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Preface

Most of the lectures in this book were first given as theoretical seminars at Cornell University during 1961-1962, and were then augmented and brought into final form for the Summer School in Theoretical Physics held at Bangalore in June 1962. Some more recent developments are discussed in two Addenda.

Regge poles enter in the latter half of the lectures after the analogies and conceptual difficulties that led to their introduction into relativistic physics have been explained. It is anticipated that some readers will be interested exclusively in the simpler aspects of Regge poles, however. Readers in this category are advised to concentrate upon the following sections of the book: the treatment of ordinary quantum mechanics in Chapters X and XI [through Equation (11-11)], and the connection of Regge poles with relativistic scattering in Chapters XIII (first two paragraphs), XIV, XV, and XVI.

The material in Chapter II is on a similar level and is intended to give some physical feeling for S-matrix theory in its simpler manifestations. The chapters on the Mandelstam representation are more weighty and will be of interest primarily to the theoretical student who wishes to work in this field.

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I. INTRODUCTION

A considerable number of strongly interacting particles is now known. For experimental reasons, the particles stable under strong interactions tended to be discovered first, and were followed by the pion-nucleon resonances and, recently, a large number of other resonances. The situation now bears some resemblance to nuclear physics: for each set of quantum numbers there appears to be a "ground level" and various unstable states with higher mass.

Early attempts to cope with these particles theoretically often followed the line of attack that had proved so successful in quantum electrodynamics. A simple Lagrangian was chosen, with renormalized couplings and masses given, and the other physical observables were calculated by a perturbation expansion. But this approach failed to give good predictions because the coupling was strong and the perturbation expansion converged slowly. In fact, when some of the particles are resonances or bound states as suggested by analogy with nuclear physics, the perturbation expansion will not converge at all. Therefore, a modified approach is needed. The S-matrix methods described in these lectures represent a modified approach which works even where perturbation expansions fail.

Since the S-matrix represents a meeting ground between theory and experiment, most of the S-matrix techniques we shall describe can be used whether or not one believes in some particular underlying structure, such as Lagrangian field theory. But, in addition to its use as a tool in evaluating the consequences of various theories, the S-matrix appears to provide hints on some fundamental questions:

i) Is there any essential difference between stable and unstable particles, other than the presence of states to decay into? In nuclear physics, the answer is no, whereas in some early treatments of Lagrangian field theory, a distinction appeared to arise because one did not know how to introduce unstable particles into the Lagrangian. In S-matrix theory, the distinction does not appear. (This is also true in recent treatments of field theory.)

ii) Can masses and coupling constants be calculated? If a particle appears only when the forces become strongly attractive, then its mass and couplings are calculable. We call such a particle a <u>bound state</u> if it is stable, a <u>resonance</u> if it is unstable, and <u>composite</u> in either case. It is also possible to introduce particles into the S-matrix, which are present independently of the strength or sign of the forces. The masses and couplings of these particles cannot be calculated -- just as masses and couplings inserted into a Lagrangian are arbitrary -- and we choose to call such particles <u>elementary</u> since we cannot explain them. Both composite and elementary particles may be either stable or unstable. The possibility of finding experimental distinctions between them is of great interest and may be provided by Regge poles as we shall see.

iii) How many arbitrary masses and coupling constants are present in strong interactions? There is no way to deduce this at present. But a study of the S-matrix as a function of energy, momentum transfer, angular momentum, etc., produces a suggestive fact -- the analytic structure of the S-matrix as a function of these variables becomes simpler as the number of independent parameters is reduced. This observation has led to the hypothesis of maximal analyticity -- the analytic structure of the S-matrix is as simple as possible.[1] If this hypothesis is correct, there are no arbitrary constants in strong interaction physics except for c, $\hbar$, and one mass, and all strongly interacting particles are composite.

The hypothesis of maximal analyticity raises many questions. It has no evident connection to previous starting points such as Lagrangian theory; it seems to call for a new axiomatic framework expressed directly in terms of the S-matrix. Stapp[2] has proposed such a framework, and the reader is referred to his lectures[2] for a detailed account. Then there is the practical question of how to make calculations when nothing is "given". For this purpose, "bootstrap calculations" based on self-consistency requirements have been devised.[3] Another question concerns the uniqueness of the solution. The actual strong interactions possess the property of maximal strength,[4,5] in the sense that high-energy total cross sections approach a constant geometrical limit. We shall take this property from experiment; it is not known whether it follows uniquely from maximal analyticity. Likewise, we shall take the usual conservation laws of isotopic spin, strangeness, electric charge, and so forth, from experiment, without knowing whether some of them can ultimately be derived.

Before plunging into details of analytic structure, let us devote a few words to the meaning of singularities. Why must there be poles and cuts in the S-matrix even when "maximal analyticity" is assumed? Above threshold at kinetic energies $\mathrm{Re}\ q^2/2m + i\epsilon$, an outgoing solution of the Schrodinger partial-wave equation acquires a phase $2\delta_\ell$ at large distances. The S-matrix in this case is defined as

$$S(q,\ell) = e^{2i\delta_\ell(q)} \quad . \tag{1-1}$$

The time-reversed solution at $\mathrm{Re}\ q^2/2m - i\epsilon$ has a reversed phase factor $\exp(-2i\delta_\ell)$, resulting in a discontinuity of S. Physically, the discontinuity arises because we are comparing two solutions related by a discontinuous transformation (time reversal).

Below threshold, q^2 becomes negative and q becomes purely imaginary. The asymptotic wave function, which had the form

$$u(q) \sim e^{-iqr} - e^{-i\pi\ell}\, S(q,\ell)\, e^{+iqr} \tag{1-2}$$

above threshold, becomes

$$u(iq_I) \sim e^{q_I r} - e^{-i\pi\ell}\, S(iq_I, \ell)\, e^{-q_I r} \quad . \tag{1-3}$$

Usually, the wave function cannot be normalized. But at a bound state, only the converging exponential is present, and this requires $S = \infty$ (normally provided by a pole) at $q_I > 0$ and $S = 0$ at $q_I < 0$. For a given bound state, both the pole and the zero occur since the Schrödinger equation is invariant under $q \to -q$.

II. EFFECTIVE RANGE THEORY OF S-WAVE SCATTERING, AND THE N/D METHOD

The effective range formula for S-wave scattering contains a good deal of low-energy physics. At the same time it is very simple, and its properties in the complex energy plane can be followed explicitly.[6] To get a physical feeling for the complex energy plane we shall consider the effective range formula in detail, relating the analyticity properties to physical properties at each step.

The S-wave elastic scattering amplitude

$$f_o = \frac{e^{i\delta_o} \sin \delta_o}{q} \tag{2-1}$$

can be rewritten:

$$f_o = \frac{\sin \delta_o}{(\cos \delta_o - i \sin \delta_o)q} = \frac{1}{q \cot \delta_o - i\, q} \quad . \tag{2-2}$$

The effective range approximation is given by

$$q \cot \delta_o = -\frac{1}{A} + \frac{R}{2}\, q^2 \quad , \tag{2-3}$$

where A is called the scattering length and R the effective range. So we have

$$f_o = \frac{1}{-\frac{1}{A} + \frac{R}{2} q^2 - i\, q} \quad . \tag{2-4}$$

This approximation is valid near threshold for short-range potentials.

Since the denominator of (2-4) is quadratic in q, f_o evidently has two poles in q. As a function of q^2, it also has a cut which can be taken along the real axis from $q^2 = 0$ to $q^2 = +\infty$. The reason for this cut at positive q^2 was already touched upon in Chapter I. Under some circumstances, one of the poles represents a bound state, a possibility that was also mentioned in Chapter I. The other pole, however, has to do with the potential, and in order to see how this comes about we must mention briefly analyticity properties one finds in a more complete treatment of f_o. We shall show that (2-4) represents a simple approximation to these analyticity properties. After studying the simple approximation in detail we shall return, at the end of this chapter and in later chapters, to a more careful consideration of the full analyticity properties of f_o.

The analyticity properties we are interested in have to do with a particle of mass M, scattering from the Yukawa potential:

$$V(r) = -\frac{\rho}{2M^2}\frac{e^{-mr}}{r} \quad . \tag{2-5}$$

This potential has a reasonable behavior at large distances and will generalize easily to relativistic scattering later on. The Yukawa potential can be Fourier-transformed to momentum space, where it gives the Born approximation f_B to the scattering amplitude:

$$f_B = \frac{\rho}{\pi}\frac{1}{\left[m^2 + 2q^2 - 2q^2\cos\theta\right]} \quad . \tag{2-6}$$

The S-wave in Born approximation, f_{oB}, can be obtained from (2-6) by the partial-wave projection

$$f_{oB} = \frac{1}{2}\int_{-1}^{1} d\cos\theta\, f_B\, P_o(\cos\theta) = \frac{\rho}{4\pi q^2}\,\ln\left(1 + \frac{4q^2}{m^2}\right) \quad . \tag{2-7}$$

Thus the Yukawa potential gives a cut in the kinetic energy variable $q^2/2M$, starting at $q^2 = -m^2/4$. The magnitude of the discontinuity increases with the strength of the potential, and the discontinuity comes closer to the physical threshold $q^2 = 0$ as the range of the potential ($1/m$) is increased. This discontinuity together with corrections, to be discussed later, from iterations of the potential, is called the "left cut" (Fig. 2-1). In Born approximation, the left cut is the only discontinuity but in higher orders a "right cut" at the physical kinetic energies $q^2/2M = 0$ to ∞ also appears due to the opposite phases at $q^2 + i\epsilon$ and $q^2 - i\epsilon$, as explained in the Introduction.

Now the simplest approximation to the left cut is to replace it by a single pole $\lambda/(q^2 + a^2)$ where, crudely speaking, λ represents the strength of the potential and $1/a$ the range of the potential. There remains the problem of finding the right cut by iteration of the potential. For this purpose, it is convenient to write the amplitude as a quotient[7)]

$$f_o = \frac{N}{D} \quad , \tag{2-9}$$

where

$$N = \frac{\lambda}{q^2 + a^2} \tag{2-10}$$

contains the approximate left cut and D contains all of the right cut with no other singularities. D can therefore be represented by a Cauchy integral

$$D = \frac{1}{\pi} \int_0^\infty \frac{dq'^2}{q'^2 - q^2} \operatorname{Im} D(q'^2) \quad , \tag{2-11}$$

where we have used the fact that the discontinuity is purely imaginary, and have assumed that the integral converges. An advantage of writing the amplitude as a quotient in this way can be seen by combining

$$\text{Im}\ D = N\ \text{Im}\ 1/f_o \qquad (q^2 \geq 0) \tag{2-12}$$

with the "unitarity condition" that tells us the phase δ is real for elastic scattering:

$$\text{Im}\ 1/f_o = \text{Im}\left[q \cot \delta - i\,q\right] = -\,q \qquad (q^2 \geq 0) \tag{2-13}$$

to give

$$\text{Im}\ D = -\,q\,N \qquad (q^2 \geq 0)\quad . \tag{2-14}$$

We now use (2-14) in the Cauchy integral (2-11) and make the normalization $D(q^2 = -\,a^2) = 1$ by a subtraction:

$$D = 1 - \frac{(q^2 + a^2)}{\pi} \int_0^\infty \frac{dq'^2\,N(q'^2)\,q'}{(q'^2 + a^2)(q'^2 - q^2)} \quad . \tag{2-15}$$

Explicit integration using the one-pole approximation to N (2-10) does converge and yields

$$D = 1 + \frac{\lambda}{2a} - \frac{\lambda}{a - i\,q} \quad , \tag{2-16}$$

$$f_o = \frac{N}{D} = \frac{1}{a^2\,(\frac{1}{\lambda} - \frac{1}{2a}) + q^2\,(\frac{1}{\lambda} + \frac{1}{2a}) - i\,q} \quad . \tag{2-17}$$

The relation of the effective range approximation (2-4) to the full analytic structure of f_o is now clear. It has the same structure as the one-pole approximation to the left cut (2-17):

i) Both have two poles in q, as well as a cut in q^2. One of the poles has now been identified with the potential term.

ii) Both contain two parameters.

iii) Both satisfy unitarity.

iv) Both are valid near threshold for a short-range potential. The reason the one-pole approximation is valid over only a limited energy range is that the left cut can look similar to a pole only over a limited region. If the force became longer range, the pole would have to come closer to threshold and the approximation would retain its validity in a more and more limited region.

For a given range of forces, one of the two poles in q is fixed at $q = i\,a$. It turns out that the other pole moves as the strength of the potential varies for a given range $1/a$. The positioning of poles in the q plane is illustrated in Fig. 2-2. Meanwhile, the q^2 plane exhibits a slightly more complicated, two-sheeted structure. A value of q,

$$q = |q|\ e^{i\theta} \quad , \tag{2-18}$$

corresponds to

$$q^2 = |q|^2\ e^{2i\theta} \quad , \tag{2-19}$$

so a rotation through $\theta = 2\pi$ in q corresponds to a rotation through 4π in q^2. The region $0 < \theta < \pi$ ($\mathrm{Im}\ q > 0$) transforms to the q^2 sheet pictured in Fig. 2-2, while $\mathrm{Im}\ q < 0$ goes onto a second sheet.

The first sheet is called the physical sheet because:

i) physical $q = |q| + i\epsilon$ is on it at ①.

ii) when the pole that moves as the potential strength varies appears on the first sheet, it has the significance of a physical bound state as described in the Introduction. Evidently, this is the sheet we were working on when we used the N/D method.

Now let us study the moving pole. The fixed pole was a pole of N (2-10); the moving pole is a zero of D (2-16). It is located at

$$D = 1 + \frac{\lambda}{2a} - \frac{\lambda}{a - i\,q} = 0 \quad , \tag{2-20}$$

which has the solution

$$q = i\,a\left[\frac{\lambda - 2a}{\lambda + 2a}\right] \quad . \tag{2-21}$$

So, as we vary the strength of interaction at fixed a, the pole moves as indicated in Fig. 2-2.

Consider the case $|\lambda| < 2a$. Here the pole lies on the second q^2 sheet. The amplitude can be rewritten

$$f_o = \frac{a^2\,(\frac{1}{\lambda} - \frac{1}{2a}) + q^2\,(\frac{1}{\lambda} + \frac{1}{2a}) + i\,q}{\left[a^2\,(\frac{1}{\lambda} - \frac{1}{2a}) + q^2\,(\frac{1}{\lambda} + \frac{1}{2a})\right]^2 + q^2}$$

$$= \frac{\cos\delta\,\sin\delta + i\,\sin^2\delta}{q} \quad . \tag{2-22}$$