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Partial Differential Equations

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FOREWORD

This volume contains a selection of papers presented at the 7th symposium on differential geometry and differential equations(=DD7), which took place at Nankai Institute of Mathematics, Tianjin, China, June 23 — July 5, 1986. The subject was partial differential equations. It was a culmination of a year-long activity in 1985-1986 at the institute. A list of the other papers presented at the symposium can be found at the end of this volume, some of which will be published elsewhere.

For the record I would like to give a list of the preceding DD-Symposia as follows:

	Subject	Date	Place	Publication
DD1	Differential geometry and differential equations	Aug. 18-Sept. 21, 1980	Beijing, China	Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Science Press, Beijing, China, 1982
DD2	Differential geometry	Aug. 20-Sept. 13, 1981	Shanghai-Hefei, China	Proceedings of the 1981 Symposium on Differential Geometry and Differential Equations, Shanghai-Hefei, Science Press, Beijing, China, 1984
DD3	Partial differential equations	Aug. 23-Sept. 16, 1982	Changchun, China	Proceedings of the 1982 Changchun Symposium on Differential Geometry and Differential Equations, Science Press, Beijing, China, 1986
DD4	Ordinary differential equations	Aug. 29-Sept. 10, 1983	Beijing, China	Proceedings of the 1983.Beijing Symposium on Differential Geometry and Differential Equations, Science Press, Beijing, China, 1986
DD5	Computation on partial differential equations	Aug. 13-aug. 17, 1984	Beijing, China	Proceedings of the 1984 Beijing Symposium on Differential Geometry and Differential Equations, Science Press, Beijing, China, 1985
DD6	Differential geometry	Jun. 21-Jul. 6, 1985	Shanghai, China	Differential Geometry and Differential Equations, Proceedings, Shanghai 1985, Lecture Notes in Mathematics 1255, Springer-Verlag 1987

S.S. Chern

June 1987

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CO-AREA, LIQUID CRYSTALS, AND MINIMAL SURFACES[1]

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Abstract. Oriented n area minimizing surfaces (integral currents) in $\mathcal{M}^{m+n}$ can be approximated by level sets (slices) of nearly m-energy minimizing mappings $\mathcal{M}^{m+n} \to \mathbf{S}^m$ with essential but controlled discontinuities. This gives new perspective on multiplicity, regularity, and computation questions in least area surface theory.

In this paper we introduce a collection of ideas showing relations between co-area, liquid crystals, area minimizing surfaces, and energy minimizing mappings. We state various theorems and sketch several proofs. A full treatment of these ideas is deferred to another paper.

Problems inspired by liquid crystal geometries.[2] Suppose Ω is a region in 3 dimensional space $\mathbf{R}^3$ and f maps Ω to the unit 2 dimensional sphere $\mathbf{S}^2$ in $\mathbf{R}^3$. Such an f is a unit vectorfield in Ω to which we can associate an 'energy'

$$\mathcal{E}(f) = \left(\frac{1}{8\pi}\right) \int_\Omega |Df|^2 \, d\mathcal{L}^3;$$

here Df is the differential of f and $|Df|^2$ is the square of its Euclidean norm—in terms of coordinates,

$$|Df(x)| = \sum_{k=1}^{3} \sum_{i=1}^{3} \left(\frac{\partial f^k}{\partial x_i}(x) \right)^2$$

for each x. The factor $1/8\pi$ which equals 1 divided by twice the area of $\mathbf{S}^2$ is a useful normalizing constant. It is straightforward to show the existence of f's of least energy for given boundary values (in an appropriate function space).

Such boundary value problems have been associated with liquid crystals.[3] In this context, a "liquid crystal" in a container Ω is a fluid containing long rod like molecules whose directions are specified by a unit vectorfield. These molecules have a preferred alignment relative to each other—in the present case the preferred alignment is parallel. If we imagine the molecule orientations along

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[1] This research was supported in part by grants from the National Science Foundation

[2] The research which led to the present paper began as an investigation of a possible equality between infimums of m-energy and the n area of area minimizing n dimensional area minimizing manifolds in $\mathbf{R}^{m+n}$ suggested in section VIII(C) of the paper, *Harmonic maps with defects* [**BCL**] by H. Brezis, J-M. Coron, and E. Lieb. Although the specific estimates suggested there do not hold (by virtue of counterexamples [**MF**][**W1**][**YL**]) their general thrust does manifest itself in the results of the present paper.

[3] See, for example, the discussion by R. Hardt, D. Kinderlehrer, and M. Luskin in [**HKL**].

$\partial\Omega$ to be fixed (perhaps by suitably etching container walls) then interior parallel alignment may not be possible. In one model the system is assumed to have 'free energy' given by our function $\mathcal{E}$ and the crystal geometry studied is that which minimizies this free energy.

If Ω is the unit ball and $f(x) = x$ for $|x| = 1$, then there is no continuous extension of these boundary values to the interior; indeed the unique least energy f is given by setting $f(x) = x/|x|$ for each x. It turns out that this singularity is representative, and the general theorem is that *least energy f's exist and are smooth except at isolated points p of discontinuity where 'tangential structure' is $\pm x/|x|$ (up to a rotation), e.g. f has local degree equal to ± 1* [**SU**] [**BCL** VII].

As a further step towards an understanding of the geometry of of energy minimizing f's one might seek estimates on the number of points of discontinuity which such an f can have—e.g. if the boundary values are not to wild must the number of points of discontinuity be not too big?[4] An alternative problem to this is to seek a lower bound on the energy when the points of discontinuity are prescribed together with the local degrees of the mapping being sought. This question has a surprisingly simple answer as follows.

THEOREM. *Suppose $p_1, \ldots, p_N$ are points in $\mathbf{R}^3$ and $d_1, \ldots, d_N \in \mathbf{Z}$ are the prescribed degrees with $\sum_{i=1}^N d_i = 0$. Let $\inf \mathcal{E}$ denote the infimum of the energies of (say, smooth) mappings from $\mathbf{R}^3 \sim \{p_1, \ldots, p_N\}$ to $\mathbf{S}^2$ which map to the 'south pole' outside some bounded region in $\mathbf{R}^3$ and which, for each i, map small spheres around p_i to $\mathbf{S}^2$ with degree d_i. Then $\inf \mathcal{E}$ equals the least mass $\mathbf{M}(T)$ of integral 1 currents T in $\mathbf{R}^3$ with*

$$\partial T = \sum_{i=1}^{N} d_i [\![p_i]\!].$$

This fact (stated in slightly different language) is one of the central results of [**BCL**]. We would like to sketch a proof in two parts: first by showing that $\inf \mathcal{E} \leq \inf \mathbf{M}$ (with the obvious meanings) and then by showing that $\inf \mathbf{M} \leq \inf \mathcal{E}$. The proof of the first part follows [**BCL**] while the second part is new. It is in this second part that the coarea formula makes its appearance.

Proof that $\inf \mathcal{E} \leq \inf \mathbf{M}$. The first inequality is proved by construction as illustrated in Figure 1. We there represent that case in which N equals 2 and p_1 and p_2 are distinct points with $d_1 = -1$ and $d_2 = +1$. We choose and fix a smooth curve C connecting these two points and orient C by a smoothly varying unit tangent vector field ς which points away from p_1 and towards p_2. The associated 1 dimensional integral current is $T = \mathbf{t}(C, 1, \varsigma)$ and its *mass* $\mathbf{M}(T)$ is the length of C since the density specified is everywhere equal to 1.[5] We now choose (somewhat arbitrarily)

[4] As it turns out, away from the boundary of Ω, the number of these points is bounded a *priori* independent of boundary values.

[5] Formally, a 1 current such as T is a linear functional on smooth differential 1 forms in $\mathbf{R}^3$. If φ is such a 1 form then

$$T(\varphi) = \int_{x \in C} \langle \varsigma(x), \varphi(x) \rangle \, d\mathcal{H}^1 x.$$

To each point p in $\mathbf{R}^3$ is associated the 0 dimensional current $[\![p]\!]$ which maps the smooth function ψ to the number $\psi(p)$. See Appendix A.4.

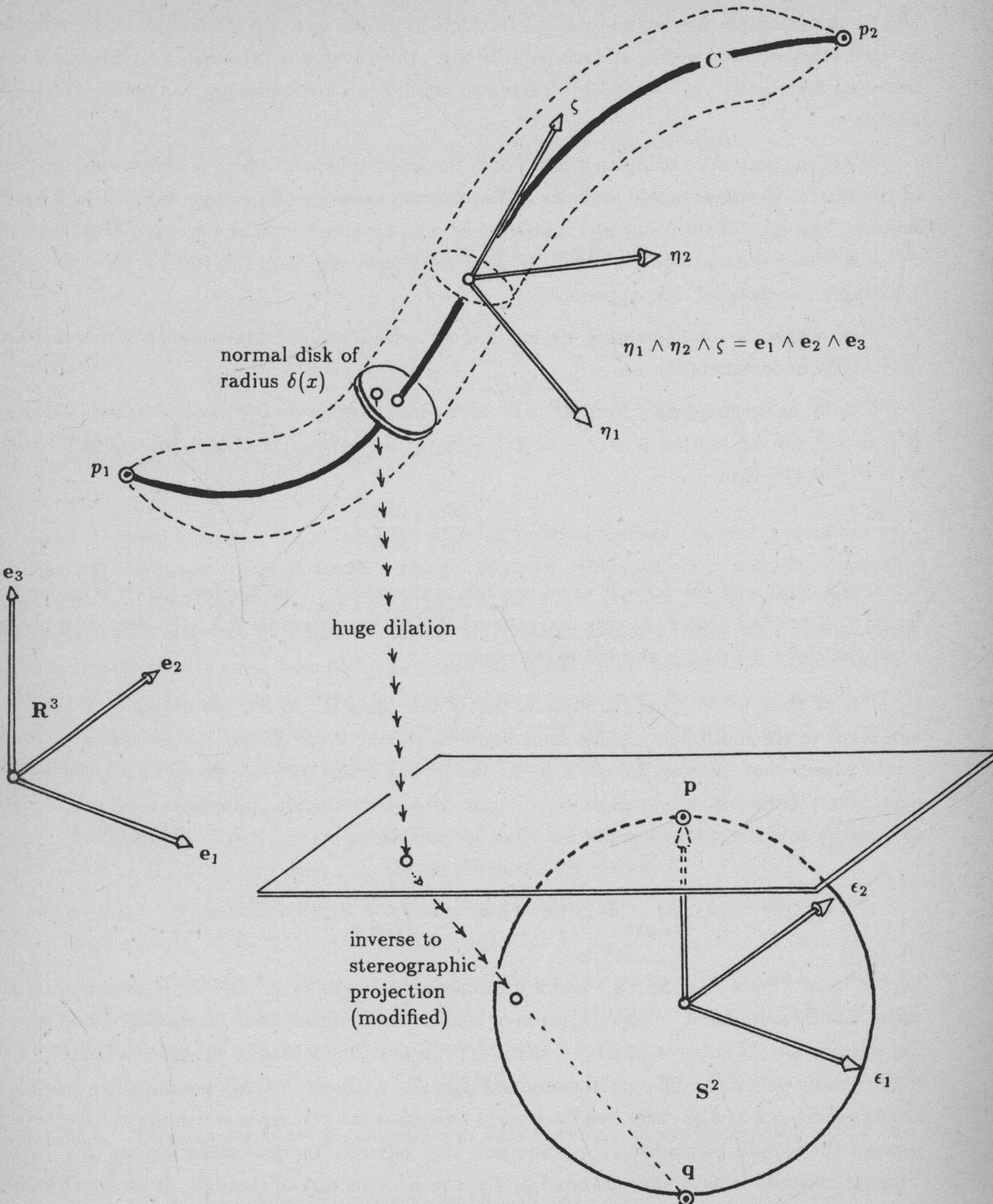

Figure 1. Construction of a mapping f (indicated by dashed arrows) from $\mathbf{R}^3$ to $\mathbf{S}^2$ having energy $\mathcal{E}(f)$ not much greater than the length of the curve C connecting the points p_1 and p_2. Small disks normal to C map by f to cover $\mathbf{S}^2$ once in a nearly conformal way. This implies that small spheres around p_1 map to $\mathbf{S}^2$ with degree -1 while small spheres around p_2 map with degree $+1$. The 1 current $\mathbf{t}(C, 1, \varsigma)$ is the slice $\langle \mathbf{E}^3, f, \mathbf{p} \rangle$ of the Euclidean 3 current $\mathbf{E}^3$ by the mapping f and the 'north pole' $\mathbf{p}$ of $\mathbf{S}^2$.

and fix two smoothly varying unit normal vector fields η_1 and η_2 along C which are perpendicular to each other and for which, at each point x of C, the 3-vector $\eta_1(x) \wedge \eta_2(x) \wedge \varsigma(x)$ equals the orienting 3-vector $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ for $\mathbf{R}^3$. These two vector fields are a 'framing' of the normal bundle of C.

We then construct a mapping γ of $\mathbf{R}^2$ onto the unit 2 sphere $\mathbf{S}^2$ which is a slight modification of the inverse to stereographic projection. To construct such γ we fix a huge radius R in $\mathbf{R}^2$ and require: (i) if $|y| \leq R$ then $\gamma(y)$ is that point in $\mathbf{S}^2$ which maps to y under stereographic projection $\mathbf{S}^2 \to \mathbf{R}^2$ from the south pole $\mathbf{q}$ of $\mathbf{S}^2$; (ii) if $|y| \geq 2R$ then $\gamma(y) = \mathbf{q}$; (iii) for $R < |y| < 2R$, $\gamma(y)$ is suitably interpolated. See Appendix A.2.

Next we choose some smoothly varying (and very small) radius function δ on C which vanishes only at the endpoints p_1 and p_2.

Finally, as our mapping f from $\mathbf{R}^3$ to $\mathbf{S}^2$ with which to estimate $\mathcal{E}(f)$ we specify the following. If p in $\mathbf{R}^3$ can be written $p = x + s\eta_1(x) + t\eta_2(x)$ for some x in C and some s and t with $s^2 + t^2 \leq \delta(x)^2$, then

$$f(p) = \gamma\left(\frac{2Rs}{\delta(x)}, \frac{2Rt}{\delta(x)}\right).$$

Otherwise, $f(p) = \mathbf{q}$. We leave it as an exercise to the reader to use the fact that γ is conformal for $|y| < R$ to check that $\mathcal{E}(f)$ very nearly equals $\mathbf{M}(T)$; see Appendix A.2. The remainder of the proof that $\inf \mathcal{E} \leq \inf \mathbf{M}$ is also left to the reader.

Proof that $\inf \mathbf{M} \leq \inf \mathcal{E}$. Suppose that f does map $\mathbf{R}^3$ to $\mathbf{S}^2$, has degree d_i at each p_i, and maps to the south pole outside some bounded region. From dimensional considerations one would expect that for most points w in $\mathbf{S}^2$ the inverse image $f^{-1}\{w\}$ would be a collection of curves connecting the various points $p_1, \dots, p_N$. H. Federer's *coarea formula* is what enables one to quantify this idea; see Appendix A.5. This formula asserts

$$\int_{w \in \mathbf{S}^2} \mathcal{H}^1(f^{-1}\{w\})\, d\mathcal{H}^2 w = \int_{x \in \mathbf{R}^3} J_2 f(x)\, d\mathcal{L}^3 x;$$

here $\mathcal{H}^1$ and $\mathcal{H}^2$ are Hausdorff's 1 and 2 dimensional measures in $\mathbf{R}^3$ and $\mathcal{L}^3$ is Lebesgue's 3 dimensional measure for $\mathbf{R}^3$. Also $J_2 f(x)$ here denotes the 2 dimensional Jacobian of f at x and a key observation (as noted in [**BCL**]) is that $J_2 f(x)$ is always less than or equal to half of $|Df(x)|^2$ with equality only if the differential mapping $Df(x)\colon \mathbf{R}^3 \to \mathrm{Tan}(\mathbf{S}^2, f(x))$ is maximally conformal; see Appendix A.1.3. Also central to the present analysis is the manner in which the curves $f^{-1}\{w\}$ connect the various points $p_1, \dots, p_N$ and how they relate to the prescribed degrees $d_1, \dots, d_N$. This connectivity is naturally measured by the current structure of these $f^{-1}\{w\}$'s which comes from the slicing theory for currents; see Appendix A.5. To set this up we regard $\mathbf{R}^3$ as the Euclidean current $\mathbf{E}^3$ (oriented by the 3 vector $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$). The *slice* of $\mathbf{E}^3$ by the map f at the point w in $\mathbf{S}^2$ is the current

$$\langle \mathbf{E}^3, f, w \rangle = \mathbf{t}(f^{-1}\{w\}, 1, \varsigma);$$

the meanings here are the same as for the current T discussed above. A check of orientations and degrees shows that

$$\partial\langle \mathbf{E}^3, f, w\rangle = \sum_{i=1}^{N} k_i [\![p_i]\!];$$

compare with our construction of η_1 and η_2 above. It follows immediately that

$$\begin{aligned} 4\pi \inf \mathbf{M}(T) &= \mathcal{H}^2(\mathbf{S}^2) \inf \mathbf{M}(T) \\ &\leq \int_{w\in \mathbf{S}^2} \mathbf{M}(\langle \mathbf{E}^3, f, w\rangle)\, d\mathcal{H}^2 w \\ &= \int_{\mathbf{R}^3} J_2 f \, d\mathcal{L}^3 \\ &= \left(\frac{1}{2}\right) \int_{\mathbf{R}^3} |Df|^2 \, d\mathcal{L}^3. \end{aligned}$$

This finishes the proof that $\inf \mathbf{M} \leq \inf \mathcal{E}$.

First Generalization. Since the methods used in the proofs of the two inequalities are quite general one might correctly suspect that considerable generalization is possible. Suppose, for example, we fix $B = \{p_1, \ldots, p_N\}$ as a general boundary set and let $\mathcal{F}_0$ be the family of those mappings f of $\mathbf{R}^3$ to $\mathbf{S}^2$ which are locally Lipschitzian except possibly on B, which map to the southpole outside some bounded region, and which have finite energy. Since deformations of mappings in $\mathcal{F}_0$ do not alter discrete combinatorial structures we are led to study properties of homotopy classes $\Pi(\mathcal{F}_0)$ of mappings in $\mathcal{F}_0$—it is most useful here if our homotopies $[0,1]\times\mathbf{R}^3 \to \mathbf{S}^2$ are permitted to have isolated point discontinuties; see Appendix A.3.

Our conditions about mapping degrees above generalize to requirements about degrees $\mathbf{d}(f, S)$ of f on general integral 2 dimensional cycles S in $\mathbf{R}^3 \sim B$. It turns out that such a degree $\mathbf{d}(f, S)$ depends only on the homotopy class of f and on the homology class of S.

It also turns out that the relative homology classes of the slices $\langle \mathbf{E}^3, f, w\rangle$ depend only on the homotopy class $[f]$ of f. We denote this homology class by $s[f]$.

The *Kronecker index* is a pairing between 2 dimensional cycles S in $\mathbf{R}^3 \sim B$ and 1 currents T having boundary in B. In general the Kronecker index $\mathbf{k}(S, T)$ is the sum over points of intersection of S and T of an index of relative orientations; see Appendix A.6

These various ideas are related in the following theorem.

THEOREM. *The diagram below is commutative. Furthermore, s is an isomorphism, and d and k are injections.*

$$\begin{array}{ccc} & & \mathbf{H}_1(\mathbf{R}^3, B; \mathbf{Z}) \\ & \nearrow s & \\ \mathbf{\Pi}(\mathcal{F}_0) & & \downarrow k \\ & \searrow d & \\ & & \mathbf{Hom}(\mathbf{H}_2(\mathbf{R}^3 \sim B, \mathbf{Z}), \mathbf{Z}) \end{array}$$

Here

$s[f] =$ "$[f^{-1}\{w\}]$" $= [\langle \mathbf{E}^3, f, w\rangle] =$ the integral homology class of the 1 current slice;

$d[f][S] = \mathbf{d}(f, S) =$ the degree of f on the 2 cycle S;

$k[T][S] = \mathbf{k}(S, T) =$ the Kronecker Index of the 2 cycle S and the 1 current T.

Our relations between energy minimization and area minimization become the following.

THEOREM. *Suppose that P is an integral 1 current in $\mathbf{R}^3$ with the support of ∂P in B. Suppose also that $T^{\mathbf{Z}}$ has least mass among all integral 1 currents which are homologous to P over the integers $\mathbf{Z}$ and that $T^{\mathbf{R}}$ has least mass among all integral 1 currents which are homologous to P over the real numbers $\mathbf{R}$. Then*

$$\mathbf{M}(T^{\mathbf{Z}}) = \inf\{\mathcal{E}(f): s[f] = [P]\}$$

and

$$\mathbf{M}(T^{\mathbf{R}}) = \inf\{\mathcal{E}(f): d[f] = k[P]\}.$$

Moreover, $\mathbf{M}(T^{\mathbf{Z}}) = \mathbf{M}(T^{\mathbf{R}})$ *(because of our special situation).*

Further generalizations. The essential ingredients of the analyses above remain, for example, if $\mathbf{R}^3$ is replaced by a general $m+n$ dimensional manifold $\mathcal{M}$ (without boundary) which is smooth, compact, and oriented (or $\mathcal{M} = \mathbf{R}^{m+n}$), and B is replaced by a sufficiently nice (possibly empty) compact subset of $\mathcal{M}$ of dimension $n-1$. To study n dimensional integral currents in $\mathcal{M}$ having boundary in B we consider mappings f of $\mathcal{M}$ to a sphere of the complementary dimension m. The spaces $\mathcal{F}$ and $\mathcal{F}_0$ of such mappings and the homotopy classes $\Pi(\mathcal{F})$ are specified in sections A.3.1 and A.3.2 of the Appendix. Some discontinuities are essential.[6] It seems worthwhile to consider three different energies $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ for mappings in $\mathcal{F}_0$. $\mathcal{E}_1$ is a normalization of the usual 'n energy' of mappings, $\mathcal{E}_3$ is a normalized Jacobian integral associated with the coarea formula, and $\mathcal{E}_2$ is an intermediate energy; see Appendix A.3.2. As indicated above, mapping degrees and the Kronecker index have general meanings which are set forth in sections A.6 and A.7 of the Appendix. These various ideas are related as the following theorem shows.

THEOREM. *The diagram of mappings below is well defined and is commutative. In particular, the images of d and k and j in $\mathbf{Hom}(\mathbf{H}_m(\mathcal{M} \sim B, \mathbf{Z}), \mathbf{Z})$ are the same. Furthermore, s is an*

---

[6] Suppose $m = 2$ and $n = 5$ and $\mathcal{M} = \mathbf{R}^7$, and B is a smoothly embedded copy of 2 dimensional complex projective space $\mathbf{CP}(2)$. Then there are no continuous mappings f from the complement of B to $\mathbf{S}^2$ such that small 2 spheres S which link B once map to $\mathbf{S}^2$ with degree one. Any f satisfying such a linking condition for general position S's near B must have interior discontinuities of dimension at least 3.

isomorphism.

$$
\begin{array}{ccccc}
 & & \mathbf{H}_n(\mathcal{M},B;\mathbf{Z}) & \xrightarrow{c} & \mathbf{H}_n(\mathcal{M},B;\mathbf{R}) \\
 & \nearrow s & & \searrow c & \uparrow i \\
\Pi(\mathcal{F}) & & \downarrow k & & c[\mathbf{H}_n(\mathcal{M},B;\mathbf{Z})] \\
 & \searrow d & & \swarrow j & \\
 & & \mathrm{Hom}(\mathbf{H}_m(\mathcal{M}\sim B,\mathbf{Z}),\mathbf{Z}) & &
\end{array}
$$

Here

$s[f] =$ "$[f^{-1}\{p\}]$" $= [\langle [\![\mathcal{M}]\!], f, p\rangle] =$ the integral homology class of the n current slice;

$d[f][S] = \mathbf{d}(f,S) =$ the degree of f on the m cycle S;

$k[T][S] = \mathbf{k}(S,T) =$ the Kronecker index of the m cycle S and the n current T;[7]

c is induced by the coefficient inclusion $\mathbf{Z} \to \mathbf{R}$;

i is the inclusion; and

j is defined by commutivity.

We defer proof of this theorem to our fuller treatment of this subject. The natural setting and generality of such relationships are still under investigation.

The relations between energy minimization and area minimization then become the following.

MAIN THEOREM. *Suppose P is an integral current in $\mathcal{M}$ with the support of ∂P contained in B so that the integral homology class $[P]$ of P belongs to $\mathbf{H}_n(\mathcal{M},B;\mathbf{Z})$. Let $T^{\mathbf{Z}}$ be an integral current of least mass among all integral currents belonging to the same integral homology class as P in $\mathbf{H}_n(\mathcal{M},B,\mathbf{Z})$, and let $T^{\mathbf{R}}$ be an integral current of least mass among all integral currents belonging to the same real homology class as P in $\mathbf{H}_n(\mathcal{M},B,\mathbf{R})$. Then*

$$\mathbf{M}(T^{\mathbf{Z}}) = \inf\{\mathcal{E}_1(f): s[f] = [P]\} = \inf\{\mathcal{E}_2(f): s[f] = [P]\} = \inf\{\mathcal{E}_3(f): s[f] = [P]\}$$

and

$$\mathbf{M}(T^{\mathbf{R}}) = \inf\{\mathcal{E}_1(f): d[f] = k[P]\} = \inf\{\mathcal{E}_2(f): d[f] = k[P]\} = \inf\{\mathcal{E}_3(f): d[f] = k[P]\}.$$

---

7 Suppose $m = 2$ and $n = 1$ and $\mathcal{M}$ is a 3 dimensional real projective space $\mathbf{RP}(3)$ and $T = \mathbf{t}(\mathcal{N}, 1, \varsigma)$; here $\mathcal{N}$ is a 1 dimensional real projective space $\mathbf{RP}(1)$ sitting in $\mathbf{RP}(3)$ in the usual way and ς is some orientation function. Since T is not a boundary while $2T$ is, we conclude that the homology class

$$[T] \in \mathbf{H}_1(\mathcal{M}, \emptyset; \mathbf{Z}) = \mathbf{Z}_2$$

is not the 0 class although $\mathbf{k}(S,T) = 0$ for each 2 cycle S in $\mathcal{M}$. In particular, the mapping k is generally not an injection.

In general, of course, $\mathbf{M}(T^{\mathbf{R}}) < \mathbf{M}(T^{\mathbf{Z}})$. Although we again defer complete proofs to our fuller treatment of this subject, it does seem useful to sketch some of the main ideas.

Proof of the inequality "$\inf \mathcal{E} \leq \inf \mathbf{M}$". The proof here is again by construction. We will indicate the main ingredients in a special case. Suppose, say, $\mathcal{M} = \mathbf{R}^{m+n}$, B is polyhedral, and T is an integral n current which is mass minimizing subject to some appropriate constraints as in the Main Theorem above. We will construct a mapping $f: \mathbf{R}^{m+n} \to \mathbf{S}^m$ in the relevant homotopy class such that $\mathcal{E}_1(f), \mathcal{E}_2(f)$, and $\mathcal{E}_3(f)$ are nearly equal and are not much bigger that $\mathbf{M}(T)$. By virtue of the Strong Approximation Theorem for integral currents [FH1 4.2.20] we can modify T slightly to become simplicial with only a slight increase in mass.

Suppose then that we can express

$$T = \sum_{\alpha=1}^{M} \mathbf{t}(\Delta_\alpha^n, z_\alpha, \varsigma_\alpha)$$

as a 'simplicial' integral current (with the obvious interpretation). For each $k = 0, \ldots, n$ we denote by K_k the collection of closed k simplexes which occur as k dimensional faces of n simplexes among the Δ_α^n's. We then choose numbers $0 < \delta_n << \delta_{n-1} << \delta_{n-2} << \ldots << \delta_0 << 1$ and define sets $N_0, N_1, \ldots, N_n$ in $\mathbf{R}^{m+n}$ by setting

$$N_0 = \{x: \operatorname{dist}(x, \cup K_0) < \delta_0\}$$

and, for each $k = 1, \ldots, n$ set

$$N_k = \{x: \operatorname{dist}(x, \cup K_k) < \delta_k\} \sim (N_{k-1} \cup N_{k-2} \cup \ldots \cup N_0).$$

We assume that $\delta_0, \ldots, \delta_n$ have been chosen so that the distinct components of each N_k correspond to distinct k simplexes in K_k.

We now define mappings $f_{n+1}, f_n, \ldots, f_0 = f$ as follows.

First, the mapping $f_{n+1}: \mathbf{R}^{m+n} \sim (N_n \cup \ldots \cup N_0) \to \mathbf{S}^m$ is defined by setting $f_{n+1}(x) = \mathbf{q}$ for each x.

Second, the mapping $f_n: \mathbf{R}^{m+n} \sim (N_{n-1} \cup \ldots \cup N_0) \to \mathbf{S}^m$ is constructed geometrically in virtually the same manner as the mapping g in the example A.8 in the Appendix. Details are left to the reader.

Third, the mapping $f_{n-1}: \mathbf{R}^{m+n} \sim (N_{n-2} \cup \ldots \cup N_0) \to \mathbf{S}^m$ is constructed geometrically in a manner virtually identical with the construction of the mapping $f_{\delta,r}$ of example A.8 of the Appendix (with δ, r replaced by $\delta_n/2, \delta_{n-1}$ respectively there). The mapping f_{n-1} is Lipschitz across parts of $n-1$ simplexes which do not lie in B and is discontinuous on those $n-1$ simplexes which contain part of ∂T.

Assuming $f_{n+1}, f_n, \ldots, f_{k+1}$ have been constructed we define

$$f_k: \mathbf{R}^{m+n} \sim (N_{k-1} \cup \ldots \cup N_0) \to \mathbf{S}^m$$

as follows. Each point v in $N_k \sim (N_{k-1} \cup \ldots \cup N_0)$ can be written uniquely in the form $v = v_0 + (v - v_0)$ where v_0 is the unique closest point in $\cup K_k$ to v and $|v - v_0| < \delta_k$. If $v \neq v_0$ we note that

$$v_1 = v_0 + \delta_k \left(\frac{v - v_0}{|v - v_0|} \right) \in \text{dmn}(f_{k+1})$$

and we set $f_k(v) = f_{k+1}(v_1)$. A direct extension of the estimates used for the example A.8 of the Appendix shows that the energies $\mathcal{E}_1(f), \mathcal{E}_2(f)$, and $\mathcal{E}_3(f)$ very nearly equal $\mathbf{M}(T)$.

Proof of the inequality "$\inf \mathbf{M} \leq \inf \mathcal{E}$". The argument here is a direct extension of the corresponding argument given above and is left to the reader.

Remarks.

(1) One of the main reasons for analyzing relations between the energy of mappings and the area of currents is that it provides a way to study n dimensional area minimizing *integral* currents (whose geometry is not specified ahead of time) by studying functions and integrals over the given ambient manifold. This seems the first such scheme which works in general codimensions. For *real* currents, however, differential forms play a role roughly analogous to that of our function spaces $\mathcal{F}_0$; in this regard see, for example, the paper of H. Federer, *Real flat chains, cochains, and variational problems* [**F2** 4.10(4), 4.11(2)]. Incidentally, in the language of [**F2** 5.12, page 400], examples show that the equation in question there is not always true under the alternative hypotheses of [**F2** 5.10].

(2) Suppose C consists of smooth simple closed curves in $\mathbf{R}^3$ oriented by ς. Suppose also for positive integers ν we have reasonable mappings f_ν from the complement of C in $\mathbf{R}^3$ to the circle $\mathbf{S}^1$ with the property that small circles which link C once are mapped to $\mathbf{S}^1$ by f_ν with degree ν. Because of the dimensions we have

$$\mathcal{E}_1(f_\nu) = \mathcal{E}_2(f_\nu) = \mathcal{E}_3(f_\nu) = \left(\frac{1}{2\pi}\right) \int |Df_\nu| \, d\mathcal{L}^3.$$

If f_ν is nearly $\mathcal{E}_1$ energy minimizing then for most w's in $\mathbf{S}^1$ the slice

$$T_\nu(w) = \langle \mathbf{E}^3 \,, f_\nu \,, w \rangle \in \mathbf{I}_2(\mathbf{R}^3)$$

will be defined with $\partial T_\nu(w) = \mathbf{t}(C \,, \nu \,, \varsigma)$ and will be nearly mass minimizing. H. Parks, in his memoir, *Explicit determination of area minimizing hypersurfaces, II* [**PH**], used a similar energy for mappings to the real numbers $\mathbf{R}$ (instead of to $\mathbf{S}^1$) and was able to exhibit an algorithm for finding area minimizing surfaces. The technique used by Parks requires that C be extreme, i.e. that it lie on the boundary of its convex hull. The analysis of our paper on the other hand applies to any collection of curves which, for example, may be knotted or linked in any way. One of our hopes is to develop a method of computation analogous to that of Parks.

(3) Suppose that C and the mappings f_ν have the same meaning as in (2) above. If θ denotes the usual (multiple-valued radian) angle function on $\mathbf{S}^1$ then $d\theta$ as a well defined closed 1-form

whose pullbacks $f_\nu^\sharp d\theta$ give closed 1 forms on the complement of C in $\mathbf{R}^3$ with $|f_\nu^\sharp d\theta| = |Df_\nu|$. For fixed x_0 in the complement of C we define functions g_ν mapping the complement of C to $\mathbf{S}^1$ by requiring that

$$\theta \circ g_\nu(x) = \theta \circ f_\nu(x_0) + \int_\gamma f_\nu^\sharp d\theta \pmod{2\pi}$$

for each x (with the obvious meanings); here $\gamma(x)$ denotes any oriented path in the complement of C starting at x_0 and ending at x. It is immediate to check that $g_\nu = f_\nu$ for each ν. If we write $\nu = \lambda \cdot \mu$ for some λ and μ and define $h_\lambda(x)$ in $\mathbf{S}^1$ by requiring

$$\theta \circ h_\lambda(x) = \int_\gamma \left(\frac{1}{\mu}\right) f_\nu^\sharp d\theta \pmod{2\pi}$$

for γ as above. The mapping h_λ maps small circles with the same degrees as does f_λ. Taking $\mu = \nu$ we readily conclude, for example, that

$$\inf\{\mathbf{M}(T): \partial T = \mathbf{t}(C, \nu, \varsigma)\} = \nu \cdot \inf\{\mathbf{M}(T): \partial T = \mathbf{t}(C, 1, \varsigma)\}$$

for each ν. This estimate implies that integral and real mass minimizing 2 currents having boundary $\mathbf{t}(C, 1, \varsigma)$ have the same masses [**F2** 5.8]; although this has been known for some time, the present proof by factoring mappings seems new and simpler. This fact (and our proof) extend to $n-1$ dimensional boundaries in general manifolds $\mathcal{M}$ of dimension $n+1$ with, for example, the property that each 1 cycle is a boundary. There are counterexamples to such equalities in higher codimensions given first by L. C. Young [**YL**] and later by F. Morgan [**MF**] and B. White [**W1**]. How *badly* such an equality can fail remains an important open question. It is not even known, for example, if the number

$$\inf\{\mathbf{M}(S)/\mathbf{M}(T): S, T \in \mathbf{I}_2(\mathbf{R}^4, \mathbf{R}^4) \text{ are mass minimizing with } 0 \neq \partial S = 2\partial T\}$$

is positive; note, however, the isoperimetric inequality [**A1** 2.6].

(4) Suppose $\mathcal{M}$ is a complex submanifold of some complex projective space $\mathbf{CP}(n)$ (or, more generally, $\mathcal{M}$ is a Kähler manifold). Then any complex analytic (meromorphic) function f from $\mathcal{M}$ to the Riemann Sphere $\mathbf{CP}(1) = \mathbf{S}^2$ has integral current slices which are absolutely mass minimizing in their integral homology classes [**F1** 5.4.19]. Such f's are thus necessarily maximally conformal and minimize each of the energies $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ among functions in the same homotopy classes.

(5) In the context of this paper, if the mass minimizing current T being sought happens to be unique then most slices of nearly minimizing mappings will be close to that current. In a sense this describes the asymptotic behavior of a sequence $\{f_k\}_k$ of mappings in $\mathcal{F}_0$ converging towards energy minimization; in particular, the real currents

$$\left\{\left(\frac{1}{(m+1)\alpha(m+1)}\right) [\![\mathcal{M}]\!] \llcorner f_k^\sharp \sigma^*\right\}_k$$

must converge to T as $k \to \infty$. If $m = 2$ then the energy $\mathcal{E}_1$ is Dirichlet's integral which is widely studied in the general theory of harmonic mappings between manifolds pioneered by J. Eells and J. Sampson.

In any codimension m each n dimensional mass minimizing integral current is a *regular* minimal submanifold except possibly on a singular set of dimension not exceeding $n-2$ as shown by F. Almgren in [**A2**]. It is not yet clear to what extent the present new setup will provide new tools for study of the regularity and singularity properties of mass minimizing integral currents. This could be one of its most important potential uses.

APPENDIX

When not otherwise specified we follow the general terminology of pages 669-671 of H. Federer's treatise, *Geometric Measure Theory* [**F1**] or the newer standardized terminology of the 1984 AMS Summer Research Institute in Geometric Measure Theory and the Calculus of Variations as summarized in pages 124-130 of F. Almgren's paper, *Deformations and multiple-valued functions* [**A1**].

A.1 Terminology.

A.1.1 We fix positive integers m and n and suppose that $\mathcal{M}$ is an $m+n$ dimensional submanifold (without boundary) of $\mathbf{R}^N$ (some N) which is smooth, compact, and oriented by the continuous unit $(m+n)$-vectorfield $\xi\colon \mathbf{M} \to \wedge_{m+n}\mathbf{R}^N$; alternatively $\mathcal{M} = \mathbf{R}^{m+n}$ with standard orthonormal basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_{m+n}$ and orienting $(m+n)$-vector $\mathbf{e}_1 \wedge \dots \wedge \mathbf{e}_{m+n}$. We also suppose that B is a finite (possibly empty) union of various (curvilinear) $n-1$ simplexes $\Delta_1, \Delta_2, \dots, \Delta_J$ associated with some smooth triangulation of $\mathcal{M}$.

A.1.2 We denote by $\mathbf{S}^m$ the unit sphere in $\mathbf{R} \times \mathbf{R}^m = \mathbf{R}^{1+m}$ with its usual orientation given by the unit m-vectorfield $\sigma\colon \mathbf{S}^m \to \wedge_m \mathbf{R}^{1+m}$; in particular, for each $w \in \mathbf{S}^m \subset \mathbf{R}^{1+m} = \wedge_1 \mathbf{R}^{1+m}$, $\sigma(w) = *w$. It is convenient to let $z, y_1, \dots, y_m$ denote the usual orthonormal coordinates for $\mathbf{R} \times \mathbf{R}^m$ and also let $\mathbf{p}, \varepsilon_1, \dots, \varepsilon_m$ be the associated orthonormal basis vectors. In particular, $\sigma(\mathbf{p}) = *\mathbf{p} = \varepsilon_1 \wedge \dots \wedge \varepsilon_m$. We regard $\mathbf{p}$ as the 'north pole' of $\mathbf{S}^m$. The 'south pole' is $\mathbf{q} = -\mathbf{p}$. We denote by σ^* the differential m form (the 'volume form') on $\mathbf{S}^m$ dual to σ.

A.1.3 If L is a linear mapping $\mathbf{R}^{m+n} \to \mathbf{R}^m$ then the polar decomposition theorem guarantees the existence of orthonormal coordinates for $\mathbf{R}^{m+n}$ and $\mathbf{R}^m$ with respect to which L has the matrix representation

$$L = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix}$$

with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0$. In these coordinates we can express the Euclidean norm $|L|$ of L as

$$|L| = \left(\lambda_1^2 + \lambda_2^2 + \dots + \lambda_m^2\right)^{\frac{1}{2}},$$