Linear Programming and Economic Analysis

ROBERT DORFMAN
Professor of Economics
Harvard University

PAUL A. SAMUELSON
Professor of Economics
Massachusetts Institute of Technology

ROBERT M. SOLOW
Associate Professor of Statistics
Massachusetts Institute of Technology

McGRAW-HILL BOOK COMPANY, INC.
New York Toronto London
1958

Foreword

Is linear programming something new or just a new name for old methods? Does it help in analyzing economic and business problems? Can it solve practical problems? The RAND Corporation, a private research corporation, whose primary contract is with the United States Air Force, has found linear programming expedient in practical problems and fruitful in analytic procedure. In part this is simply the result of the fact that much of standard economic analysis is linear programming.

In 1951, RAND and the Cowles Foundation for Research in Economics jointly sponsored *Activity Analysis of Production and Allocation,* a book that dealt with the mathematical and computational features of linear programming. Now RAND presents a book emphasizing the economic interpretation of linear programming.

While this book is intended not as a text but as a general exposition of the relationship of linear programming to standard economic analysis, it has been successfully used for graduate classes in economics. It is hoped that it will satisfy the curiosity of all economists, from first-year graduate students learning the "old" classical principles to their teachers who want to know what is "new."

Preface

Linear programming has been one of the most important postwar developments in economic theory. Its growth has been particularly rapid, thanks to the joint efforts of mathematicians, business and defense administrators, statisticians, and economists. Yet the economist who wants to learn how linear programming is related to traditional economic theory can nowhere find a comprehensive treatment of its many facets. The present book hopes to give the economist, who knows existing economic theory but who does not pretend to be an accomplished mathematician, a broad introduction to the theory of linear programming, or, as it is sometimes called, activity analysis. It hopes also to be useful to the practitioner of managerial economics, and possibly to provide the growing body of mathematicians interested in programming problems with insights into the vast body of modern economic theory.

When asked by The RAND Corporation to undertake the book, we agreed to avoid higher mathematics. We planned to stress the economic aspects of the problem, paying attention to practical problems of computation and giving important concrete applications but laying no stress on them. So vast has the theory become that we have had to be selective, reluctantly deciding to omit many interesting topics and applications. Thus, we have not dealt with the important role of linear-programming concepts in statistical decision theory.

On the other hand, we have gone into the extensive interrelations between the celebrated von Neumann theory of games and linear programming, particularly since every economist will want to know the interrelations between game theory and traditional economic theories of duopoly and bilateral monopoly. And modern economists will be interested in the interrelations between linear programming and modern welfare economics and the insights that linear programming gives into the determinateness of Walrasian equilibrium—as perfected by the recent works of K. Arrow, G. Debreu, L. W. McKenzie, and others.

This book can also serve as an expository introduction to the student interested in the Leontief theory of input-output, which has played so important a role in the last twenty years. Similarly, we have treated extensively problems of dynamic linear programming, not only because of their intrinsic interest but also because of their vital connections with the economist's theory of capital—that most difficult field of modern economic theory. Had we more space and time at our disposal we might have added some material summarizing the related "dynamic programming" methods of Richard Bellman, also developed at RAND. This new theory is of considerable interest to economists but mathematically more difficult than what we have attempted here. Fortunately, Bellman has just published a full exposition of his own.

Our task took more time than we had expected, primarily because we found ourselves in somewhat the same situation as the friend of Dr. Samuel Johnson who explained that he had hoped to become a philosopher but "cheerfulness kept breaking in." Our task of quickly providing an explanation has been frustrated because originality kept breaking in—as gaps were discovered in the existing theory or as whole new fields for analysis suggested themselves. The RAND Corporation has been extraordinarily patient in putting up with our explorations and extraordinarily generous in providing interested scholars with our research memoranda for a period of more years than we dare recall. However, in a field characterized by such intimate cooperation among numerous individuals from diverse disciplines, there is no need to stake out claims for new results. And needless to say, the book is the joint work of the three authors, with each taking responsibility for all.

Our acknowledgments can be brief, since footnotes within the text and a selective annotated bibliography at the end will relate our work to the literature. Yet we cannot fail to mention the names of George B. Dantzig, A. W. Tucker, H. W. Kuhn, David Gale, Tjalling C. Koopmans, A. Charnes, A. Wald, and John von Neumann, who laid the foundations of the theory of linear programming.

And within The RAND Corporation itself we must give our thanks to many people. First, to Professor Armen Alchian of UCLA whose many suggestions in theoretical interpretation have improved the work. Second, to Charles J. Hitch, the head of the RAND Economics Division, who had the original idea for such a work. Third, to Joseph A. Kershaw. Fourth, to Melvin Dresher, Reuben Kessel, and Russell Nichols, who read and improved parts of the manuscript. Finally, to a number of others at RAND for countless favors over a long period of time.

We alone take responsibility for all flaws, but we dare to hope that this group operation may be a minor exception to the view of those who, forgetting that the King James Bible was the work of a committee, categorically deny value to any work not produced by a single, isolated individual.

Robert Dorfman
Paul A. Samuelson
Robert M. Solow

Linear Programming and Economic Analysis

Contents

1

Introduction

1-1. HISTORICAL SKETCH

At any time, an economy has at its disposal given quantities of various factors of production and a number of tasks to which those factors can be devoted. These factors of production can be allocated to the different tasks, generally, in a large number of different ways, and the results will vary. There is no more frequent problem in economic analysis than the inquiry into the characteristics of the "best" allocation in situations of this kind.

We have just outlined a rudimentary problem in welfare economics or in the theory of production. It is also a problem in *linear* economics, the word "linear" being introduced to call attention to the fact that the basic restrictions in the problem take the form of the simplest of all mathematical functions. In this case the restrictions state that the total amount of any factor devoted to all tasks must not exceed the total amount available; mathematically each restriction is a simple sum.

This illustration suggests that many familiar problems in economics fall within the scope of linear economics. Like Molière's M. Jourdain and his prose, economists have been doing linear economics for more than forty years without being conscious of it. Why, then, a book on the subject at this date? Because until recently economists have passed over the linear aspects of their problems as obvious, trivial, and uninteresting. But in the last decade the stone which the builders rejected has become the headstone of the corner. New methods of analysis have been developed that depend heavily on the linear characteristics of economic problems and, indeed, accentuate them. The most flourishing of these methods are linear programming, input-output analysis, and game theory.

These three branches of linear economics originated separately and only gradually grew together. The first to be developed was game theory, the central theorem of which was announced by John von Neumann[1] in 1928. The main impact of game theory on economics was delayed, however,

[1] "Zur Theorie der Gesellschaftsspiele," *Mathematische Annalen*, **100**:295–320 (1928).

until the publication of *Theory of Games and Economic Behavior*[1] in 1944. Briefly stated, the theory of games rests on the notion that there is a close analogy between parlor games of skill, on the one hand, and conflict situations in economic, political, and military life, on the other. In any of these situations there are a number of participants with incompatible objectives, and the extent to which each participant attains his objective depends upon what all the participants do. The problem faced by each participant is to lay his plans so as to take account of the actions of his opponents, each of whom, of course, is laying his own plans so as to take account of the first participant's actions. Thus each participant must surmise what each of his opponents will expect him to do and how these opponents will react to these expectations.

It was von Neumann's remarkable achievement to demonstrate that something definite can be said about such a welter of cross-purposes and psychological interactions. He showed that under certain assumptions, which we shall have to examine, each participant can act so as to be guaranteed at least a certain minimum gain (or maximum loss). When each participant acts so as to secure his minimum guaranteed return, then he prevents his opponents from attaining any more than their minimum guaranteeable gains. Thus the minimum gains become the actual gains, and the actions and returns for all participants are determinate.

The implications of this theory for economics are evident. It holds out the hope of banishing oligopolistic indeterminacy from economic situations in which von Neumann's assumptions are satisfied. The military implications are also evident. And, it turns out, there are important implications for statistical theory as well. Since 1944 the development of these three fields of application of game theory has gone forward actively.

Input-output analysis was the second of the three branches of linear economics to appear. Leontief published the first clear statement of the method in 1936[2] and a full exposition in 1941.[3] Input-output analysis is based on the idea that a very considerable proportion of the effort of a modern economy is devoted to the production of intermediate goods, and the output of intermediate goods is closely linked to the output of final products. A change in the output of any final product (say automobiles) implies changes in the outputs of the intermediate goods (copper, glass, steel, etc., including automobiles) used in producing that final product

[1] John von Neumann and Oskar Morgenstern, Princeton University Press, Princeton, N.J., 1944. Third edition, 1953.

[2] W. W. Leontief, "Quantitative Input and Output Relations in the Economic System of the United States," *Review of Economic Statistics*, **18**:105–125 (August, 1936).

[3] W. W. Leontief, *The Structure of American Economy, 1919–1929*, Harvard University Press, Cambridge, Mass., 1941. Second edition, Oxford University Press, New York, 1951.

and, indeed, in producing goods used in producing that final product, and so on.

In its original version, input-output analysis dealt with an entirely closed economic system—one in which all goods were intermediate goods, consumables being regarded as the intermediate goods needed in the production of personal services. Equilibrium in such a system exists when the outputs of the various products are in balance in the sense that just enough of each is produced to meet the input requirements of all the others. The specification of this balance and its pricing implications was Leontief's first objective.

Beginning with World War II, interest has shifted to a different view of Leontief's model. In this view final demand is regarded as being exogenously determined, and input-output analysis is used to find levels of activity in the various sectors of the economy consistent with the specified pattern of final demand. For example, Cornfield, Evans, and Hoffenberg have computed employment levels in the various sectors and, hence, total employment consequent upon a presumed pattern of final demand,[1] and Leontief has estimated the extent to which fluctuations in foreign trade influenced activity in various domestic sectors.[2] The input-output model, obviously, lends itself well to mobilization planning and planning for economic development.[3]

The last of the three branches of linear economics to originate was linear programming. Linear programming was developed by George B. Dantzig in 1947 as a technique for planning the diversified activities of the U.S. Air Force.[4] The problem solved by Dantzig has important similarities to the one studied by Leontief. In any operating period the Air Force has certain goals to achieve, and its various activities of procurement, recruitment, maintenance, training, etc., are intended to serve those goals. The relationship between goals and activities in an Air Force plan is analogous to the relationship between final products and industrial-sector outputs in Leontief's model; in each case there is an end-means connection. The novelty in Dantzig's problem arises from the fact that in Leontief's scheme there is only a single set of sector output levels that is consistent with a specified pattern of final products, while in Air Force planning, or

[1] J. Cornfield, W. D. Evans, and M. Hoffenberg, "Full Employment Patterns, 1950," *Monthly Labor Review*, **64**:163–190 (February, 1947), 420–432 (March, 1947).

[2] W. W. Leontief, "Exports, Imports, Domestic Output, and Employment," *Quarterly Journal of Economics*, **60**:171–193 (February, 1946).

[3] See, for example, H. B. Chenery and K. S. Kretschmer, "Resource Allocation for Economic Development," *Econometrica*, **24**:365–399 (October, 1956).

[4] The fundamental paper was circulated privately for several years and published as G. B. Dantzig, "Maximization of a Linear Function of Variables Subject to Linear Inequalities," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, pp. 339–347, John Wiley & Sons, Inc., New York, 1951.

in planning for any similar organization, there are generally found to be several different plans that fulfill the goals. Thus a criterion is needed for deciding which of these satisfactory plans is best, and a procedure is needed for actually finding the best plan.

This problem is an instance of the kind of optimizing that has long been familiar to economics. Traditionally it is solved by setting up a production function and determining that arrangement of production which yields the desired outputs at lowest cost or which conforms to some other criterion of superiority. This approach cannot be applied to the Air Force, or to any other organization made up of numerous components, because it is impossible to write down a global production function relating the final products to the original inputs.[1] Instead it is necessary to consider a number (perhaps large) of interconnected partial production functions, one for each type of activity in the organization. The technique of linear programming is designed to handle this type of problem.

The solution of the linear-programming problem for the Air Force stimulated two lines of development. First was the application of the technique to managerial planning in other contexts. A group at the Carnegie Institute of Technology took the lead in this direction.[2] Second, a number of economists, with T. C. Koopmans perhaps in the forefront, began exploring the implications of the new approach for economic theory generally.[3] The present volume belongs to this general direction of effort. We shall regard linear programming as a flexible and powerful tool of economic analysis and hope that the applications to be presented below will justify our position.

These are the three major branches of linear economics. The relationship between input-output analysis and linear programming is evident. Input-output analysis may be thought of as a special case of linear programming in which there is no scope for choice once the desired pattern of final outputs has been determined.

The connection of these two with game theory is more obscure. Indeed,

[1] This statement is a little too strong. A global production function *can* be constructed, but its construction presupposes that the relationships among the levels of operation of the different parts of the organization have already been determined, i.e., that the hardest part of the problem has been solved. In other words, the heart of the problem is the construction of the over-all production function with which the usual economic analysis starts.

[2] For a typical application, see A. Charnes, W. W. Cooper, and B. Mellon, "Blending Aviation Gasolines: A Study in Programming Interdependent Activities in an Integrated Oil Company," *Econometrica*, **20**:135–159 (April, 1952).

[3] For work in this spirit see the symposium volume: T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley and Sons, Inc., New York, 1951, particularly chap. 3 by Koopmans, chap. 7 by Samuelson, and chap. 10 by Georgescu-Roegen.

after the sketches we have given of the problems handled by the three techniques, it may seem surprising that there is any relationship, and, as a matter of history, the connection was not perceived for some time after the three individual problems and their solutions were well known. The connection resides in the fact that the mathematical structures of linear programming and of game theory are practically identical. Is this a pure coincidence?[1] Probably it does not pay to search for an economic interpretation. It may make the connection seem less mysterious if we put it this way: Both game theory and linear programming are applications of the same branch of mathematics—the analysis of linear inequalities—a branch which has many other applications as well, both in and out of economics. The connection is analogous with the connection between the growth of investments at compound interest and Malthusian population theory.

1-2. OUTLINE OF THE BOOK

Linear programming is the core of linear economics, and we take it up first. Chapter 2 sets forth the basic concepts and assumptions of linear programming and illustrates them by two examples, one from home economics and one from the theory of international trade. The truism that the problem of allocation and the problem of valuation are inseparable applies as well to linear programming as to other modes of economic analysis. The valuation aspect of linear programming is explored in Chap. 3.

Chapters 2 and 3 together take up the leading ideas of linear programming; Chap. 4 goes on to the mathematical properties of linear-programming problems and practical methods of solution. This latter chapter is somewhat technical and may be omitted since it adds no new economic concepts. Readers who are interested in actual solutions will find it indispensable, however.

Chapter 5 presents a particularly simple and important application of linear programming. It deals with this problem: Suppose that a homogeneous commodity is produced at a number of places and consumed at a number of places, and suppose also that the total demand at each point of consumption and total supply at each point of production are known. How much should each consuming point purchase from each producing

[1] Applied mathematics abounds in such coincidences. To take an example from physics: the well-known "parallelogram of forces" is used both (1) in mechanics to find the resultant of a number of forces, and (2) in electricity to find the current and phase (i.e., timing) of an alternating current affected by resistances, inductances, etc. This is just coincidence; these two problems have no physical connection in spite of their mathematical identity.

point so that all demands are satisfied and total costs of transportation are kept as small as possible? This "transportation" or "assignment" problem is interesting not only for its own sake but because it has useful generalizations.

In Chap. 6 the linear-programming approach is applied to the theory of the competitive firm. The conclusions are consistent with those of the marginalist theory of production. But, as we noted earlier, the marginalist theory invokes the concept of a global production function comprehending all the activities of the firm, while in a multiproduct or multistage firm it may be more convenient to work with a number of partial production functions. Chapter 7 covers the imputation of values to the resources used by a competitive firm.

Chapters 6 and 7 were restricted to competitive firms because of one of the linearity assumptions. In a competitive firm, gross revenue is a linear function of the physical volume of sales, namely, the sum over all the kinds of commodity sold by the firm of price times quantity sold. In a firm not in perfect competition the relationship between revenue and physical sales volume is more complicated; it is, in fact, nonlinear. Chapter 8 discusses the analysis of such firms and the problem of relaxing some of the linearity assumptions in linear programming.

Input-output analysis is taken up next. The basic input-output system is set forth, illustrated, and discussed in Chap. 9. Chapter 10 is a more technical discussion of the system and may be omitted by readers who wish to avoid the more mathematical aspects of the subject. It deals with more difficult questions of interpretation than does Chap. 9, including an examination of Leontief's strongest assumption—that there is a unique combination of factor and material inputs for the product of each economic sector.

Chapters 11 and 12 extend the input-output model dynamically, i.e., to a sequence of time periods, and link it up with the theory of capital. In this pair of chapters, again, the earlier chapter is primarily conceptual and the later is devoted to the more difficult and technical problems. Here, almost uniquely in this volume, our presentation takes issue with previously published results. We have mentioned above that in Leontief's static system there is only one set of levels of sector outputs that will produce a specified pattern of final products. There is therefore no room for choice once the pattern of final output has been determined. Leontief has extended his system dynamically in a way that preserves this fully determined character. Our position is that the possibility of holding intermediate and final products in inventory makes choices inevitable, so that Leontief's analysis ignores an important aspect of economic dynamics. But we cannot pursue the issues here; the reader will have to wait until Chap. 11. These chapters also arrive at some new criteria for economic efficiency in a dynamic context and some new conclusions concerning the operation of competitive markets in a dynamic context.

Rather surprisingly, linear programming has turned out to be the most powerful method available for resolving the problems of general equilibrium left unsolved by Walras and his immediate successors. Under what conditions will there exist an equilibrium position for an economy in which all prices and all outputs are nonnegative? Under what conditions is this equilibrium unique? The techniques at Walras' disposal did not permit him to reach satisfactory answers to these questions. Solutions by means of linear programming are given in Chap. 13. Linear programming has also proved to be an easy and powerful method for deriving the basic theorems of welfare economics and is used for this purpose in Chap. 14.

The final two chapters deal with game theory. Chapter 15 deals with the basic concepts of game theory as applied to economic problems and discusses some methods of practical solution of game situations. Chapter 16 explores thoroughly the mathematical connections between game theory and linear programming.

The crucial dependence of game theory on the measurability of utility warrants some discussion, particularly in view of the old issue of the relevance of the measurability of utility for economics. Appendix A is devoted to this issue.

The reader will shortly become aware that linear economics makes liberal use of the results of matrix algebra. The text is nearly, but not completely, free of matrices. Nevertheless, to help readers who wish to gain some insight into matrix methods we have added Appendix B on matrix algebra, which, it is hoped, despite being called an appendix, will not be a useless appendage.

2

Basic Concepts of Linear Programming

2-1. INTRODUCTION

Since at least the time of Adam Smith and Cournot, economic theory has been concerned with maximum and minimum problems. Modern "neoclassical marginalism" represents the culmination of this interest.

In comparatively recent times mathematicians concerned with the complex problems of internal planning in the U.S. Air Force and other large organizations have developed a set of theories and procedures closely related to the maximization problems of economic theory. Since these procedures deal explicitly with the problem of planning the activities of large organizations, they are known as "linear programming." The mathematical definition of linear programming is simple. It is the analysis of problems in which a linear function of a number of variables is to be maximized (or minimized) when those variables are subject to a number of restraints in the form of linear inequalities. That definition is a bit arid, to be sure, but there is nothing difficult about it.

The difficulties begin to enter when we raise the question of applying methods derived from linear programming to economic problems. Notice that the word "linear" occurred twice in stating the mathematical definition of linear programming. Can economic problems be cast in this strict format without doing them mortal violence? On the surface it may not seem so. The U-shaped cost curves, the gently curving isoquants, the nests of indifference lines on which so much of economic theorizing depends seem to stand in the way of expressing meaningful economic problems in terms of strictly linear relationships.

Yet it can be done, and with advantage. That is the theme of this and the following five chapters. We shall develop, in some detail, the way in which economic problems have to be reformulated in order to be amenable to the methods of linear programming. The gain from this reformulation will be seen to be twofold. First, we shall be able to bring to bear on economic problems the powerful computational and solution methods developed for handling linear-programming problems. Second, by looking at familiar problems from an unfamiliar point of view we shall gain some new insights of economic importance.

A word of caution before we embark. The linear-programming models we shall develop will, of course, not be strictly accurate representations of the economic situations with which they deal. Strict descriptive faithfulness is an unreasonable demand to make of any conceptualization. The most completely accepted of economic concepts—the production function, the demand curve, or whatnot—would fail if held up to that standard. What we have a right to ask of a conceptual model is that it seize on the strategic relationships that control the phenomenon it describes and that it thereby permit us to manipulate, i.e., think about, the situation.

In the present chapter we shall illustrate the application of linear programming to economic problems by discussing two examples. The first of these—the so-called diet problem—was brought into prominence in recent years by mathematical linear programmers, who used it as a kind of trial run for their new methods. The second example—the theory of comparative advantage—was devised a long time ago by economists, who had no thought of linear programming in mind. Both bring out important aspects of the concepts and uses of linear programming.

2-2. THE DIET PROBLEM

The diet problem is famous in the literature of linear programming because it is the first economic problem ever solved by the explicit use of this method.[1] It was originally intended merely to serve as an illustration and test of the use of the method, but, like so many toy models, it has turned out to have unexpected but important practical applications. The essential issue in this problem is that a diet to be acceptable must meet certain quality specifications; e.g., it must contain so many calories, so many units of riboflavin, etc. Moreover, the quality of a diet in terms of these specifications is the mathematical sum of the qualities of its component parts, i.e., of the foods that comprise it. These characteristics—attention to quality specifications derived by addition from the qualities of components—are the structural elements on which the solution to the problem depends.

Do problems with this structure have any important place in economics? They do. They occur in such industries as livestock feeding, gasoline and textile blending, and ice-cream manufacturing, to name a few. Thus they enter into many significant business decisions and play

[1] History of the problem: 1941—independently formulated and approximately proposed by Jerome Cornfield in an unpublished memorandum. 1945—solved by G. J. Stigler, not using linear programming; published in *Journal of Farm Economics*, **27**:303–314 (May, 1945). 1947—solved by G. B. Dantzig and J. Laderman by use of linear programming; not published.

a role in determining the shapes of supply and demand curves in many industries.

Now consider a hyperscientific and hard-pressed housewife who desires to provide an adequate diet for her family at the minimum possible cost. What foods shall she buy, and how much of each? To answer this question she must take into account the data we now outline.

2-2-1. Health Standards. The National Research Council (NRC) has published a table purporting to show, on the basis of present scientific knowledge, the minimum (annual) amounts of different nutritional elements—calories, niacin, vitamin D, etc.—that a typical adult should have. Opinions change rapidly in this field, and no claim can be made for great accuracy in such a specification. Moreover, the penalties for having less than these amounts are known only for extreme cases of unbalanced diet; and there is the further point that too much of some elements, such as calories, may be as harmful as too little. But for our purposes we may take the table as definitive and write it symbolically as shown in Table 2-1.

TABLE 2-1. MINIMUM STANDARDS OF NUTRITIONAL ELEMENTS

Nutritional elements	*Minimum standards*
1	C_1
2	C_2
3	C_3
.	.
.	.
.	.
i	C_i
.	.
.	.
.	.
m	C_m

Each of the requirements $C_1, \ldots, C_m$ is, naturally, positive.

2-2-2. Nutritional Composition of Foods. Our second bit of information comes from biologists and chemists. It analyzes the nutritional content of a large number of common foods (cooked in some agreed-upon way). We may call these foods, measured in their appropriate units, X_1, $X_2, \ldots, X_n$. We shall make the (somewhat doubtful) assumption that there is a constant amount of each nutritional element in *each* unit of any given food; so that if 10 units of X_1 gives us 100 calories, 20 units will give us 200, and 100 units will give us 1,000 calories—all this independently of the other X's that are being simultaneously consumed. This "constant-return-to-scale" and "independence" assumption helps to keep the problem within the simpler realms of linear-programming theory. It also permits us to summarize our second type of information in one rectangular table (Table 2-2).

TABLE 2-2. NUTRITIONAL CONTENT OF UNITS OF VARIOUS FOODS

Nutritional element	Food							Minimum standards
	X_1	X_2	X_3	...	X_k	...	X_n	
Element 1	a_{11}	a_{12}	a_{13}	...	a_{1k}	...	a_{1n}	C_1
Element 2	a_{21}	a_{22}	a_{23}	...	a_{2k}	...	a_{2n}	C_2
....	...	...	...	...	...	...	...	...
Element i	a_{i1}	a_{i2}	a_{i3}	...	a_{ik}	...	a_{in}	C_i
....	...	...	...	...	...	...	...	...
Element m	a_{m1}	a_{m2}	a_{m3}	...	a_{mk}	...	a_{mn}	C_m

In words, the amount of the third nutritional element contained in the seventh food is a_{37}. If we think of one slice of toast as having 50 calories, we could say $a_{\text{calories, toast}} = 50$ (calories per slice), etc.

Usually the number of foods will be much greater than the known number of nutritional elements, so that $n > m$. (But this need not be the case; indeed it would not be the case on a desert island or for a community subject to many taboos.) So long as each prescribed element is actually present in at least one food, it is clear that the given standard of nutrition can somehow be reached. (This means that we must not have *all* the a's zero in any row.) Ordinarily, the prescribed standard of nutrition $(C_1, C_2, \ldots, C_i, \ldots, C_m)$ can be reached and surpassed in a variety of different ways or diets; but the different diets will not all be equally tasty or cheap.

How do we test whether a given diet, say

$$(x_1, x_2, \ldots, x_k, \ldots, x_n) = (100, 550, \ldots, 3.5, \ldots, 25{,}000)$$

is adequate? Here x_k denotes the quantity of X_k. We must test each nutritional element in turn. Since each unit of the first food contains a_{11} units of the first element, we get altogether $a_{11}x_1$ of such an element from the first food. Similarly we get $a_{12}x_2$ of this first element from the second food. We must compare the sum of this element *from all foods in the diet* with the prescribed minimum C_1 to make sure that

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k + \cdots + a_{1n}x_n \geq C_1$$

and similarly for the second element, we must have

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k + \cdots + a_{2n}x_n \geq C_2$$

and so forth, for the ith or mth element.

We have not yet introduced the cost of food into the picture, but when we do it will become apparent that it is desirable not to have to pay for any excess consumption of food. In the above equations we should like, if possible, to have the equality signs hold rather than the inequalities, to