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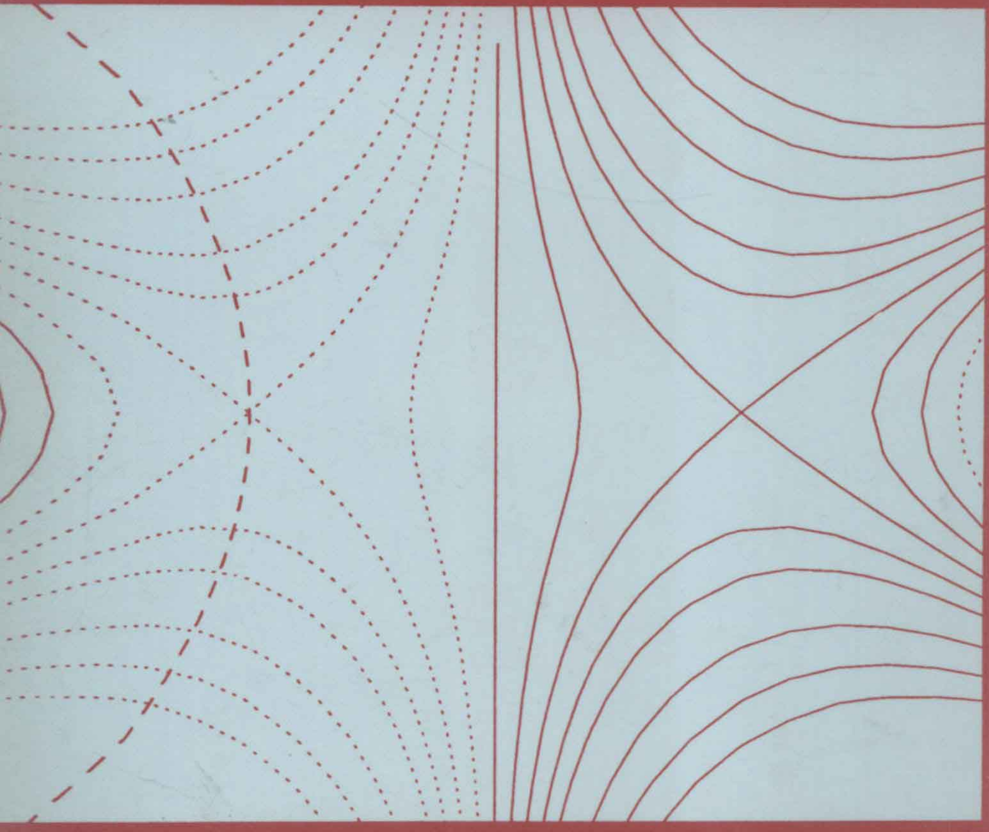
Perturbation Methods



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Preface

Making precise approximations to solve equations is an occupation of applied mathematicians which distinguishes them from pure mathematicians, physicists and engineers. A precise approximation is not a contradiction in terms but rather an approximation with an error which is understood and controllable; in particular the error could be made smaller by some rational procedure. There are two methods for obtaining precise approximations to the solutions of an equation, numerical methods and analytic methods, and this book is about the latter. The analytic approximations are obtained when some parameter of the problem is small, and hence the name *perturbation methods.* The perturbation and numerical methods are not however in competition but rather complement one another as the following example illustrates.

The van der Pol oscillator is governed by the equation

$$\ddot{x} + k\dot{x}(x^2 - 1) + x = 0$$

In time the solution tends to an oscillation with a particular amplitude which does not depend on the initial conditions. The period of this limit oscillation is of interest and is plotted in figure 1 as a function of the strength of the nonlinear friction, k. The circles give the numerical results obtained by a Runge–Kutta method. The dashed curves give the first and second order perturbation approximations

$$\text{Period} = \begin{cases} 2\pi\left(1 + \frac{1}{16}k^2 + O(k^4)\right) & \text{as } k \to 0 \\ k(3 - 2\ln 2) + 7.0143k^{-1/3} + O(k^{-1}\ln k) & \text{as } k \to \infty \end{cases}$$

At intermediate values of the parameter k, from 2 to 6, the numerical method is most useful. At extreme values however the numerical method loses its accuracy rapidly, for example by $k = 10$ the time-step must be reduced to 0.01 in order to obtain 5 figure accuracy. The analytic approximations take over in the extreme conditions. Further they give an explicit dependence on the parameter k rather than the isolated results

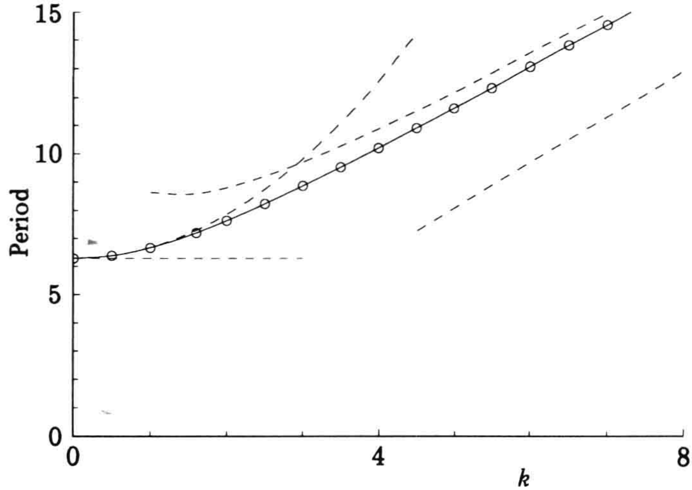

Fig. 1 The period of the limit oscillation of the van der Pol oscillator as a function of the strength of the nonlinear friction k.

at particular values from the numerical method. But the most important feature of the figure is the satisfying agreement between the numerical approximation and the two independent perturbation approximations – such checks are essential in research.

Obtaining good numerical values for the solution is not the only quest of a perturbation approximation. One can hope that the analysis will reveal some physical insights through the simplified physics of the limiting problem. In this book I will however suppress the physics in the problems discussed.

Finding perturbation approximations is an art rather than a science. In research it is useful to be responsive to suggestions from the physics. There is certainly no routine method appropriate to all problems, or even classes of problems. Instead one needs a determination to exploit the smallness of the parameter. This book attempts to present many of the weapons which have been found useful, but they should not be viewed as exhaustive.

While this book is mathematical, no attempt has been made to make the arguments fully rigorous. In general I have tried to explain why the results are correct. Often these reasons can be turned into strict theorems, albeit with some difficulty in the case of singular problems. My own opinion is that such superficial rigour rarely adds to the understanding of the problem, and that of greater use is a numerical statement about the range of applicability achieving some specified accuracy.

This book is based on a course of lectures which I gave for a number of years to first year graduate students in the University of Cambridge. In its turn it was based on my own education from a course of lectures by L. E. Fraenkel and from the book on the subject by M. Van Dyke. These two inspiring teachers asked many interesting questions which I have attempted to answer in this book; questions such as why are some results convergent whilst others only asymptotic, why is matching possible, what selection criterion should be used with strained co-ordinates, and what characterises problems to be tackled by multiple scales.

While no previous knowledge of perturbation methods is assumed, some previous experience is probable. The students who attended my lecture course would have seen several examples (small friction on projectiles, perturbed energy levels in quantum mechanics, adiabatic invariants in Hamiltonian systems, Watson's lemma, and viscous boundary layers in fluid mechanics) usually presented in an informal way relying heavily on physical insight. They would not however have seen a formal and organised approach to a perturbation problem.

The eventual goal of this book is to present the method of matched asymptotic expansions and the method of multiple scales, progressing to an advanced level in considering the more difficult issues such as the occurrence of logarithms and the occurrence of more than two scales. Tackling differential equations with such singular perturbation problems is certainly not easy. Fortunately many of the essential concepts can be presented in the simpler context of algebraic equations and later with integrals. Thus issues such as iterations and expansions, singular problems and rescaling, non-integral powers and logarithms will be presented well before the difficult singular differential equations are encountered. Finally I should observe that most of the chapters follow the basic method with an advanced application whose understanding is not essential to the following chapters – thus §§ 1.6, 3.5, 5.3, 5.4, 5.5, 5.6, 6.3, 7.3, 7.4 and 7.6 should be viewed as optional.

E.J. Hinch
Cambridge, 1990

Contents

1

Algebraic equations

Many of the techniques of perturbation analysis can be introduced in the simple setting of algebraic equations. By starting with some particularly easy algebraic equations, three quadratics, we can benefit from the luxury of the existence of exact answers, taking useful hints from them to overcome difficulties.

1.1 Iteration and expansion

We start with the equation for x which contains the parameter ϵ,

$$x^2 + \epsilon x - 1 = 0$$

This has exact solutions

$$x = -\tfrac{1}{2}\epsilon \pm \sqrt{1 + \tfrac{1}{4}\epsilon^2}$$

which can be expanded for small ϵ as

$$x = \begin{cases} +1 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 - \frac{1}{128}\epsilon^4 + O(\epsilon^6) \\ -1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{128}\epsilon^4 + O(\epsilon^6) \end{cases}$$

These binomial expansions converge if $|\epsilon| < 2$.

More important than converging, the truncated series give a good approximation if ϵ is small. The first few terms give a result within 3% of the exact result if

$$\begin{array}{ccccccccccc} |\epsilon| & < & 0.05 & & 0.5 & & 1.2 & & 1.6 & & \\ & & & \vdots & & \vdots & & \vdots & & \vdots & \\ x & = & 1 & - & \frac{1}{2}\epsilon & + & \frac{1}{8}\epsilon^2 & - & \frac{1}{128}\epsilon^4 & + & O(\epsilon^6) \end{array}$$

The last 1.6 being not too far from the convergence boundary. Alternatively for the fixed value of $\epsilon = 0.1$ the first few terms give

$$
\begin{aligned}
x &\sim 1.0 \\
& \quad 0.95 \\
& \quad 0.95125 \\
& \quad 0.95124921\ldots \\
& \quad \ldots \\
\text{exact} &= 0.95124922\ldots
\end{aligned}
$$

Often the numerical summation of these short expansions involves less computer time than the evaluation of the exact answer with its costly surds.

We started by finding the exact solution of the quadratic equation and then we expanded the exact solution. In most problems, however, it is not possible to find the exact solution. We must therefore develop techniques which first make the approximations and then, only afterwards, involve a calculation. There are two distinct methods of first approximating and then calculating, the iterative method and the expansion method. Each method has its own advantages and disadvantages.

Iterative method

We start with the iterative method, because it is a method which is often overlooked although it has much to offer.

The first step of the iterative method is to find a *rearrangement* of the original equation which will become the basis of an iterative process. This first step involves a certain amount of inspiration which must therefore count as a major drawback of the method. A suitable rearrangement of our present quadratic is

$$x = \pm\sqrt{1 - \epsilon x}$$

Any solution of the original equation is a solution of this rearrangement and vice versa.

Working with just the positive root, we thus adopt the iterative process

$$x_{n+1} = \sqrt{1 - \epsilon x_n}$$

The iterative process needs a starting point, the value of the root when $\epsilon = 0$, $x_0 = 1$.

Making the first iteration, we find

$$x_1 = \sqrt{1-\epsilon}$$

which can be expanded in a binomial series

$$x_1 = 1 - \tfrac{1}{2}\epsilon - \tfrac{1}{8}\epsilon^2 - \tfrac{1}{16}\epsilon^3 + \cdots$$

Looking at the exact answer, we see that the ϵ^2 and higher terms are erroneous. We therefore truncate the series for x_1 after the second term:

$$x_1 = 1 - \tfrac{1}{2}\epsilon + \cdots$$

Proceeding to the next iteration, we find

$$x_2 = \sqrt{1-\epsilon(1-\tfrac{1}{2}\epsilon)}$$

which can be expanded, this time retaining only terms up to ϵ^2:

$$\begin{aligned} x_2 &= 1 - \tfrac{1}{2}\epsilon(1-\tfrac{1}{2}\epsilon) - \tfrac{1}{8}\epsilon^2(1+\cdots)^2 + \cdots \\ &= 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon^2 + \cdots \end{aligned}$$

We note that the ϵ^2 term is now correct after two iterations. Iterating again, we find

$$\begin{aligned} x_3 &= \sqrt{1-\epsilon(1-\tfrac{1}{2}\epsilon+\tfrac{1}{8}\epsilon^2)} \\ &= 1-\tfrac{1}{2}\epsilon(1-\tfrac{1}{2}\epsilon+\tfrac{1}{8}e^2)-\tfrac{1}{8}\epsilon^2(1-\tfrac{1}{2}e+\cdots)^2-\tfrac{1}{16}\epsilon^3(1-\cdots)^3+\cdots \\ &= 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon^2 + 0\epsilon^3 + \cdots \end{aligned}$$

It is clear that progressively more work is required to obtain the higher order terms by the iterative method. The method also has the undesirable feature that in the early iterations it gives erroneous values to the higher terms. One can only check that a term is correct by making one more iteration, which of course is usually convincing but no rigorous proof (but see §1.5).

Expansion method

The first step of the expansion method is to set $\epsilon = 0$ and find the unperturbed roots $x = \pm 1$. Then one poses an expansion about one of these roots, say $x = +1$, expanding in powers of ϵ, i.e.

$$x(\epsilon) = 1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots$$

This expansion is formally substituted into the governing quadratic equation.

$$\begin{array}{rlllll} 1 & +\ \epsilon 2x_1 & +\ \epsilon^2(x_1^2+2x_2) & +\ \epsilon^3(2x_1x_2+2x_3) & +\cdots \\ & +\ \epsilon & +\ \epsilon^2 x_1 & +\ \epsilon^3 x_2 & +\cdots \\ -1 & & & & \\ =0 & & & & \end{array}$$

Here one ignores potential difficulties in making the substitution such as the limitations in multiplying series term by term. The coefficients of the powers of ϵ on the two sides of the equation are now compared.

At ϵ^0: $\quad 1-1 = 0$

This level is satisfied automatically because we started the expansion from the correct value $x = 1$ at $\epsilon = 0$.

At ϵ^1: $\quad 2x_1 + 1 = 0 \qquad$ i.e. $x_1 = -\frac{1}{2}$

At ϵ^2: $\quad x_1^2 + 2x_2 + x_1 = 0 \qquad$ i.e. $x_2 = \frac{1}{8}$

Here the previously determined value of x_1 has been used.

At ϵ^3: $\quad 2x_1x_2 + 2x_3 + x_2 = 0 \qquad$ i.e. $x_3 = 0$

again using the previously determined values of x_1 and x_2.

The expansion method is much easier than the iterative method when working to higher orders. To use the expansion method, however, it is necessary to assume that the result can be expanded in powers of ϵ and that the formal substitution and associated manipulations are permitted.

Exercise 1.1. Find four terms in the expansion of the root near $x = -1$, using both the iterative and expansion methods.

1.2 Singular perturbations and rescaling

In this section we study the quadratic

$$\epsilon x^2 + x - 1 = 0$$

If $\epsilon = 0$ there is just one root at $x = 1$, whereas when $\epsilon \neq 0$ there are two roots. This is an example of a *singular perturbation* problem, in which the limit point $\epsilon = 0$ differs in an important way from the approach to the limit $\epsilon \to 0$. Interesting problems are often singular. Problems which are not singular are said to be *regular*.

To resolve the paradox of the behaviour of the second root we take the exact solutions to the quadratic and expand them for small ϵ (convergent if $|\epsilon| < \frac{1}{4}$). The two roots are

$$x = \begin{cases} 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + \cdots \\ -1/\epsilon - 1 + \epsilon - 2\epsilon^2 + 5\epsilon^3 + \cdots \end{cases}$$

Thus the singular second root evaporates off to $x = \infty$ in the limit $\epsilon = 0$.

Iterative method

To set up an iterative process for the singular root we argue as follows. In order to retain the second solution of the governing quadratic, it is necessary to keep the ϵx^2 term as a main term rather than as a small correction. Thus x must be large. Hence at leading order the -1 term in the equation will be negligible when compared with the x term, i.e.

$$\epsilon x^2 + x \sim 0 \qquad \text{with solution} \qquad x \sim -1/\epsilon$$

Hence we are led to the rearrangement of the quadratic

$$x = -\frac{1}{\epsilon} + \frac{1}{\epsilon x}$$

and the iterative process

$$x_{n+1} = -\frac{1}{\epsilon} + \frac{1}{\epsilon x_n}$$

with a starting point $x_0 = -1/\epsilon$.

Iterating once we find

$$x_1 = -\epsilon^{-1} - 1$$

and iterating again

$$\begin{aligned} x_2 &= -\epsilon^{-1} - \frac{1}{1+\epsilon} \\ &= -\epsilon^{-1} - 1 + \epsilon + \cdots \end{aligned}$$

A further iteration is needed to obtain the ϵ^2 term correctly.

Expansion method

The expansion method can be applied to the singular root by posing a power series in ϵ which starts with an ϵ^{-1} term instead of the usual ϵ^0. The way in which one determines the correct starting point is left until later in this section. Thus substituting

$$x(\epsilon) = \epsilon^{-1} x_{-1} + x_0 + \epsilon x_1 + \cdots$$