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TRANSCENDENTAL & ALGEBRAIC NUMBERS

BY A. O. GELFOND

Translated from the First Russian Edition

LEO F. BORON

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This translation is dedicated to

Professor Dr. A. O. Gelfond

Philadelphia, 1959 L.F.B.

FOREWORD

IT WAS NOT until the twentieth century that the theory of transcendental numbers was formulated as a theory having its own special methods and a sufficient supply of solved problems. Isolated formulations of the problems of this theory existed long ago and the first of them, as far as we know, is due to Euler. The problem of approximating algebraic numbers by rational fractions or, more generally, by algebraic numbers may also be included in the theory of transcendental numbers, regardless of the fact that the study of approximations to algebraic numbers by rational fractions was stimulated by problems in the theory of Diophantine equations. The object of the present monograph is not only to point out the content of the modern theory of transcendental numbers and to discuss the fundamental methods of this theory, but also to give an idea of the historical course of development of its methods and of those connections which exist between this theory and other problems in number theory.

Since the proofs of the fundamental theorems in the theory of transcendental numbers are rather cumbersome and depend on a large number of auxiliary propositions, each such proof is prefaced by a brief discussion of its scheme, which, in our opinion, should facilitate the understanding of the basic ideas of the corresponding method.

The author's articles *Approximation of algebraic irrationalities and their logarithms* [11], *On the algebraic independence of transcendental numbers of certain classes* [15] are included in this monograph in their entirety, and use was made of the author's article *The approximation of algebraic numbers by algebraic numbers and the theory of transcendental numbers* [17].

Siegel's method is discussed in this monograph in the form given by Siegel in his book *Transcendental Numbers*, Princeton, 1949 [5].

Moscow A. O. GELFOND

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CHAPTER I

The Approximation of Algebraic Irrationalities

§1. Introduction

An *algebraic number* is a root of an algebraic equation with rational integral coefficients; in other words, it is any root of an equation of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \tag{1}$$

where all the numbers $a_0, a_1, \ldots, a_n$ are rational integers and $a_0 \neq 0$. A number which is not algebraic is said to be *transcendental*.

If equation (1) is irreducible, i.e. its left member is not the product of two polynomials with rational integral coefficients, then its degree will be the *degree of the algebraic number* α which satisfies it. A root of equation (1) in the case $a_0 = 1$ is called an *integral algebraic number* or an *algebraic integer*.

The reader can find the elementary arithmetic properties of algebraic numbers which are required for understanding the following material in any book on algebraic numbers, for example the books *Vorlesungen über die Theorie der algebraischen Zahlen* by Hecke [1] and *The Theory of Algebraic Numbers* by Pollard [1]. Here we shall be occupied only with the problem of the approximation of algebraic irrationalities and various applications of this theory.

All methods of proof of the transcendence of a number in either the explicit or implicit form depend on the fact that algebraic numbers cannot be very well approximated by rational fractions or, more generally, by algebraic numbers. Therefore, the approximation of algebraic numbers by algebraic numbers will be considered in this chapter. This problem, as will be shown, is closely related to the problem of solving algebraic and transcendental equations in integers, and to other problems in number theory. Analytic methods in transcendental number theory may be utilized, in turn, in integral solutions of equations, and in the sequel certainly in the solution of problems dealing with the approximation of algebraic irrationalities.

We note first of all that the existence of transcendental numbers may also be proved without knowledge of the nature of the approximation of algebraic numbers by algebraic numbers. In fact, since the coefficients of equation (1) can be rational integers only, there can be only a countable number of equations of type (1) with prescribed degree n. From this it follows that there exists only a countable set of algebraic numbers of degree n inasmuch as every equation of degree n has only n roots. Therefore, the set of all algebraic numbers is countable. But the set of all complex numbers (or real numbers) is not countable, from which it follows that the transcendental numbers form the major part of all complex and real numbers. Despite this fact, the proof of the transcendence of any concrete, prescribed numbers, for example π or $2^{\sqrt[3]{2}}$, is rather difficult.

The question of the arithmetic nature of an extensive class of numerical expressions was first formulated by Euler. In his book *Introductio in analysin infinitorum* [1], 1748, he makes the assertion that for rational base a the logarithm of any rational number b which is not a rational power of a cannot be an irrational number (in modern terminology, algebraic) and must be counted among the transcendentals. Besides this assertion, which was proved only recently, he also formulated other problems dealing directly with transcendental number theory. Almost a century after Euler, Liouville [2] was the first, in 1844, to give a necessary condition that a number be algebraic and, by the same token, a sufficient condition that the number be transcendental. He showed that if α is a real root of an irreducible equation of degree $\nu \geqslant 2$, and p, q are any rational integers, then the inequality

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^{\nu}}, \quad c > 0,$$

is satisfied, where the constant C does not depend on p and q.

The proof of this inequality is quite straightforward. Suppose α is a real root of an irreducible equation

$$f(x) = a_0 x^{\nu} + \cdots + a_{\nu} = 0,$$

where all the a_i ($i = 0, 1, \ldots, \nu$) are rational integers. Then, using the mean value theorem, we get

$$\left|f\left(\frac{p}{q}\right)\right| = \left|\alpha - \frac{p}{q}\right| |f'(\xi)| \geqslant \frac{1}{q^\nu}; \quad \xi = \alpha + \tau\left(\frac{p}{q} - \alpha\right), \quad |\tau| \leqslant 1,$$

from which the Liouville theorem follows directly. This criterion for the transcendence of a number permitted the first construction of examples of transcendental numbers. In fact, it follows from the Liouville transcendence criterion, for example, that the number

$$\eta = \sum_{n=1}^{\infty} \frac{1}{2^{n!}}$$

is transcendental.

Thus, Liouville established that algebraic numbers cannot be very well approximated by rational fractions. In connection with this fact, the problem arose of determining a constant $\vartheta = \vartheta(\nu)$ such that for an arbitrary algebraic number α of degree ν the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\vartheta+\varepsilon}}, \tag{2}$$

where p, q are integers, will have only a finite number of solutions when $\varepsilon > 0$ and an infinite number of solutions when $\varepsilon < 0$. We remark that the numbers α for which inequality (2) has an infinite number of solutions for arbitrary ϑ are called *Liouville numbers*. Thue [1] was the first, at the beginning of the present century, to be able to decrease the magnitude of this constant. He showed that $\vartheta \leqslant \nu/2 + 1$. To prove this proposition, Thue constructed a polynomial in two variables x and y with rational integral coefficients having the form

$$f(x, y) = (y - \alpha) f_1(x, y) + (x - \alpha)^m f_2(x, \alpha), \tag{3}$$

where $f_1(x, y)$ and $f_2(x, \alpha)$ are polynomials.

Assuming that inequality (2) has two solutions p_1/q_1 and p_2/q_2, with sufficiently large denominators q_1 and q_2, then setting $m \approx \dfrac{\ln q_2}{\ln q_1}$ in relation (3) and proving that the left member of (3) does not vanish for a suitable choice of $f(x, y)$ when $x = p_1/q_1$ and $y = p_2/q_2$, he obtained his assertion in a manner analogous to the way

Liouville's theorem was proved. This method, which enabled one to essentially decrease the Liouville constant, is inseparably related to the assumption that there exist two sufficiently large solutions of inequality (2). Therefore, this method enables one to establish only a bound for the number of solutions of inequality (2) and not for the magnitude of their denominators.

In fact, it follows from Thue's line of reasoning that if inequality (2) has a sufficiently large number of solutions for $\vartheta = \nu/2+1$ and $\varepsilon > 0$ with denominators $q_1 > q_1'(\alpha, \varepsilon)$, then there are no solutions with denominators $q_2 \geqslant q_2'(\alpha, \varepsilon, q_1)$. This at once enables one to establish, in particular, the finiteness of the number of solutions of the equation

$$(4) \qquad y^n f\left(\frac{x}{y}\right) = c_0 y^n + c_1 y^{n-1} x + \cdots + c_n x^n = c, \quad n \geqslant 3$$

in integers x and y if the coefficients $c, c_0, c_1, \ldots, c_n$ are rational integrals.

In fact, equation (4) implies the relations

$$(5) \qquad f\left(\frac{x}{y}\right) = \left(\alpha - \frac{x}{y}\right) f'(\xi) = \frac{c}{y^n},$$
$$\xi = \alpha + \tau\left(\frac{x}{y} - \alpha\right), \quad 0 \leqslant \tau \leqslant 1,$$

from which, under the condition that the polynomial $f(t)$ is irreducible in the rational field, it follows immediately that we have a contradiction with inequality (2) when $\vartheta + \varepsilon < n$, provided only that we assume the existence of an infinite number of solutions of equation (4).

This method was generalized and made precise by Siegel [1] who showed, using, as did Thue, the existence of two sufficiently large solutions, that the inequality

$$(6) \qquad \vartheta \leqslant \min_{1 \leqslant s \leqslant \nu - 1} \left[\frac{\nu}{s+1} + s\right] < 2\sqrt{\nu}$$

holds. Not only did Siegel make Thue's method more precise; he generalized it to the case of the approximation of an algebraic number α by another algebraic number ζ of height H and degree n. The height of an algebraic number ζ is the maximum of the absolute values of the coefficients of that equation, irreducible in the rational field, which is satisfied by ζ where all the coefficients of this equation are integers and their greatest common divisor equals 1. He showed that the inequality

$$(7) \qquad |\alpha-\zeta| < H^{-n(\vartheta+\varepsilon)}, \quad \vartheta = \min_{1\leqslant s\leqslant \nu-1}\left[\frac{\nu}{s+1}+s\right], \quad \varepsilon > 0$$

has only a finite number of solutions in algebraic numbers ζ if α is an algebraic number of degree ν.

Furthermore, he also gave other variants of inequality (7). Further attempts by Siegel [2] and his students to decrease the magnitude of the constant ϑ in inequalities (2) and (7), assuming the existence not only of two, but of an arbitrary number of sufficiently large solutions of inequalities (2) and (7), led Siegel to a theorem which was sharpened by his student Schneider [1] and which in the sharpened form reads as follows: If $q_1, q_2, \ldots, q_n, \ldots$ are the denominators of all sequences of solutions of inequality (2) for $\vartheta=2$ and $\varepsilon>0$, then either $\overline{\lim_{n\to\infty}}\dfrac{\ln q_{n+1}}{\ln q_n}=\infty$ or $n<n_0$. This so-called Siegel-Schneider theorem, as we see, not only does not make it possible to establish a bound for the magnitudes of the denominators of the solutions of inequality (2) for $2<\vartheta<\vartheta_0$, $\vartheta_0=\min\limits_{1\leqslant s\leqslant \nu}\left[\dfrac{\nu}{s+1}+s\right]$, but it also does not even assert their finiteness.

The last theorem given above generalizes naturally to the case of inequality (7). From the first generalization of the Thue theorem, based on the consideration of two sufficiently large solutions, it follows, in particular, that the equation

$$(8) \qquad c_0y^n+c_1y^{n-1}x+\cdots+c_nx^n = P_m(x,y), \quad n \geqslant 3,$$

for rational integers $c_0, c_1, \ldots, c_n$ and $P_m(x, y)$ a polynomial with rational integral coefficients of degree m, has only a finite number of solutions in rational integers x and y when

$$n-m > \min_{1\leqslant s\leqslant n-1}\left[\frac{n}{s+1}+s\right]$$

and the left member of the equation is irreducible. From the Siegel-Schneider theorem it only follows that for $n>m+2$ the integral solutions of equation (8) are very rare. We note that the question whether the number of solutions of equation (8) with $n\geqslant m+1$ is finite or infinite is answered completely by another means.

Further generalizations of the Siegel-Schneider theorem and its applications can be found in the works of Mahler [2–5]. One ought also to note that some results in the area of approximations of algebraic irrationalities were obtained by Morduhai-Boltovskoi [1, 4–6], Kuzmin [2], Gelfond [10, 11], and other authors.

Results, analogous to the Thue-Siegel theorem, dealing with the problem of the simultaneous approximation of several algebraic numbers by rational fractions with the same denominators were obtained by Hasse [1].

The most interesting direct application of theorems of Siegel-Schneider type in the theory of transcendental numbers is the following. Suppose $p(x)$ is an integral polynomial which is positive for $x \geqslant 1$. We write down its values for $x = 1, 2, 3, \ldots$ in the number system with radix q. We write the infinite q-nary fraction as

$$\eta = 0.q_1 q_2 \ldots q_{\nu_1} \ldots q_{\nu_2} \ldots,$$

where $q_1, q_2, \ldots, q_{\nu_1}$ are the "digits" in the q-nary expansion of $p(1)$, $q_{\nu_1+1}, \ldots, q_{\nu_2}$ are the "digits" in the q-nary expansion of $p(2)$, and so on. Then the number η will be transcendental but it is not a Liouville number. In particular, for $p(x) = x$ and $q = 10$, it will be the transcendental number

$$\eta = 0.123456789101112\ldots.$$

This theorem was proved by Mahler [5] with the aid of the theorem on the approximation of algebraic irrationalities by rational fractions, which was a sharpening of Schneider's theorem for the case when the numerators and denominators of the approximating fractions are of a special sort. It also follows from this theorem that the numbers

$$\eta = \sum_{0}^{\infty} \frac{a_k}{a^{\lambda_k}}, \quad \lambda_{k+1} > (1+\varepsilon)\lambda_k + \frac{\ln a_{k+1}}{\ln a}, \quad \varepsilon > 0,$$

where $a > 1$, $a_0, a_1, \ldots, \lambda_1, \lambda_2, \ldots$ are positive rational integers, are transcendental. In particular, this assertion holds for the number

$$\eta = \sum_{0}^{\infty} \frac{1}{a^{[2^{\alpha n}]}}, \quad \alpha > 0.$$

In connection with the status of the problem of the approximation of algebraic irrationalities which was briefly discussed above, the first question that naturally arises is whether it is possible to decrease the magnitude of the constant ϑ in comparison with the quantity obtained by Siegel using only two solutions of inequality (2). Further, taking into consideration the noneffectiveness of the results, obtained by Thue's method, noneffectiveness in the sense that it is impossible to establish by this method the bounds of the magnitudes of the denominators of the solutions of inequality (2) for $\vartheta<\nu$, the problem how the theorem on the approximation of algebraic numbers which would be a limiting case in the sense of effectiveness, using two solutions of inequality (2), should be worded, also arises naturally. In this formulation of the problem, one must speak of only two solutions inasmuch as by using a larger number of solutions one encounters difficulties which have not been eliminated up to the present time and which are related to the general theory of elimination.

We shall now formulate the theorem, which will give the answer to the above question, by introducing, in anticipation, the concept of *measure of an algebraic number*. Suppose ζ is a number in an algebraic field K of degree σ, and let the numbers $\omega_1, \omega_2, \ldots, \omega_\sigma$ be a basis for the ring of integers in this field. The number ζ we have taken can be represented in an infinite number of ways in the form

$$(9)\quad \zeta=\frac{p_1\omega_1+\cdots+p_\sigma\omega_\sigma}{q_1\omega_1+\cdots+q_\sigma\omega_\sigma},\quad q[p_1,\ldots,q_\sigma]=\max[|p_1|,\ldots,|q_\sigma|],$$

where $p_1, p_2, \ldots, p_\sigma, q_1, \ldots, q_\sigma$ are rational integers with greatest common divisor 1. We shall call the number q the measure of the number ζ if it is defined by the relation

$$(10)\qquad q=\min q[p_1,\ldots,p_\sigma,q_1,\ldots,q_\sigma]$$

where the minimum in the right member is taken over all possible representations of the number ζ. It is not difficult to note that when $\zeta=p/q$ is a rational number then its measure equals $\max[|p|,|q|]$, i.e. to within a nonessential constant factor, it coincides with its denominator q if ζ is an element of the sequence of fractions which converges to the number $\alpha\neq 0, 1$ as q increases. We can now formulate our general theorem, which we shall call Theorem I in the sequel. Suppose α and β are two arbitrary numbers in the algebraic field K_0 of degree ν (where the case $\alpha=\beta$ is not excluded). Suppose, further, that ζ and ζ_1 are numbers in an algebraic field K, whose measures with respect to a fixed integral basis $\omega_1, \omega_2, \ldots, \omega_\sigma$ of this field are q and q_1, respectively, and that ϑ and ϑ_1 are two real numbers subjected to the conditions $\vartheta \leqslant \vartheta_1 \leqslant \nu$, $\vartheta\vartheta_1 = 2\nu(1+\varepsilon)$, where $\varepsilon > 0$ is an arbitrarily small, fixed number. Then, if the inequality

$$(11) \qquad |\alpha-\zeta| < q^{-\sigma\vartheta}$$

has the solution ζ with measure $q > q'[K_0, K, \alpha, \beta, \varepsilon, \delta]$, the inequality

$$(12) \qquad |\beta-\zeta_1| < q_1^{-\sigma\vartheta_1}$$

cannot have solutions with measure q_1 under the condition that

$$(13) \qquad \ln q_1 \geqslant \left[\frac{\vartheta-1}{2(\sqrt{1+\varepsilon}-1)}+\delta\right] \ln q,$$

where δ is any arbitrarily small positive constant. [The special case of this theorem, when $\alpha=\beta$, ζ a rational fraction, $\vartheta=\vartheta_1$ and without inequality (13), was proved independently by Dyson.]

The $\wp$-adic analogue of the above theorem is formulated in a similar manner. It also follows from the above theorem, setting $\vartheta=\vartheta_1$ in it, that inequality (2) has only a finite number of solutions for $\varepsilon > 0$, when $\vartheta=\sqrt{2\nu}$. That our general theorem is the best possible from the point of view of effectiveness can be directly established in the case when ζ and ζ_1 are rational fractions and $\alpha=\beta$. In fact, if one could replace $\varepsilon > 0$ by $-\varepsilon < 0$ in the condition $\vartheta\vartheta_1 = 2\nu(1+\varepsilon)$ of the theorem, then it would have the form $\vartheta\vartheta_1 = 2\nu(1-\varepsilon)$ and we could have set $\vartheta = 2\sqrt{1-\varepsilon} < 2$ and $\vartheta_1 = \nu\sqrt{1-\varepsilon} < \nu$. But inequality (11) would indeed have an infinite number of solutions for ζ rational, $\sigma=1$ and $\vartheta < 2$, which means that for solutions of inequality (12) with rational denominators, we should find an effective bound in the form of a function of K_0, α, ε. It would already follow directly from this that there exists an effective bound for the magnitudes of the solutions of equation (4). Finally, one can say that our general theorem retains its validity if the measures of the numbers ζ are replaced by the heights of the numbers ζ.

The proof of this theorem is based on a somewhat stronger form of Thue's theorem. Using our general Theorem I, with the aid of some additional considerations, one can prove Theorem II: Suppose $\alpha, \zeta_1, \zeta_2, \ldots, \zeta_s$ are algebraic numbers in the field K.

Suppose also that the product of any integral powers of the numbers $\zeta_1, \zeta_2, \ldots, \zeta_s$ cannot be equal to 1. Then the inequality

$$(14) \qquad |\alpha - \zeta_1^{x_1}\zeta_2^{x_2} \ldots \zeta_s^{x_s}| < e^{-\varepsilon x}, \quad x = \max_{1 \leqslant i \leqslant s} |x_i|$$

and the congruence

$$(15) \qquad \alpha \equiv \zeta_1^{y_1}\zeta_2^{y_2} \ldots \zeta_s^{y_s} \bmod \wp^m, \quad m = [\delta y], y = \max_{1 \leqslant i \leqslant s} |y_i|$$

can have only a finite number of solutions in rational integers $x_1, x_2, \ldots, x_s$ and $y_1, y_2, \ldots, y_s$ provided the numbers $\varepsilon > 0$ and $\delta > 0$ are small; $\wp$ is a prime ideal in the field K. We now state two corollaries to Theorems I and II. We first of all introduce an application of Theorem I to the theory of algebraic equations. Suppose the system of homogeneous forms $P_1(x, y), P_2(x, y), \ldots, P_n(x, y)$ possess the following properties: their degrees are greater than one, all the coefficients of the polynomials $P_1(x, y), \ldots, P_n(x, y)$ are rational integers, these polynomials do not have linear divisors in the rational field, every real zero of the polynomial $t^{-m_k}P_k(t, tx) = R_k(x)$ belongs to the algebraic field K of degree not greater than ν and all such zeros are distinct. We shall also say that the degree of the polynomial $P(x_1, y_1, \ldots, x_n, y_n)$ in $2n$ variables, having rational integral coefficients, is the set of numbers $(s_1, s_2, \ldots, s_n)$, where s_i is the degree of the polynomial P in the variables x_i, y_i. Then the following theorem holds: The equation

$$(16) \qquad P_1(x_1, y_1) \ldots P_n(x_n, y_n) = P(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$$

has only a finite number of solutions in rational integers $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$, provided the inequalities $m_k - s_k > \sqrt{2\nu}$, $k = 1, 2, \ldots, n$; $\nu \geqslant 3$, are satisfied simultaneously.

One may also obtain a number of corollaries to Theorem II, but these are already for exponential functions. Suppose, for example, that the numbers $\zeta_1, \ldots, \zeta_n, \psi_1, \ldots, \psi_m, \eta_1, \ldots, \eta_p$ are integers in the field K; none is an algebraic unit; A, B, C, $ABC \neq 0$, are numbers in the same field K; and the numbers

$$\zeta = \zeta_1 \ldots \zeta_n, \quad \psi = \psi_1 \ldots \psi_m, \quad \eta = \eta_1 \ldots \eta_p$$

are relatively prime. Then the equation

$$(17) \qquad A\zeta_1^{x_1} \ldots \zeta_n^{x_n} + B\psi_1^{y_1} \ldots \psi_m^{y_m} + C\eta_1^{z_1} \ldots \eta_p^{z_p} = 0$$

can have only a finite number of nonnegative solutions in rational integers $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$. Further, almost directly from Theorem II, one can obtain a theorem on the