_n > 0,\ n = 0, 1, 2, \ldots$$

By construction, the first order recurrence (0-3) contains the same amount of information about the function G(x, y) as the second order differential equation (0-2). If (0-3) is to be useful, however, it is necessary to know how to find its relevant solution. Because of the simplicity of both (0-2) and (0-3), this solution can be found by trial and error :

$$x_n = x_o e^{-2cn} + (1 - e^{-2cn}) / (1 - e^{-c}) \qquad \text{(0-4)}$$

without any need of a preliminary abstract analysis characterizing its nature, and the functional space in which it is located. A known explicit solution is, of course, more informative than an abstract theorem affirming its existence, uniqueness and continuity, but unfortunately in the general context of non linear recurrences such a favourable situation occurs only exceptionally. From an examination of x_n in (0-4) as a function of n, it is easily seen that the dynamic system (0-2) admits a unique periodic solution, whose closed phase plane trajectory G_∞ crosses the x-axis at

$$x_\infty = \lim_{n\to\infty} x_n = 1/(1 - e^{-c}) \quad . \qquad \text{(0-5)}$$

The uniqueness of G_∞ results from the fact that x_∞ is independent of x_o. All other trajectories G are spirals which approach G_∞ asymptotically. The periodic solution is therefore asymptotically stable (in the sense of Liapunov) and its influence domain is the whole phase plane x, y, i.e. it is reached from arbitrary $\bar{x}$, $\bar{y}$. The point ($x_o \neq a$, 0), excluded in the determination of (0-3), is an unstable constant solution of (0-2).

It is known that in certain cases (example : problem of the climate, [L 9]), the trajectory structure of a differential equation is described by a recurrence whose order is two units lower. The reduction of dimensionality by one or more units is obviously not a general property of an intrinsic structural equivalence between a differential equation and a recurrence. Consider in fact the first order autonomous differential equation :

$$\dot{x} = g(x, c), \; x = x(t), \; t > 0, \; x(0) = x_o \quad , \qquad \text{(0-6)}$$

where x_o, c are parameters and g is a single-valued continuously differentiable function of its arguments. Let

$$x(t) = H(x_o, t, c) \qquad \text{(0-7)}$$

be the general solution of (0-6). Replacing x(t) by the finite difference (x(t+h) - x(t)) / h, h > 0 and letting $t_n = nh$, $x_n = x(t_n)$, equation (0-6) turns into the recurrence

$$x_{n+1} = x_n + hg(x_n, c) \qquad \text{(0-8)}$$

which is of the same order as the differential equation (0-6). For $h \ll 1$ the solution structure of (0-6) and (0-8) is known to be equivalent (Euler's method of discretization, but this equivalence does not persist for larger h, [G 9], p. 46).

The two examples (0-2), (0-3) and (0-6), (0-8), raise the following question : how is it possible to decide whether for a specific f the recurrence (0-1) is structurally equivalent to a differential equation of order one, two or higher ? A meaningful answer to this question is impossible unless the notions "solution of a recurrence" and "structural equivalence", used above in a rather self-evident fashion, have been given unambiguous definitions. The detailed nature of these definitions appears to have far-reaching consequences with respect to what constitutes a characteristic property of a recurrence, because such a characterization is to be consistent with the properties of associated differential equations, regardless of a possible difference of order. Moreover, since there exists a very strong link (c.f. [K 6]) between the recurrence (0-1), the functional iterates

(0-9) $$f_{n+1}(x, c) = f(f_n(x, c)),\ f_1(x, c) = f(x, c),\ f_o(x, c) = x,$$

$n = 0, 1, 2, \ldots$ and some functional equations, like for example

(0-10)
$$w(f(x, c)) = s\, w(x) \qquad \text{(Schröder)}$$
$$w(f(x, c)) = w(x) \qquad \text{(automorphic functions)}$$
$$w(f(x, c)) = (w(x))^m \qquad \text{(Böttcher)}$$
$$w(f(x, c) = w(x) + a \qquad \text{(Abel)}$$
$$\sum_i u_i\, w(v_i) = w(x) \qquad \text{(Perron-Frobenius, special case)}$$

where m, s, a, are parameters, $u_i = u_i(f(x, c))$, $v_i = v_i(f(x, c)$ known functions of f, and f is the same as in (0-1), any characteristic property of a recurrence sheds considerable light on the properties of iterations and functional equations. It should be stressed at this point that x is a discrete dependent variable in (0-1), whereas in (0-9), (0-10), it is usually a continuous independent one.

Due to the assumed discreteness of x, the recurrence (0-1) is not equivalent to a nominally identical difference equation. In a difference equation x is assumed to be a continuous variable. This apparently minor distinction has major consequences. For example, when (0-1) is considered as a recurrence, a completely and unambiguously defined initial data (i.e. Cauchy) problem is formulated for $n > 0$ by specifying an isolated initial value x_o of x, whereas when (0-1) is considered as a difference equation, it is necessary to specify x on a continuum of values (for example, on the interval $x_o < x < x_1$). In the first case the solution is a sequence of real numbers $\{x_n\}$, $n = 0, 1, 2, \ldots$, and in the second a rather complicated mathematical entity (a functional of the initial function). In the case of a linear recurrence (with respect to x_n), this theoretically fundamental distinction dissolves into practical insignificance (see for example eq. (0-3) and (0-4)), but for non linear f no deeper analysis of (0-1) is possible without it.

Even a cursory examination of the literature on non linear recurrences shows that their properties are extremely complex. The simplest possible "generic" example of (0-1) :

(0-11) $$x_{n+1} = x_n^2 + c, \qquad -2 \leq c \leq \frac{1}{4}$$,

has been studied intensely during at least two decades [M 11]-[M 15], but its basic solution structure has been identified only recently. This recurrence, examined in detail in Chapter I, constitutes therefore a yardstick with which to measure the efficiency of arguments and the amount of progress made on other first order recurrences.

Keeping in mind the fact that recurrences appear fundamentally as natural descriptions of observed evolution phenomena, because most measurements of time-evolving variables (except for short-period continuous "analog"-recordings) are discrete, and only incidentally via differential or functional equations (which from a fundamental physical point of view are more far-fetched abstractions), it is possible to refer to them as dynamic systems in their own right. Whenever a distinction is required between a dynamic system expressed in terms of differential equations and one in terms of recurrences, the former will be called continuous and the latter discrete. In these designations the adjectives continuous and discrete refer only to the independent variables t and n, respectively. The dynamic system point of view presents the advantage of giving access to an efficient and widely known terminology, which should be able to cover at least in part the variety of situations expected to arise in recurrences. In fact, there exists a hard-core part of dynamic system terminology, which possesses a physically transparent phenomenological meaning, and which has withstood the wear of prolonged theoretical as well as experimental usage. All efforts will be made to stick to this "naturally selected" part.

The complex nature of non linear dynamic processes requires some comments on what is to be understood by the notion "relevant solution" of a recurrence. A formal, or perhaps better a formalistic definition is easy enough. Similarly to a continuous dynamic system, the recurrence (O-1) is considered as an implicit definition of a function

(0-12) $$x_n = F(x_o, n, c),$$

which leads to an identity in n, c, and x_o after insertion into (O-1). While n takes only integer values, the parameters x_o, c are essentially continuous. Since x_o plays in F and H, eq. (O-7), an analoguous role, it is natural to define (O-12) as the general solution of (O-1). The definition (O-12) is theoretically quite satisfactory, but "operationally" (i.e. practically) it is almost useless, because in general the function F is unknown. The physical, biological, etc. information content of (O-12) is therefore very low. In fact, since the function F is known to be in general extremely complicated (in all non-contrived cases it cannot be expressed explicitly in terms of known elementary and transcendental functions), it is illusory to study F by means of, say, series expansions, integral representations, etc. Such particular expressions of F possess at most a local significance and cannot be used to test whether all globally relevant properties of (0-1) have been obtained or not.

Passing to an opposite extreme, motivated by the existence of numerical computation facilities, the expression (0-12) can be thought of as being merely a shorthand notation for a set of qualitatively different (real number) sequences $\{x_n\}$, $n = 0, \pm 1, \pm 2, \ldots$, generated by a suitable chosen set of initial values x_o. This definition is fully operational for $n > 0$, because having fixed the set of x_o, the sequences $\{x_n\}$, $n > 0$, called for brevity sequences of consequents of the x_o, are unambiguously defined and can be straightforwardly computed for any fixed x_o and c. The definition of (0-12) in terms of the $\{x_n\}$ is therefore widely used. A "graphical" display in phase space of the points $\{x_n\}$, $n = 0, 1, 2, \ldots$ is called a discrete half-trajectory of (0-1), and whenever it is permissible to omit the qualification $n = 0, 1, 2, \ldots$, simply a trajectory of (0-1). The numerical and graphical knowledge of a finite number of discrete half-trajectories of (0-1) has obviously a very low theoretical information content. In fact, there is no simple way of knowing whether a given finite, or even enumerable set of $\{x_n\}$, $n = 0, 1, 2, \ldots$ is qualitatively exhaustive, i.e. whether it contains a sufficient number of relevant "samples" permitting to establish all characteristic properties of the function F in (0-12). The doubt about qualitative exhaustivity is reinforced by the observation that the "complementary" discrete half-trajectories $\{x_n\}$, $n = 0, -1, -2, \ldots$, called for brevity sequences of antecedents of x_o, are rarely, if ever examined. The reason for the practical avoidance of the $\{x_n\}$, $n < 0$ lies in the operational difficulty of inverting (0-1), i.e. in finding the x_{n-1} corresponding to a known x_n. It is obvious that simple single-valued smooth functions $f(x_n, c)$ may have complicated multi-valued and not necessarily smooth inverse functions $f^{-1}(x_n, c)$. The computational determination of antecedents requires thus the use of (real-valued) root-finding algorithms. As a rule, the non-uniqueness of f^{-1} triggers a complex branching process, whose existence undermines seriously the presumed practical usefullness of the operational definition of (0-12).

A third definition of the solution of (0-1), complementary to the analytical one, is essentially indirect. It is based on the notion of a set of singularities of the function F in (0-12). Both F and its singularities are assumed to be defined implicitly by the function f in (0-1). Following an idea introduced by Poincaré in connection with continuous dynamic systems, a meaningful characterization of F consists in the identification of its singularities, and in the description of the behaviour of the latter as x_o and c vary. As in the case of continuous dynamic systems, any change of the singularities, or of their properties, is called a bifurcation. The function F in (0-12) is said to be known, i.e. fully characterized, when all its singularities, and all bifurcations of the latter in the admissible range of x_o and c, have been described. This characterization of F is similar to that of a meroporphic function by means of poles and zeros in the complex plane.

The indirect definition is used extensively in this monograph, the implicitly defined singularity structure serving as a conceptual skeletton to which all properties of a recurrence are related. Such an ordering of otherwise isolated particular properties (microscopic, macroscopic or collective ones) discloses many of the intrinsic interrelationships of the latter. Once the decision has been taken to use the singularity structure as a key tool, the remaining basic problem consists in discovering what constitutes a relevant, and possibly a complete set of singularities of F. For firts order recurrences with continuous and at least piecewise differentiable f, a substantial set of fundamental, i.e. building block-like, singularities is already known (c.f. chapter I). The problem of completeness is still open, because the number of all possible qualitatively distinct singularities has not yet been determined. Accumulations of "elementary" singularities occur frequently, giving rise to composite singularities ; the number of the latter is not necessarily finite, which gives rise to new accumulations and thus to "higher-order" composite singularities, etc. Some singularities are due to the form of f in (O-1), i.e. they exist even if f is a polynomial in x_n ; others are due to the limited smoothness of f, i.e. they exist only when f lacks a sufficient number of continuous derivatives, but is otherwise arbitrarily close to a polynomial (in the sense of some norm consistent with the limited smoothness). In the case of a specific recurrence, i.e. with a given f in (O-1), the singularities of F in (O-12) can only be determined one by one, or at most set by set, and then ordered into sequences of similar elements. If these sequences converge (generally non uniformly), their limits may constitute additional qualitatively distinct singularities. In order to avoid unnecessarily awkward sentences in what follows, it is understood that whenever some parameters or functions are mentioned, they belong to their respective admissible spaces. All instances to the contrary are explicitly mentioned.

Similarly to the case of continuous dynamic systems, a value or a set of values of x_o is called a singularity of the recurrence (O-1), if and only if, for this value or set the function F in (O-12) describes a stationary state, or a constitutuent part of the latter. A stationary state is a dynamic system-term for an invariant manifold of (O-1), which in some sense, to be specified in each case, is independent of n. A necessary, but not always a sufficient condition for an x_o of (O-1) to be singular, is the violation of uniqueness of F at x_o. Since the recurrence (O-1) contains a single dependent variable, its singularities are either zero- or one-dimensional, i.e. they consist either of points or of segments of the x-axis. The simplest point-singularity $x_n = \bar{x}$ is given by an isolated finite root of the "algebraic" equation :

$$(O\text{-}13) \qquad f(x, c) - c = 0 \quad .$$

Eq. (O-13) may possess, of course, more than one root. No loss of generality occurs by considering one isolated root at a time. The corresponding stationary state is : $x_n \equiv \bar{x} = \bar{x}(c)$, which is invariant with respect to (O-1) and independent of n in an obvious manner. It is also a point of non-uniqueness of F. A constant stationary state of (0-1) constitutes a fixed point of the mapping, defined by f, of the x-axis onto itself. The successive iterates x_{n+k}, k = 1, 2, ..., of x_n are called for conciseness *consequents* of x_n *of rank* k, with respect to the recurrence (0-1). The statement about the rank is usually omitted unless necessity dictates otherwise. All consequents of a fixed point $\bar{x}$ coincide, of course, with $\bar{x}$, but this is not necessarily so for all antecedents. Suppose that $\bar{x}$ admits a non-zero neighbourhood X_ε : $-\varepsilon_1 < x_n - \bar{x} < \varepsilon$, $0 < \varepsilon_1$, $0 < \varepsilon$, free from other singularities of (0-1). The existence of such an X_ε is not always guaranteed *a priori*, except when the root $\bar{x}$ is known to be isolated, but it is useful to make the existence assumption provisionally, subject to an *a posteriori* confirmation. Consider a point $x_\varepsilon \neq \bar{x}$ inside X_ε, and the set of its consequents $x_{\varepsilon n}$, n = 1, 2, ... Three cases are possible as n increases indefinitely :

a) the (Euclidian) distance d_n between $x_{\varepsilon n}$ and $\bar{x}$ diminishes and approaches zero,
b) d_n increases till one of the $x_{\varepsilon n}$ reaches the boundary of X_ε or leaves X_ε entirely,
c) d_n remains strictly bounded below and above, and all $x_{\varepsilon n}$ remain inside X_ε.
In the first case the fixed point $\bar{x}$ is said to be attractive, in the second repulsive, and in the third neutral. In the terminology of dynamic systems the equivalent statement is : the constant stationary state (or static equilibrium) $\bar{x}$ is asymptotically stable, unstable, and (simply or indifferently) stable, respectively. Stability is understood to be in the sense of Liapunov, extended to discrete dynamic systems, except when specified otherwise. The singularity $x = x_\infty$, eq. (0-5) of the recurrence (0-3) is an example of an attractive fixed point, whose X_ε is the whole positive x-axis, minus the point $x_n = a$, excluded from (0-3)-(0-4) by construction. Since (0-3) is a linear recurrence, the singularity x_∞ is of course unique (when the singularity at infinity is omitted).

Isolated roots of eq. (0-13) are not the only possible point-singularities of the recurrence (0-1). In fact, no information is added to (0-1) by determining successively k > 1 consequents of x_n. Eliminating the "intermediate" variables $x_{n+1}, \ldots, x_{n+k-1}$, it is possible to construct from (0-1) the *iterated recurrence*

(0-14) $$x_{n+k} = f_k(x_n, c), \quad k > 1, \; f_1 = f, \; n = 0, 1, 2, \ldots$$

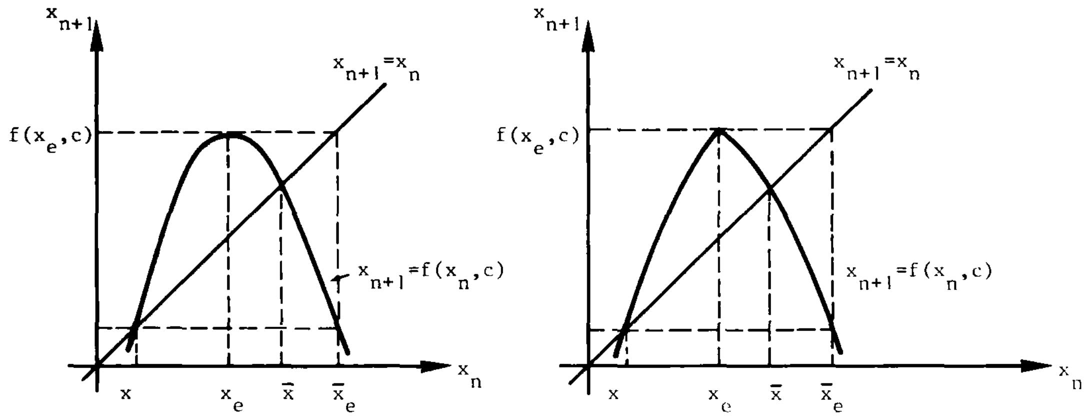
where the *function* f_k is unambiguously defined and single-valued. The construction of (0-14) from (0-1) may be tedious, but it is straightforward. The inverse problem of constructing f from the sole knowledge of f_k is not straightforward at all ; its general solution is still unknown. It is related to the more general problem of determining fractional iterates of a given function ; for example, determining the "half-iterate" f(x) of f(f(x)) = g(x) when g is given. The recurrence (0-14) is not different in principle from the recurrence (0-1), and it may also admit isolated point-singularities, defined by the roots of the algebraic equation

(0-15) $$f_k(x, c) - x = 0, \quad k = 2, 3, \ldots$$

The case k = 1 is of no interest because eq. (0-15) is then the same as eq. (0-3) . Traditionally a root x of (0-15) is said to be a cycle (or a periodic point) of (0-1), provided x is not simultaneously a root of (0-15) when k is replaced by one of its divisors, unity included. Every x defines $k - 1 > 0$ distinct consequents, obtained by means of (0-1), which are also roots of (0-15). The k-th consequent of any root coincides with itself. The set of k points forming a cycle (of order k) is invariant with respect to the iterated recurrence (0-15) in the same way as a fixed point (a root of (0-13)) is invariant with respect to (0-1). In a physical, biological, etc., content, when a fixed point of the recurrence (0-1) describes a periodic solution ("main" resonance) of the corresponding continuous dynamic system (example : (0-5), (0-3) and (0-2), respectively), a cycle describes a subharmonic periodic solution (subharmonic resonance or frequency division). A fixed point or a cycle of a fractional iterate, which is not simultaneously a fixed point or a cycle for k = integer, describes a harmonic or fractionally periodic solution (a harmonic resonance or frequency multiplication, and a combination resonance or rational frequency conversion, respectively).

The situation of a cycle with regard to neighbouring points, or equivalently, its stability, is therefore the same as that of a fixed point : a cycle (if isolated) is either stable, asymptotically stable or unstable. The points of a cycle represent a stationary state of (0-1), independent of n, modulo-k. A cycle constitutes the simplest possible non-constant stationary state of the recurrence (0-1). Constant stationary states of a continuous dynamic system like (0-6), are defined by real roots x of $g(x, c) = 0$. With the assumed smoothness of g, eq. (0-6) has no continuous periodic solutions.

Unless the form of f in (0-1) is severely restricted (to smooth monotonic functions, for example), the number of different cycles of (0-1) is not finite, and accumulations of point-singularities are possible inside finite parts of the x-axis (cf. [S 6]). This is so for the quadratic recurrence (0-11) ; its cycles and bifurcations as a function of c are described in Chapter 1. Recurrences of form (0-1) possess also invariant segments, provided f(x, c) has at least one local extremum. Two examples of an invariant segment $X_i : x < x_n < \bar{x}_e$ are shown in Fig. 0-1, where x, $\bar{x}$, are unstable fixed points, x_e is the abscissa of the extremum, and $\bar{x}_e$ a consequent of x_e. Fig. 0-1 constitutes an illustration of a rather well known geometrical method of analyzing real-valued one-dimensional recurrences. From an inspection of Fig. 0-1, it is obviously that, except for a point set X_p of zero measure, the consequents of any internal point x_n of X_i will remain inside X_i, without any possibility of escaping or settling down as n increases. The segment X_i does not contain any stable point singularity ; it contains cycles, but these are all unstable. The excluded set X_p consists of points (and antecedents) of the unstable cycles and of the unstable fixed point x. In the case of the quadratic recurrence (0-14), the situation of Fig. 0-1-a occurs for several values of c. One example is : $k = 1$, $c = -2$, $x = 2$, $x_e = 0$, $\bar{x}_e = f(x_e, c) = -2$, $X_i : -2 < x_n < 2$. An invariant segment like X_i represents a "complex","disorderly",



a) f and $\frac{\partial f}{\partial x_n}$ continuous

b) f continuous $\frac{\partial f}{\partial x_n}$ piecewise continuous

Fig. 0-1. One type of invariant segment of the recurrence (0-1), c = constant.

"chaotic" or "stochastic" stationary state. If X_i is attractive with respect to neighbouring external points, then it is also called a "strange" attractor, but of course all strange attractors need not be of this very special type. The adjectives complex, disorderly, chaotic, stochastic and strange are used synonimously to express the fact that within the stationary state the behaviour of consequents of an individual initial "state" x_o is "intuitively" (i.e. without the knowledge of (0-1)) unpredictable : the function $F(x_o, n, c)$, eq. (0-12), possesses some "random" features. This randomness of F is routinely exploited in digital computers for (pseudo-) random number generation. The best known recurrence used for this purpose is "Lehmer's congruential algorithm" :

(0-16)

$$x_{n+1} = f(x_n, a, b, m) = \mathrm{mod}_m(a+bx_n), \quad n = 0, 1, 2, \ldots$$

where a, b, m, are suitably chosen positive constants (due to a finite computer word length they are all integers). In contrast to (0-1), the function f in (0-16) is not continuous with respect to x_n.

In a systematic study of functions H(t), defined implicitly by continuous dynamic systems, Birkhoff and Andronov (see p. 108-109 of [A 4]) have proposed a classification of stationary states in the order of increasing complexity (decreasing orderliness), each class containing the preceding one : I) constant, II) periodic, III) quasiperiodic, i.e. $H(t) = \sum_m a_m \cos(b_m t + c_m)$, max $m < \infty$, all angular frequences b_m mutually incommensurable, IV) almost periodic, i.e. H(t) the same as in class III), except $m \to \infty$, V) recurrent and stable in the sense of Poisson, VI) recurrent and unstable in the sense of Poisson, and if a presently popular limit-class is added, VII) (pseudo-) random. If the same classification is used for the stationary states of an autonomous recurrence of order one, then a correct description of an invariant segment requires a function F, eq. (O-12), whose Birkhoff's class is not less than III). The problem of determining the exact Birkhoff class of a specific F is not only still open, but the study of several particular (generic) recurrences suggests that Birkhoff's classification is too coarse, and probably incomplete. This conjecture is reinforced by the study of stationary states of second order autonomous recurrences (cf. Chapters III-V).

Consider a continuous dynamic system of form (O-6), where x and g are no longer scalars but m-vectors, $m = 1, 2, \ldots$ If the functions g are single-valued and sufficiently smooth (without any substantial loss of generality : analytic), then it is well known that a stationary state belonging to Birkhoff's class II) is impossible unless $m > 1$; a state belonging to class III), or eventually class IV), is impossible unless $m > 2$. Since an autonomous recurrence of order one may possess a stationary state of class III), or higher, functions defined implicitly by recurrences are intrinsically more complex than those defined by differential equations. Hence, a global structural equivalence between a recurrence and a differential equation can exist only when the stationary states of both belong to the same maximal Birkhoff class. This is the case for the example described by eq. (O-2) and (O-3). A local structural equivalence may exist under less stringent conditions. This is so in all "computationally stable" discretization schemes currently used in digital computers, the simplest example of which is the recurrence (O-8). It is well known that a global structural equivalence between (O-6) and (O-8) is in general (excluding linear equations and those reducible to linear ones) impossible, no matter how small $h > 0$ is chosen.

The main reason for the greater complexity of functions defined by recurrences is the limited invertibility of the latter, already refered to briefly in connection with the definition of F in (O-12) in terms of discrete half-trajectories. In fact, when the function g in (O-6) is single-valued, then the function H in (O-7) can be determined in principle for $t > 0$ and $t < 0$ with equal facility. An analoguous situation exists only exceptionally in the case of the recurrence (O-1), as was also mentioned before.

The single-valuedness of f for $n = 0, 1, 2, \ldots$ does not imply anything about the existence and uniqueness of x_n for $n = -1, -2, \ldots$, because the inverse recurrence

(O-17) $$x_n = f_{-1}(x_{n+1}, c), \qquad n = -1, -2, \ldots$$

where $f_1(f_{-1}(x)) = f_{-1}(f_1(x)) = x$, may not exist, or it may be multi-valued. For example the inverse recurrence of (O-11) is

(O-18) $$x_n = \pm\sqrt{x_{n+1} - c}$$

and there are no real x_n for $x_{n+1} < c$, while for $x_{n+1} > c$ the x_n are double-valued. In analogy with the rank of consequents, the x_{n+m}, $m < 0$, of x_n are called its