Lecture Notes in Mathematics

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Introduction

The National Science Foundation (NSF) and the Cosiglio Nazionale delle Ricerche (CNR) are supporting the collaboration between a group of American harmonic analysts and a group of Italian harmonic analysts. Among the many activities involved in this collaboration is an annual conference. These are the proceedings of the second conference held by these two groups (the first conference was held in April 1980 at the Scuola Normale Superiore, Pisa, Italy), and the Proceedings appeared as a Supplemento ai Rendiconti del Circolo Matematico di Palermo , n. 1, 1981. Harmonic analysts from all over the world are encouraged to attend these meetings whose main purpose is to bring the various participants up to date on the most recent research in their field. Both meetings have been very succesfull and the topics ranged through most of harmonic analysis and related subjects. These proceedings include many original research articles and three very timely expository articles by A. Baernstein (on the Bieberbach conjecture), A.W. Knapp and B. Speh (on the present status of the theory of the irreducible unitary representations) and O.C. McGehee(a discussion concerning the recently solved Littlewood conjecture).

We wish to thank the members of the Department of Mathematics, University of Minnesoty, and, in particular, Eugene Fabes, who assumed the principal responsibilities for the organization.

Fulvio Ricci
Guido Weiss

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Status of Classification of Irreducible Unitary Representations

By A. W. Knapp* and B. Speh*

One of the first questions that one would like to answer for Fourier analysis with a particular group is: "What are all the irreducible unitary representations of the group?" For semisimple groups this problem remains unsolved——in fact, very far from solved. Our intention here is to give a survey of some aspects of what is known about the problem for semisimple Lie groups. For an earlier survey of this kind, see [23].

Most of the survey will be of old results, but we shall include some new facts as well:

1) a useful reformulation of the known criterion [23] for unitarity of an irreducible admissible representation. This is given as Theorem 1.2. Progress to date in applying this or some equivalent criterion to settle concrete unitarity questions is summarized in §2.

2) a description, given in a diagram in §3, of some representations of $SU(N,2)$ that we can prove are unitary. The diagram is complicated enough to illustrate the difficulty of the general problem yet simple enough to suggest a number of inductive approaches to a solution. In §4 we summarize briefly some techniques, including those needed for our result about $SU(N,2)$, for applying the unitarity criterion to determine whether a particular irreducible admissible representation is unitary.

3) an extension in §5 of one of the techniques listed in §4,

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namely use of explicit scalar formulas obtained from intertwining operators. We have already applied this extended technique to our own classification [19] of the irreducible unitary representations of $SU(2,2)$.

§1. Unitarity criterion

Let G denote a connected semisimple Lie group with a faithful matrix representation, let K be a maximal compact subgroup, and let θ be the corresponding Cartan involution. Fix a minimal parabolic subgroup P_{min}, and let $P_{min} = M_{min}A_{min}N_{min}$ be its Langlands decomposition. Here M_{min} is compact, A_{min} is a vector group, N_{min} is simply-connected nilpotent, and $G = KA_{min}N_{min}$ is an Iwasawa decomposition of G. A <u>standard parabolic subgroup</u> P of G is any closed subgroup containing P_{min}. There are finitely many such subgroups P, and each has a Langlands decomposition $P = MAN$. Here M is noncompact unless $P = P_{min}$, and also $A \subseteq A_{min}$ and $N \subseteq N_{min}$. The group P is called <u>cuspidal</u> if $\operatorname{rank} M = \operatorname{rank}(K \cap M)$.

Let ω be an irreducible unitary representation of G. A vector v is <u>K-finite</u> if the span of $\omega(K)v$ is finite-dimensional. Then ω defines an irreducible admissible representation of the Lie algebra of G on the space of K-finite vectors, by [11]. (We shall abuse notation and speak of an irreducible admissible representation of G.) We recall the statement of the Langlands classification [27] of irreducible admissible representations; the statement below has been sharpened by the incorporation of a result of Miličić [28].

<u>Langlands classification</u> [27]. The (equivalence classes of) irreducible admissible representations of G stand in one-one correspondence with all triples (P,π,ν), where

$P = MAN$ is a standard parabolic subgroup

π is an irreducible "tempered" unitary representation (equivalence class) of M

ν is a complex-valued linear functional on the Lie algebra of A with $\operatorname{Re}\nu$ in the open positive Weyl chamber.

The Langlands representation $J(P,\pi,\nu)$ is the unique irreducible quotient of the induced representation

$$U(P,\pi,\nu) = \operatorname{ind}_{MAN}^{G}(\pi \otimes e^{\nu} \otimes 1) \tag{1.1}$$

and is given as the image of an explicit intertwining operator $A(\theta P:P:\pi:\nu)$ applied to $U(P,\pi,\nu)$.

In (1.1) we have arranged parameters so that unitary representations induce to unitary representations, and we adopt the convention that G acts on the left. The intertwining operator $A(\theta P:P:\pi:\nu)$ is given by a convergent integral in the context of the theorem; its general definition and properties may be found in §§6-7 of [22]. The representation π is assumed "tempered" in the sense that $(\pi(m)\varphi,\psi)$ is in $L^{2+\epsilon}(M)$ for every $\epsilon > 0$ and for all $(K\cap M)$-finite vectors φ and ψ. The irreducible tempered representations were classified in 1976, with details appearing in [24]; their classification will be combined with the Langlands classification in Theorem 1.1 below.

An irreducible admissible representation comes from the space of K-finite vectors of a unitary representation if and only if it is infinitesimally unitary (in the sense of admitting a Hermitian inner product such that the Lie algebra of G acts in skew-Hermitian fashion), and in this case the unitary representation is unique (up to unitary equivalence) and irreducible.

Corollary [23]. $J(P,\pi,\nu)$ is infinitesimally unitary if and only if

(i) the formal symmetry conditions hold: there exists w in K normalizing A with $wPw^{-1} = \theta P$, $w\pi \cong \pi$, and $w\nu = -\bar{\nu}$, and

(ii) the Hermitian intertwining operator $\pi(w)R(w)A(\theta P:P:\pi:\nu)$, where $R(w)$ denotes right translation of functions by w, is positive or negative semidefinite.

For connected linear semisimple groups, it is proved in [24] that the irreducible tempered representations are all induced from cuspidal parabolic subgroups $M_1A_1N_1$ with a discrete series or limit of discrete series representation on M_1 and a unitary character on A_1; moreover, the limit of discrete series representation may be assumed to be given with nondegenerate data. Conversely such an induced representation is always tempered, and it is irreducible if and only if a certain finite group, known as the R group, is trivial.

Most of the steps needed to extend this result to handle an irreducible tempered representation π of the (possibly disconnected) group M obtained from a standard parabolic subgroup of G are already present in [24], and it is easy to complete the argument. Then we can substitute for π in the Langlands classification, and we arrive at Theorem 1.1 below (Theorem 5 of [23]). The information from the R group ensuring that π is irreducible needs to be built into the statement, and we accordingly recall some definitions from [23]. Let MAN be a cuspidal parabolic subgroup of G, let $W(A:G)$ be the Weyl group of A, let $\mathfrak{a}$ be the Lie algebra of A, and let σ be a discrete series or limit of discrete series of M with nondegenerate data. For each $\mathfrak{a}$ root α, let $\mu_{\sigma,\alpha}(\nu)$ be the Plancherel factor of §7 of [24]. Define

$$\Delta' = \{\text{useful } \mathfrak{a} \text{ roots } \alpha \mid s_\alpha \nu = \nu \text{ and } \mu_{\sigma,\alpha}(\nu) = 0\} \qquad (1.2)$$

and

$$W'_{\sigma,\nu} = \text{Weyl group of root system } \Delta' . \qquad (1.3)$$

The group $W'_{\sigma,\nu}$ is a subgroup of

$$W_{\sigma,\nu} = \{w \in W(A:G) \mid w\sigma \cong \sigma \text{ and } w\nu = \nu\} . \qquad (1.4)$$

We can then reformulate the completeness of the Langlands classification as Theorem 1.1. The idea is that the R group of the concealed tempered representation π is isomorphic to $W_{\sigma,\nu}/W'_{\sigma,\nu}$.

Theorem 1.1 [23]. Let $P = MAN$ be a cuspidal standard parabolic subgroup of G, let σ be a discrete series or limit of discrete series representation of M with nondegenerate data, and let ν be a complex-valued linear functional on $\mathfrak{a}$ with $\operatorname{Re} \nu$ in the closed positive Weyl chamber. Suppose that $W_{\sigma,\nu} = W'_{\sigma,\nu}$. Then the induced representation $U(P,\sigma,\nu)$ has a unique irreducible quotient $J'(P,\sigma,\nu)$, and every irreducible admissible representation of G is of the form $J'(P,\sigma,\nu)$ for some such triple (P,σ,ν).

The effect of Theorem 1.1 is to rewrite the completeness of the Langlands classification in terms of more manageable representations.

What is lost is the simple criterion for equivalences, but equivalences can always be sorted out by going back to the earlier statement. If we take these matters into account, then we can translate into the present language the unitarity criterion given in the corollary stated earlier.

Theorem 1.2. Let (P,σ,ν) be such that the irreducible admissible representation $J'(P,\sigma,\nu)$ is defined. Then $J'(P,\sigma,\nu)$ is infinitesimally unitary if and only if

(i) there exists w in $W(A:G)$ such that $w^2 = 1$, $w\sigma \cong \sigma$, and $w\nu = -\bar{\nu}$, and

(ii) the standard intertwining operator $\sigma(w)A_P(w,\sigma,\nu)$ of §§7-8 of [22], when normalized to be pole-free and not identically zero as

$$\sigma(w)\,\mathcal{A}_P(w,\sigma,\nu)\ , \tag{1.5}$$

is positive or negative semidefinite.

If $J'(P,\sigma,\nu)$ is infinitesimally unitary, then every w satisfying (i) is such that the operator (1.5) is positive or negative semidefinite.

Proof. By way of preliminaries let us introduce notation that makes clear how to regard $J'(P,\sigma,\nu)$ as a Langlands quotient. With $P = MAN$, let $\mathfrak{m}$, $\mathfrak{a}$, and $\mathfrak{n}$ be the Lie algebras of M, A, and N. Define $\mathfrak{a}_*$ to be the span in $\mathfrak{a}$ of the vectors H_α such that the $\mathfrak{a}$ root α is orthogonal to $\operatorname{Re}\nu$. Let $\mathfrak{a}_1$ be the orthocomplement of $\mathfrak{a}_*$ in $\mathfrak{a}$. Define $\mathfrak{n}_*$ to be the centralizer of $\mathfrak{a}_1$ in $\mathfrak{n}$, $\mathfrak{n}_1$ to be the natural complement of $\mathfrak{n}_*$ in $\mathfrak{n}$, and $\mathfrak{m}_1$ to be

$$\mathfrak{m}_1 = \mathfrak{m} \oplus \mathfrak{a}_* \oplus \mathfrak{n}_* \oplus \theta\mathfrak{n}_*\ .$$

Then we can form a corresponding standard parabolic subgroup $P_1 = M_1A_1N_1$ of G with

$$M_1A_1N_1 \supseteq MAN$$

and with MA_*N_* a parabolic subgroup of M_1. These definitions are arranged so that $\nu|_{\mathfrak{a}_*}$ is imaginary and so that $\mathrm{Re}(\nu|_{\mathfrak{a}_1})$ is in the open positive Weyl chamber of $\mathfrak{a}_1'$. The representation

$$\pi = \mathrm{ind}_{MA_*N_*}^{M_1}(\sigma \otimes \exp(\nu|_{\mathfrak{a}_*}) \otimes 1)$$

is tempered and, by assumption, irreducible. Then we have

$$J'(P,\sigma,\nu) \cong J(P_1,\pi,\nu|_{\mathfrak{a}_1}).$$

Now we come to the proof of the theorem. The main step will be to prove that $J'(P,\sigma,\nu)$ infinitesimally unitary implies that (i) holds. Once this is done, we can argue as follows: If (i) holds for some w, then (1.5) is defined (by Lemma 7.9 of [22]), and Corollary 8.7 of [22] shows that the sesquilinear form

$$\langle u,v\rangle = \int_K (\sigma(w)\mathcal{Q}_P(w,\sigma,\nu)u(k),v(k))\, dk \tag{1.6}$$

is invariant (in the sense that the Lie algebra of G acts by skew-Hermitian operators) and Hermitian. Since $w\nu = -\bar{\nu}$, we have

$$w(\mathrm{Re}\,\nu) = -\mathrm{Re}\,\nu\,. \tag{1.7}$$

From (1.7) it follows that $w\mathfrak{a}_* = \mathfrak{a}_*$ and therefore that $w\mathfrak{a}_1 = \mathfrak{a}_1$. Another application of (1.7) then shows that

$$w(\mathrm{Re}\,\nu|_{\mathfrak{a}_1}) = -\mathrm{Re}\,\nu|_{\mathfrak{a}_1}\,.$$

Since $\operatorname{Re} \nu|_{\mathfrak{a}_1}$ is in the open positive Weyl chamber of $\mathfrak{a}_1'$, $w\mathfrak{n}_1 = \theta\mathfrak{n}_1$. Thus $wP_1w^{-1} = \theta P_1$. From this equality and Corollary 7.7 of [22], we see that $\mathcal{A}_P(w,\sigma,\nu)$ can be regarded as a composition of the Langlands operator $A(\theta P_1 : P_1 : \pi : \nu|_{\mathfrak{a}_1})$ followed by another operator. Since the image of the Langlands operator is irreducible, the image of (1.5) must be equivalent with $J'(P,\sigma,\nu)$. Consequently (1.6) descends to a nonzero invariant Hermitian form on $J'(P,\sigma,\nu)$. By irreducibility of $J'(P,\sigma,\nu)$, such a form is unique up to a scalar, and $J'(P,\sigma,\nu)$ is infinitesimally unitary if and only if a nonzero such form is semidefinite.

Thus the theorem will be proved if we show that $J'(P,\sigma,\nu)$ infinitesimally unitary implies that (i) holds. Thus suppose $J'(P,\sigma,\nu)$ is infinitesimally unitary. Then so is the equivalent representation $J(P_1,\pi,\nu|_{\mathfrak{a}_1})$. By the corollary above, there exists w_1 in the normalizer $N_K(\mathfrak{a}_1)$ such that

$$w_1P_1w_1^{-1} = \theta P_1, \quad w_1\pi \cong \pi, \quad \text{and} \quad \operatorname{Ad}(w_1)\nu|_{\mathfrak{a}_1} = -\bar{\nu}|_{\mathfrak{a}_1}. \tag{1.8}$$

We shall apply the equivalence criterion for irreducible tempered representations to the formula $w_1\pi \cong \pi$. (See Theorem 4 of [23] or Theorem 14.2 of [24]. These theorems are stated in the connected case, but they extend to groups like M without difficulty.) The criterion says that the equivalence of

$$\pi = \operatorname{ind}_{MA_*N_*}^{M_1} (\sigma \otimes \exp \nu|_{\mathfrak{a}_*} \otimes 1)$$

and

$$w_1\pi \cong \operatorname{ind}_{w_1(MA_*N_*)w_1^{-1}}^{M_1} (w_1\sigma \otimes \exp(\mathrm{Ad}(w_1)\nu)|_{\mathrm{Ad}(w_1)\mathfrak{a}_*} \otimes 1)$$

implies there is an element w_2 in $K \cap M_1$ with

$$w_1Mw_1^{-1} = w_2Mw_2^{-1} \tag{1.9a}$$

$$w_1A_*w_1^{-1} = w_2A_*w_2^{-1} \tag{1.9b}$$

$$w_1\sigma \cong w_2\sigma \tag{1.9c}$$

$$(\mathrm{Ad}(w_1)\nu)|_{\mathrm{Ad}(w_1)\mathfrak{a}_*} = (\mathrm{Ad}(w_2)\nu)|_{\mathrm{Ad}(w_2)\mathfrak{a}_*} . \tag{1.9d}$$

We shall list some properties of $w_2^{-1}w_1$. Since w_2 is in M_1, (1.8) gives

$$(w_2^{-1}w_1)P_1(w_2^{-1}w_1)^{-1} = \theta P_1 . \tag{1.10a}$$

Also w_1 in $N_K(\mathfrak{a}_1)$ and w_2 in the centralizer $Z_K(\mathfrak{a}_1)$ imply $w_2^{-1}w_1$ is in $N_K(\mathfrak{a}_1)$, and (1.9b) shows $w_2^{-1}w_1$ is in $N_K(\mathfrak{a}_*)$. Thus

$$w_2^{-1}w_1 \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_*) . \tag{1.10b}$$

From (1.9c) we have

$$w_2^{-1}w_1\sigma \cong \sigma . \tag{1.10c}$$

By (1.9d) and (1.10b), we have $\mathrm{Ad}(w_2^{-1}w_1)(\nu|_{\mathfrak{a}_*}) = (\nu|_{\mathfrak{a}_*})$, which is imaginary. Hence

$$\mathrm{Ad}(w_2^{-1}w_1)(\nu|_{\mathfrak{a}_*}) = -(\bar{\nu}|_{\mathfrak{a}_*}) ,$$

and (1.8) gives

$$\mathrm{Ad}(w_2^{-1}w_1)\nu = -\bar{\nu} . \tag{1.10d}$$

Dropping "Ad" for simplicity, let us observe that $w_2^{-1}w_1$ normalizes the system Δ' of (1.2). [In fact, α in Δ' implies $w_2^{-1}w_1\alpha$ useful, and we have

$$s_{w_2^{-1}w_1\alpha}\nu = (w_2^{-1}w_1)s_\alpha(w_2^{-1}w_1)^{-1}\nu = -\ w_2^{-1}w_1 s_\alpha\bar{\nu}$$

$$= -\ w_2^{-1}w_1\bar{\nu} = \nu$$

by two applications of (1.10d). Also

$$\mu_{\sigma, w_2^{-1}w_1\alpha}(\nu) = \mu_{(w_2^{-1}w_1)^{-1}\sigma,\alpha}((w_2^{-1}w_1)^{-1}\nu)$$

$$= \mu_{\sigma,\alpha}((w_2^{-1}w_1)^{-1}\nu) \qquad \text{by (1.10c)}$$

$$= \mu_{\sigma,\alpha}(\nu)$$

since $\mu_{\sigma,\alpha}$ depends only on the $\mathfrak{a}_*$ component and since $w_2^{-1}w_1$ fixes $\nu|_{\mathfrak{a}_*}$.] Then we can choose w_3 in $K \cap M_1$ representing a member of $W'_{\sigma,\nu}$ such that

$$w_3^{-1}w_2^{-1}w_1\Delta'^{+} = \Delta'^{+} . \tag{1.11}$$

Then it is clear that

$$(w_3^{-1}w_2^{-1}w_1)P_1(w_3^{-1}w_2^{-1}w_1)^{-1} = \theta P_1 \tag{1.12a}$$

and

$$w_3^{-1}w_2^{-1}w_1 \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_*) . \tag{1.12b}$$

Since $W'_{\sigma,\nu} \subseteq W_{\sigma,\nu}$, (1.10c) and (1.10d) give

$$w_3^{-1}w_2^{-1}w_1\sigma \cong \sigma \tag{1.12c}$$

and

$$\mathrm{Ad}(w_3^{-1}w_2^{-1}w_1)\nu = -\bar{\nu} . \tag{1.12d}$$

Let w be the class of $w_3^{-1}w_2^{-1}w_1$ in $W(A:G)$. Then w^2 fixes σ and ν, by (1.12c) and (1.12d), and so is in $W_{\sigma,\nu}$. Since π is irreducible (in order to have $J'(P,\sigma,\nu)$ defined), we have $W_{\sigma,\nu} = W'_{\sigma,\nu}$. Thus w^2 is in $W'_{\sigma,\nu}$. From (1.11), $w^2\Delta'^{+} = \Delta'^{+}$, and thus $w^2 = 1$. This identity and formulas (1.12c) and (1.12d) together prove (i) and complete the proof of the theorem.

§2. Progress

The problem of classifying irreducible unitary representations comes down to deciding which parameters (P,σ,ν) in Theorem 1.2 satisfy (i) and (ii) of the theorem. Here (i) is easy to decide, but (ii) is often hard. There are several sufficient conditions for deciding one way or the other, and we shall list a number of them in §4. It is unlikely that the final answer will be a group-by-group investigation, but it does give some idea of the nature of the problem to tell what simple noncompact matrix groups have been completely settled.

The groups handled so far are the following.