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A. G. Vitushkin (Ed.)

Several Complex Variables I

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I. Remarkable Facts of Complex Analysis

A.G. Vitushkin

Translated from the Russian
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Introduction

The present article gives a short survey of results in contemporary complex analysis and its applications. The material presented is concentrated around several pivotal facts whose understanding enables one to have a general view of this area of analysis.

§1. The Continuation Phenomenon

The most impressive fact from complex analysis is the phenomenon of the continuation of functions (Hartogs, 1906; Poincaré, 1907). We elucidate its significance by an example. If a function f is defined and holomorphic on the boundary of a ball B in n-dimensional complex space $\mathbb{C}^n (n \geq 2)$, then it turns out that f may be continued to a function holomorphic on the whole ball B. Analogously, for an arbitrary bounded domain whose complement is connected, any function holomorphic on the boundary of such a domain admits a holomorphic continuation to the domain itself. Let us emphasize that this holds only for $n \geq 2$. In the one dimensional case, this phenomenon clearly does not occur. Indeed, for each set $E \subset \mathbb{C}^1$ and each point $z_0 \in \mathbb{C} \backslash E$, the function $1/(z-z_0)$ is holomorphic on E but cannot be holomorphically continued to the point z_0.

This discovery marked the beginning of the systematic study of functions of several complex variables. Two fundamental notions, originating in connection with this property of holomorphic functions, are "envelope of holomorphy" and "domain of holomorphy". Let D be a domain or a compact set in $\mathbb{C}^n$. The *envelope of holomorphy* $\tilde{D}$ of the set D is the largest set to which all functions holomorphic on D extend holomorphically. The envelope of holomorphy of a domain in $\mathbb{C}^n$ is a domain which in general "cannot fit" into $\mathbb{C}^n$, but rather is a multi-sheeted domain over $\mathbb{C}^n$ (Thullen, 1932). A domain $D \subset \mathbb{C}^n$ is called a *domain of holomorphy* if $\tilde{D} = D$, i.e. if there exists a holomorphic function on D which cannot be continued to any larger domain. Domains of holomorphy are also sometimes called *holomorphically convex domains*.

The theorem on discs (Hartogs, 1909) gives an idea helpful in constructing the envelope of holomorphy of a domain: if a sequence of analytic discs, lying in the domain D, converges towards a disc whose boundary lies in D, then this entire limit disc lies in the envelope of holomorphy of D. An *analytic disc* is the biholomorphic image of a closed disc. The technique of construction of the envelope of holomorphy of a compact set and in particular of a surface relies on a conglomeration of "attached" discs whose boundaries lie on the given surface (Bishop, 1965).

Closely related to the notion of envelope of holomorphy is the notion of hull with respect to some class or other of functions, for example, the polynomial hull, the rational hull, etc. The *polynomial hull* of a set $D \subset \mathbb{C}^n$ is the set of all $z \in \mathbb{C}^n$ for which the following condition holds: for each polynomial $P(\zeta)$,

$$|P(z)| \leq \sup_{\zeta \in D} |P(\zeta)|.$$

Every smooth curve is *holomorphically convex*, i.e. its envelope of holomorphy coincides with the curve itself. The polynomial hull of a curve is in general non-trivial. For example, if a smooth curve is closed and without self-intersections, then its polynomial hull is either trivial or it is a one-dimensional complex analytic set whose boundary coincides with the given curve (Wermer, 1958; Bishop, 1962). We recall that a *set* in $\mathbb{C}^n$ is called *analytic* provided that in the vicinity of each of its points it is defined by a finite system of equations $\{f_j(\zeta) = 0\}$, where $\{f_j\}$ are holomorphic functions.

An unsolved problem. Is a set in $\mathbb{C}^n$ consisting of a finite number of pairwise disjoint balls polynomially convex? If the number of balls is at most 3, then the answer is positive; their union is polynomially convex (Kallin, 1964).

Another variant of the continuation phenomenon is the theorem of Bogolyubov, nicknamed the edge-of-the-wedge theorem (S.N. Bernstein, 1912; N.N. Bogolyubov, 1956; . . . , V.V. Zharinov, 1980). Let C^+ be an acute convex cone in $\mathbb{R}^n$ consisting of rays emanating from the origin. Let C^- be the cone symmetric to C^+ with respect to the origin. Let Ω be a domain in $\mathbb{R}^n$, and D^+ and D^- two wedges, i.e. domains in $\mathbb{C}^n$ of the type

$$D^+ = \{z \in \mathbb{C}^n\colon \operatorname{Re} z \in \Omega,\ \operatorname{Im} z \in C^+\}$$

and

$$D^- = \{z \in \mathbb{C}^n\colon \operatorname{Re} z \in \Omega,\ \operatorname{Im} z \in C^-\}.$$

Suppose f is a function holomorphic on $D^+ \cup D^-$ and suppose the functions $f|_{D^+}$ and $f|_{D^-}$ have boundary values which agree in the sense of distributions along the edge of these cones, i.e. on the set $D^0 = \{z \in \mathbb{C}^n\colon \operatorname{Re} z \in \Omega, \operatorname{Im} z = 0\}$. Then, f has a holomorphic extension to some neighbourhood of the set D^0. The theorem on C-convex hull (V.S. Vladimirov, 1961) gives an estimate on the size of this neighbourhood. For example, if $\Omega = \mathbb{R}^n$, then $(D^+ \cup D^0 \cup D^-)^{\tilde{}} = \mathbb{C}^n$ (Bochner, 1937).

The theorem of Bogolyubov has been used to establish several relations in axiomatic quantum field theory. This theorem also laid the foundations of the theory of hyperfunctions (Sato, 1959; Martineau, 1964; . . . , V.V. Napalkov, 1974). For more details, see articles II, III and volume 8, article IV.

§2. Domains of Holomorphy

Domains of holomorphy are of interest because in such domains one can solve traditional problems of analysis. In certain of these domains holomorphic functions have integral representations and admit approximation by polynomials. In domains of holomorphy the Cauchy–Riemann equations are solvable; it turns out to be possible to interpolate functions; the problem of division is solvable; etc.

Two of the simplest types of domains of holomorphy are polynomial polyhedra and strictly pseudoconvex domains. A *polynomial polyhedron* is a domain given by a system of the type $|P_j(z)| < 1$, $j = 1, 2, \ldots, k$, where each $P_j(z)$ is a polynomial in z. Polynomial polyhedra were introduced by Weil (1932) and are also called *Weil polyhedra*. A domain is called *strictly pseudoconvex* if in the neighbourhood of each of its boundary points the domain is strictly convex for a suitable choice of coordinates. Suppose the hypersurface bounding a domain is given by an equation $\rho(z, \bar{z}) = 0$. If in each point of the hypersurface the Levi form of the hypersurface is positive definite, then the domain in question is strictly pseudoconvex (E. Levi, 1910). The *Levi form* is the form $\sum_{i,k} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_k} dz_i \cdot d\bar{z}_k$ restricted to the complex tangent space to the hypersurface at the point z_0.

The solution of various forms of the problem of Levi concerning the holomorphic convexity of strictly pseudoconvex domains remained the central problem of complex analysis for several decades. Oka (1942) showed that each strictly pseudoconvex domain is holomorphically convex and conversely each domain of holomorphy can be exhausted from the interior by domains of this type. Polynomial polyhedra are easily seen to be polynomially convex and consequently holomorphically convex.

Boundary points of a domain of holomorphy are not equivalent. A particularly important role is played by that part of the boundary which is called the distinguished boundary or the Shilov boundary. The *Shilov boundary* of a bounded domain is the smallest closed subset $S(D)$ of the boundary of D such that, for each function f continuous on the closure of D and holomorphic in D and for each point $z \in D$ the inequality $|f(z)| \leq \max_{\zeta \in S(D)} |f(\zeta)|$ holds. For a ball the Shilov boundary coincides with its topological boundary. The Shilov boundary of the polydisc $|z_j| < 1$, $j = 1, 2, \ldots, n$, is the n-dimensional torus $|z_j| = 1$, $j = 1, 2, \ldots, n$. For domains whose boundary is C^2, the Shilov boundary is the closure of the set of strictly pseudoconvex points (Basener, 1973).

For domains of holomorphy, a strong maximum principle holds. If D is a domain of holomorphy and f is non-constant, continuous on the closure of D, holomorphic in D, and attains a local maximum at some point, then that point lies in $S(D)$ (Rossi, 1961). In simple cases the non-Shilov part of the boundary has analytic structure, it folliates into analytic sets. This was shown, for example, for domains in $\mathbb{C}^2$ having C^1 boundary (N.V. Shcherbina, 1982).

Concerning the topology of domains of holomorphy, it is known that the homology groups H_k of order k are trivial for all $k > n$. For polynomially convex domains, the n-th homology group is also trivial (Serre, 1953; Andreotti and Narasimhan, 1962).

Several classical problems of analysis are solvable only for domains of holomorphy. For example a domain is a domain of holomorphy if and only if each function holomorphic on a complex submanifold of the domain is the restriction of some function holomorphic on the whole domain (Oka, H. Cartan, 1950). Analogously, a domain is a domain of holomorphy if and only if the problem of division is solvable (Oka, H. Cartan, 1950). The *problem of division* is said to be solvable in the domain D if for any functions $f_1, \ldots, f_k$ holomorphic in D, and any holomorphic function f in D whose zero set contains (taking into account multiplicities) the set of common zeros of the functions $f_1, \ldots, f_k$, there exist functions $g_1, \ldots, g_k$, holomorphic in D, such that $\sum_j f_j g_j = f$. We recall that on account of the Weierstrass preparation Theorem (1885), the local problem of division is always solvable.

One can define the notion of holomorphic convexity in terms of plurisubharmonic functions. A function is called *plurisubharmonic* if its restriction to each complex line is a subharmonic function. A domain D is a domain of holomorphy if and only if the function $-\ln \rho(z)$ is plurisubharmonic on D, where $\rho(z)$ is the distance from the point z to the boundary of D (Lelong, 1945).

For further details see article II and Volume 8, article II.

§3. Holomorphic Mappings. Classification Problems

By the Riemann Mapping Theorem, in $\mathbb{C}^1$ any two proper simply-connected domains are holomorphically equivalent. In the multidimensional case, the situation is substantially different. For example, a ball and a polydisc are not equivalent (Reinhardt, 1921). Moreover, almost any two randomly chosen domains turn out to be non-equivalent (Burns, Shnider, Wells, 1978).

Let us consider the class of strictly pseudoconvex domains having analytic boundary. In this situation any biholomorphic mapping from one domain onto another extends to a biholomorphic correspondence between the boundaries (Fefferman, 1974; S.I. Pinchuk, 1975), and by the same token, the classification problem for such domains reduces to that of classifying hypersurfaces. There are two approaches to this problem. The first is geometric; the hypersurface is characterized by a system of differential-geometric invariants (E. Cartan, 1934; Tanaka, 1967; Chern, 1974). In the second approach, the characterization is by a special equation, the so-called normal form (Moser, 1974). Both of these constructions enable one to distinguish the infinite-dimensional space of pairwise non-equivalent analytic hypersurfaces.

In connection with the classification problem, a description of mappings realizing the equivalence between two surfaces has been obtained. The results for mappings are described as for the case of functions by properties of continuation. In the case of mappings a new variant of this phenomenon appears. For example, it turns out that a holomorphic mapping of a sphere to itself given in a small neighbourhood of some point of the sphere can be holomorphically extended to the entire sphere and moreover, is in fact a fractional linear transformation (Poincaré, 1907; Alexander, 1974). If the surface is *not spherical*, i.e. cannot, by a local change of coordinates, be transformed into the equation of a sphere, then the germ of such a mapping of the surface into itself can be continued, not only along the surface, but also, in a direction normal to the surface. Namely, if a strictly pseudoconvex analytic hypersurface is not spherical, then the germ of any holomorphic mapping of this surface into itself has a holomorphic continuation (with an estimate on the norm) to a "large" neighbourhood of the center of the germ. Moreover, a guaranteed size, for both the neighbourhood as well as for the constant estimating the norm, is determined by the two characteristics of the surface, namely, the parameters of analyticity of the surface and its constant of non-sphericity (A.G. Vitushkin, 1985). In particular, a surface of the indicated type has a neighbourhood to which all automorphisms of the surface extend. It is worth emphasizing that in both examples we have presented, the mappings, in contrast to functions, extend not only to the envelope of holomorphy of the domain on which they are defined, but also to some domain lying outside the domain of holomorphy. The theorem on germs of mappings concludes a lengthy chain of works on holomorphic mappings of surfaces (Alexander, 1974; Burns and Shnider, 1976; S.I. Pinchuk, 1978; V.K. Beloshapka and A.V. Loboda, 1980; V.V. Ezhov and N.G. Kruzhilin, 1982).

From the Theorem on Germs, it follows that a stability group of a surface (group of its automorphisms which leave a certain point fixed) is compact. Hence, by Bochner's theorem on the linearization of a compact group of automorphisms (1945), one obtains that a stability group of a non-spherical surface can be linearized, i.e. by choosing appropriate coordinates, every automorphism can be written as a linear transformation (N.G. Kruzhilin and A.V. Loboda, 1983). Together with the theorem of Poincaré, this means that for each pair of locally given strictly pseudoconvex analytic hypersurfaces, every mapping sending one hypersurface into the other can be written as a fractional-linear transformation by an appropriate choice of coordinates in the image and preimage. The problem on the linearization of mappings of surfaces having a non-positive Levi form remains open. For further details see article IV and Volume 9, articles V and VI.

We have considered here only one aspect of the problem of classification. Large sections of complex analysis are concerned with the study of invariant metrics (Kähler, 1933; Carathéodory, 1927; Bergman, 1933; Kobayashi, 1967; Fefferman, 1974, . . .); classification of manifolds (Hodge, Kodaira, 1953; Yau, Siu, 1980; . . .); description of singularities of complex surfaces (Milnor, 1968; Brieskorn, 1966; Malgrange, 1974; A.N. Varchenko, 1981; . . .).

§4. Integral Representations of Functions

A smooth function in a closed domain $\bar{D} \subset \mathbb{C}$ can be expressed using the Cauchy-Green formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{D} \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{1}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta.$$

The first term on the right side is the formula which reproduces a holomorphic function in a domain in terms of its boundary values. The second term isolates the non-holomorphic part of f and yields a solution to the $\bar{\partial}$-equation $\frac{\partial f}{\partial \bar{z}} = g$. For functions of several variables, there does not exist such a simple and universal formula, and hence it is suitable to consider the problem of integral formulas for holomorphic functions and the solvability of the $\bar{\partial}$-equations separately.

For some classes of domains in $\mathbb{C}^n$, there are explicit formulas which reproduce a holomorphic function in terms of its boundary values. For polynomial polyhedra such a formula was obtained by A. Weil (1932); for strictly pseudoconvex domains, by G.M. Khenkin (1968). Such a formula was given for the polydisc by Cauchy (1841) and for the ball, by Bochner (1943). There is a formula of Bochner–Martinelli (1943) for smooth functions on arbitrary domains having smooth boundary. In this formula, in contrast to the previous ones, the kernel is not holomorphic, and this often makes it difficult to apply. For polynomial polyhedra there is still another formula which distinguishes itself from the Weil formula and other formulas in that its kernel is not only holomorphic but also integrable (A.G. Vitushkin, 1968).

Let us introduce the formulas for the polydisc and the ball. If f is holomorphic on the closure of the polydisc D^n, then

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{(\partial D)^n} \frac{f(\zeta)}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)}\, d\zeta_1 \wedge \ldots \wedge d\zeta_n.$$

If f is holomorphic on the closed ball $\bar{B}\colon |z| \leq 1$, then inside the ball,

$$f(z) = \frac{1}{V} \int_{\partial B} \frac{f(\zeta)}{(1 - \bar{\zeta}_1 z_1 - \ldots - \bar{\zeta}_n z_n)^n}\, dV,$$

where V is the $(2n-1)$-dimensional volume of the sphere ∂B and dV is its element of volume.

All of the formulas which we have mentioned above differ from one another in appearance. The appearance of the formula depends on the type of domain. There is a formula due to Fantappiè–Leray (1956) which gives a general scheme for writing such formulas. Let D be a domain in $\mathbb{C}^n_z$, where $z = (z_1, \ldots, z_n)$ is a set of coordinate functions, and let f be holomorphic on the closure of D. Then

$$f(z) = \frac{(n-1)!}{(2\pi i)^n} \int_\gamma \frac{f(\zeta)}{[\eta_1(\zeta_1 - z_1) + \ldots + \eta_n(\zeta_n - z_n)]^n} \cdot \sum_{k=1}^{n} (-1)^k \eta_k \wedge d\eta_1 \wedge \ldots \wedge d\eta_{k-1} \wedge d\eta_{k+1} \wedge \ldots \wedge d\eta_n \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n,$$

where γ is a $(2n-1)$-dimensional cycle in the space $\mathbb{C}^n_\zeta \times \mathbb{C}^n_\eta$ lying over the boundary of the domain $D \subset \mathbb{C}^n_\zeta$ and covering it once. By choosing suitably the form of the cycle γ, having chosen η as a function of ζ, one can obtain any of the preceding integral formulas.

One of the applications of integral formulas is in solving the problem of interpolation with estimates. If a complex submanifold M of the ball B crosses the boundary of the ball transversally, then every function holomorphic and bounded on M can be continued to a function holomorphic and bounded in the entire ball (G.M. Khenkin, 1971). The extension is constructed as follows. The function $f(z)$ for $z \in M$ can be written as an integral $I(z)$ of f on the boundary of M. Moreover, it turns out that the function $I(z)$ is defined for all $z \in B$, and from the explicit formula for $I(z)$, one obtains that the extended function $f(z) = I(z)$ is holomorphic and bounded on B.

The problem on the possibility of division with uniform estimates remains open. Namely, it is not known whether for each set of functions $f_1, \ldots, f_k$, holomorphic and bounded in the ball $B \subset \mathbb{C}^n$ and such that $\inf\limits_{\zeta \in B} \sum\limits_{j=1}^{k} |f_j(\zeta)| \neq 0$, there exist functions $g_1, \ldots, g_k$ bounded and holomorphic on B such that $\sum\limits_{j=1}^{k} f_j g_j \equiv 1$. This is a modified formulation of the famous "corona" problem. In the one dimensional case, this problem was solved by Carleson (1962). The answer is positive: in the maximal ideal space for the algebra of bounded holomorphic functions in the one-dimensional disc, the set of ideals, corresponding to points of the disc, is everywhere dense.

The above enumerated formulas are for bounded domains. In the present time analysis on unbounded domains is also flourishing. In particular, integral formulas have been constructed for such domains. There are explicit formulas for tubular domains over a cone (Bochner, 1944), on Dyson domains (Jost, Lehmann, Dyson, 1958; V.S. Vladimirov) and Siegel domains (S.G. Gindikin, 1964). Weighted integral representations for entire functions have also been constructed (Berndtsson, 1983). For further results see article II and Volume 8, articles I, II and IV.

§5. Approximation of Functions

Let us denote by $CH(E)$ the set of all continuous functions on the compact set $E \subset \mathbb{C}^n$ which are holomorphic at interior points of E. It is clear that functions which can be uniformly approximated on E with arbitrary accuracy by complex polynomials or by functions holomorphic on E belong to the class $CH(E)$. When we speak of the possibility of approximating functions on the compact set E, we shall mean the following: each function in $CH(E)$ can be approximated uniformly with arbitrary precision by functions holomorphic on E.

If a compact set E in C^1 has a connected complement, then each function holomorphic on E can be approximated by polynomials (Runge, 1885). This is equivalent to a theorem of Hilbert (1897): on each polynomial polyhedron in $\mathbb{C}^1$, any holomorphic function can be represented as the sum of a series of polynomials. Runge's Theorem reduces the question of the possibility of approximating functions by polynomials to that of constructing holomorphic approximations of functions. The criterion for the possibility of approximation by holomorphic functions (A.G. Vitushkin, 1966) is formulated as follows. The assertion that each function in $CH(E)$, where $E \subset \mathbb{C}^1$, can be uniformly approximated with arbitrary accuracy by functions holomorphic on E is equivalent to the following condition on the compact set E: for each disc K, $\alpha(K \setminus E) = \alpha(K \setminus \mathring{E})$, where $\mathring{E}$ denotes the interior of E, and $\alpha(M)$ is the *continuous analytic capacity* of a set M. By definition

$$\alpha(M) = \sup_{M^*; f} \left| \lim_{z \to \infty} z f(z) \right|.$$

The supremum is taken over all compact sets $M^* \subset M$ and all functions f which are everywhere continuous on $\mathbb{C}^1$, bounded in modulus by 1 and holomorphic outside of M^*. In particular, approximation is possible if the inner boundary of E is empty, i.e. each boundary point of E belongs to the boundary of some complementary component of E. For example, all compact sets with connected complement belong to this class. The above criterion emerged as a result of a long series of works on approximation (Walsh, 1926; Hartogs and Rosenthal, 1931; M.A. Lavrentiev, 1934; M.V. Kelc'ysh, 1945; S.N. Mergelyan, 1951 and others).

The notion of analytic capacity is useful not only in approximation. It appears along with its analogues in integral estimates (M.S. Mel'nikov, 1967). Such capacities are used for describing the set of removeable singularities of a function (Ahlfors, 1947; . . . E.P. Dolzhenko, 1962; . . . Mattila, 1985). Among the unsolved problems, we draw attention to the problem of the subadditivity of analytic capacity: is it true that for any two compact sets, the capacity of their union is no greater than the sum of their capacities?

The integral formula of Weil is a generalization of Hilbert's construction. Using this formula, A. Weil (1932) showed that on any polynomially convex compact set in $\mathbb{C}^n$, each holomorphic function can be approximated by polynomials. Thus in $\mathbb{C}^n$ as in $\mathbb{C}^1$, polynomial approximation reduces to holomorphic approximation. The integral formula of G.M. Khenkin emerged as a result of attempting to construct holomorphic approximations on arcs. While developing such approximations, the technique of integral formulas found various applications. Nevertheless, the initial question on the possibility of approximating continuous functions on polynomially convex arcs by polynomials remains open.

The possibility of holomorphic approximation has been established for the following cases: arcs having nowhere dense projection on the coordinate planes (E.M. Chirka, 1965); strictly pseudoconvex domains (G.M. Khenkin, 1968); non degenerate Weil polyhedra (A.I. Petrosyan, 1970); and C.R.-manifolds (Baouendi and Trèves, 1981). There are several examples of compact sets on which approximation is not possible. Diederich and Fornaess (1975) constructed a domain of holomorphy in $\mathbb{C}^2$, with C^∞-boundary, whose closure is not a compact set of holomorphy, i.e. it cannot be represented as the intersection of a decreasing sequence of domains of holomorphy. Moreover, on this domain one can define a holomorphic function, infinitely differentiable up to the boundary of the domain, which cannot be approximated by functions holomorphic on the closure of the domain.

For related results, see papers II and III.

Above we discussed only the possibility of approximation. There is a lengthy series of works devoted to the explicit construction of approximating functions. In recent years in connection with applications, there has been a renewed interest in classical rational approximation (continuous fractions, Padé approximation, etc.). We mention one example concerning rational approximation in connection with the holomorphic continuation of functions. Let f be holomorphic on the ball $B \subset \mathbb{C}^n$, and set $r_k(f) = \inf_{\varphi} \sup_{z \in B} |f(z) - \varphi(z)|$, where the infimum is taken over all rational functions φ of degree k. Then, if for each $q > 0$, $\lim_{k \to \infty} r_k(f) q^{-k} = 0$, then the global analytic function, generated by the element f, turns out to be single-valued, i.e., its domain of existence is single-sheeted over $\mathbb{C}^n$ (A.A. Gonchar, 1974). See Vol. 8, paper II.

§6. Isolating the Non-Holomorphic Part of a Function

Sometimes in order to construct a holomorphic function with given properties, one proceeds as follows. One constructs some smooth function φ with the desired properties and then one breaks up φ as the sum of two functions the first of which is holomorphic while the second is in some sense small. In this situation, the first function may turn out to be the function we require. The second term is sought in the form of a solution to the equation $\bar{\partial} f = g$, where $\bar{\partial} = \frac{\partial}{\partial \bar{z}_1} d\bar{z}_1 + \dots + \frac{\partial}{\partial \bar{z}_n} d\bar{z}_n$, and $g = \bar{\partial}\varphi$. This scheme is used for constructing functions with prescribed zeros, in approximation, etc. Equations of the type $\bar{\partial} f = g$ are called the Cauchy–Riemann equations or $\bar{\partial}$-equations.

Let us consider a more general case of the equation $\bar{\partial} f = g$, namely, we shall take for g a differential (p, q)-form, i.e., a form having degree $p \geq 0$ in dz and degree $q \geq 1$ in $d\bar{z}$. A necessary condition for the solvability of this equation is that the form g be $\bar{\partial}$-closed, i.e. $\bar{\partial} g = 0$. This is a necessary compatibility condition and so it is always assumed to be satisfied. The Cauchy–Riemann equations are solvable on each domain of holomorphy (Grothendieck, Dolbeault, 1953). If the domain is bounded and $g \in L_2$, then there exists a solution to the C.-R. equations which lies in L_2 and is orthogonal to the subspace of $\bar{\partial}$-closed $(p, q-1)$-forms (Morrey, Kohn, Hörmander, 1965). For strictly pseudoconvex domains there are explicit formulas for the solution of these equations and estimates on the solution in the uniform norm and in several other metrics (G.M. Khenkin, Grauert, Lieb, 1969).

For some simple domains, the question of the possibility of solving the $\bar{\partial}$-equations with uniform estimates remains open. For example there are no such estimates on a Siegel domain, also called a generalized unit disc. This is the domain, in the n^2-dimensional space, of square matrices Z determined by the condition $E - Z \cdot Z^* \gg 0$, i.e., consisting of matrices Z, for which the indicated expression is a positive definite matrix.

To every complex manifold is associated a system of cohomology groups called the *Dolbeault cohomology* (1953). The Dolbeault group of type (p, q) is the quotient of the group of $\bar{\partial}$-closed (p, q)-forms by the group of $\bar{\partial}$-exact (p, q)-forms. In many cases (for example, for compact Kähler manifolds), these groups can be calculated using de Rham cohomology. However, on domains of holomorphy, the Dolbeault cohomology is trivial while the de Rham cohomology may be non-trivial.

Interest in the $\bar{\partial}$-equations is also connected to the phenomenon that there is a wide class of differential equations which by a change of variables are transformed to the $\bar{\partial}$-equations, and in many cases this yields the possibility of characterizing the solutions of the initial equations in one form or the other. In the general situation, this change of variables leads to the $\bar{\partial}$-equations on a surface (the tangential Cauchy–Riemann equations). In these situations the $\bar{\partial}$-equations are to be understood as follows: f is called a solution to the equation $\bar{\partial} f = g$ on the surface M if this equation is fulfilled for all vectors lying in the complex tangent space to M. Each system of linear differential equations in general position, with analytic coefficients, and one unknown function, can be transformed by an analytic change of coordinates to the $\bar{\partial}$-equations (of type $(0, 1)$) on an analytic surface (Rossi, Andreotti, Hill, 1970). Such equations satisfying the natural compatibility conditions, are locally solvable (Spencer, V.P. Palamodov, 1968). If the right-hand side is not analytic, then, such equations are, generally speaking, not solvable. For example, on the sphere in $\mathbb{C}^2$, one can give an infinitely differentiable $(0, 1)$-form such that the equation $\bar{\partial} f = g$ turns out to be not locally solvable (H. Lewy, 1957). Explicit integral formulae for solutions to the $\bar{\partial}$-equations yield criteria for solvability (G.M. Khenkin 1980). Systems of equations with smooth coefficients are, generally speaking, not reduceable to $\bar{\partial}$-equations (Nirenberg, 1971). If we extend the class of transformations acting on these equations, namely, by adding homogeneous simpletic transformations in the cotangent bundle, then almost all linear systems of equations with analytic coefficients can be reduced locally to $\bar{\partial}$-equations on standard surfaces (Sato, Kawai, Kashiwara, 1973). For further results see paper II.