# Source Coding Theory

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by

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*à mes amies*

# Preface

Source coding theory has as its goal the characterization of the optimal performance achievable in idealized communication systems which must code an information source for transmission over a digital communication or storage channel for transmission to a user. The user must decode the information into a form that is a good approximation to the original. A code is *optimal* within some class if it achieves the best possible fidelity given whatever constraints are imposed on the code by the available channel. In theory, the primary constraint imposed on a code by the channel is its rate or resolution, the number of bits per second or per input symbol that it can transmit from sender to receiver. In the real world, complexity may be as important as rate.

The origins and the basic form of much of the theory date from Shannon's classical development of noiseless source coding and source coding subject to a fidelity criterion (also called rate-distortion theory) [73] [74]. Shannon combined a probabilistic notion of information with limit theorems from ergodic theory and a random coding technique to describe the optimal performance of systems with a constrained rate but with unconstrained complexity and delay. An alternative approach called asymptotic or high rate quantization theory based on different techniques and approximations was introduced by Bennett at approximately the same time [4]. This approach constrained the delay but allowed the rate to grow large.

The goal of both approaches was to provide unbeatable bounds to the achievable performance using realistic code structures on reasonable mathematical models of real-world source coding systems such as analog-to-digital conversion, data compression, and entropy coding. The original theory dealt almost exclusively with a particular form of code—a block code or, as it is sometimes called in current applications, a vector quantizer. Such codes operate on nonoverlapping blocks or vectors of input symbols in a memoryless fashion, that is, in a way that does not depend on previous blocks. Much of the theory also concentrated on memoryless sources or sources with very simple memory structure. These results have since been extended to a variety of coding structures and to far more general sources. Unfortunately, however, most of the results for nonblock codes have not appeared in book form and their proofs have involved a heavy dose of measure theory and ergodic theory. The results for nonmemoryless sources have also usually been either difficult to prove or confined to Gaussian sources.

This monograph is intended to provide a survey of the Shannon coding theorems for general sources and coding structures along with a treatment of high rate vector quantization theory. The two theories are compared and contrasted. As perhaps the most important special case of the theory, the uniform quantizer is analyzed in some detail and the behavior of quantization noise is compared and contrasted with that predicted by the theory and approximations. The treatment includes examples of uniform quantizers used inside feedback loops. In particular, the validity of the common white noise approximation is examined for both Sigma-Delta and Delta modulation. Lattice vector quantizers are also considered briefly.

Much of this manuscript was originally intended to be part of a book by Allen Gersho and myself titled *Vector Quantization and Signal Compression* which was originally intended to treat in detail both the design algorithms and performance theory of source coding. The project grew too large, however, and the design and applications-oriented material eventually crowded out the theory. This volume can be considered as a theoretical companion to *Vector Quantization and Signal Compression*, which will also be published by Kluwer Academic Press.

Although a prerequisite graduate engineering level of mathematical sophistication is assumed, this is not a mathematics text and I have been admittedly somewhat cavalier with the mathematical details. The arguments always have a solid foundation, however, and the interested reader can pursue them in the literature. In particular, a far more careful and detailed treatment of most of these topics may be found in my manuscript *Mathematical Information Theory.*

The principal existing text devoted to source coding theory is Berger's book on rate-distortion theory [5]. Gallager's chapter on source coding in [28] also contains a thorough and oft referred-to treatment. Topics treated here that are either little treated or not treated at all in Berger and Gallager include sliding-block codes, feedback and finite-state codes, trellis encoders, synchronization of block codes, high rate vector quantization theory, process definitions of rate-distortion functions, uniform scalar quantizer noise theory, and Sigma-Delta and Delta modulation noise theory in scalar quantization (or PCM), in Sigma-Delta modulation, and in Delta modulation. Here the basic source coding theorem for block codes is proved without recourse to the Nedoma decomposition used by Berger and Gallager [5], [28]. The variational equations defining the rate-distortion function are developed using Blahut's approach [6], but both Gallager's and Berger's solutions are provided as each has its uses. The Blahut and Gallager/Berger approaches are presented in some detail and the proofs given take advantage of both viewpoints. In particular, the use of calculus optimizations is minimized by repeated applications of the divergence inequality.

The primary topic treated by Berger and Gallager and not included here is that of continuous time source coding theory. Noiseless coding is also not treated here as it is developed in the companion volume as well as in most information theory texts (as well as texts devoted entirely to noiseless coding, e.g., [77].)

This book is intended to be a short but complete survey of source coding theory, including rate-distortion theory, high rate quantization theory, and uniform quantizer noise theory. It can be used in conjunction with *Vector Quantization and Signal Compression* in a graduate level course to provide either background reading on the underlying theory or as a supplementary text if the course devotes time to both the theory and the design of vector quantizers. When I teach an "Advanced Topics" course on data compression, I proceed from a brief review of the prerequisite random process material into Part III of *Vector Quantization and Signal Compression*, that is, directly into the development of code design algorithms for vector quantizers. I treat the design material first as this is usually the preferred material for term papers or projects. The second half of the course is then devoted to Chapters 3, 4, and 5 of this book. Portions of Chapter 6 are treated if time permits and are used to point out the shortcomings of the asymptotic approximations as well as to provide an introduction into the theory of oversampled analog-to-digital converters.

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July 1989

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# SOURCE CODING THEORY

# Contents

# Chapter 1

# Information Sources

An information source is modeled mathematically as a discrete time random process, a sequence of random variables. This permits the use of all of the tools of the theory of probability and random processes. In particular, it allows us to do theory with probabilistic averages or expectations and relate these to actual time average performance through laws of large numbers or ergodic theorems. Such theorems describing the long term behavior of well behaved random systems are crucial to such theoretical analysis. Relating expectations and long term time averages requires an understanding of stationarity and ergodic properties of random processes, properties which are somewhat difficult to define precisely, but which usually have a simple intuitive interpretation. These issues are not simply of concern to mathematical dilettants. For example, stationarity can be violated by such commonly occurring phenomena as transients and variable length coding, yet sample averages may still converge in a useful way. In this chapter we survey some of the key ideas from the theory of random processes. The chapter strongly overlaps portions Chapter 2 of Gersho and Gray [30] and is intended to provide the necessary prerequisites and establish notation. The reader is assumed to be familiar with the general topics of probability and random processes and the chapter is intended primarily for reference and review. A more extensive treatment of basic random processes from a similar point of view may be found in [40] and [37].

## 1.1 Probability Spaces

The basic building block of the theory of probability and random processes is the *probability space* or *experiment*, a collection of definitions and axioms which yield the calculus of probability and the basic limit theorems relating expectations and sample averages. For completeness we include the basic definitions and examples. A *probability space* $(\Omega, \mathcal{F}, P)$ is a collection of three things:

$\Omega$ An abstract space $\Omega$ called the *sample space*. Intuitively this is a listing of all conceivable outcomes of the experiment.

$\mathcal{F}$ A nonempty collection of subsets of $\Omega$ called *events* which has the following properties:

1. If $F \in \mathcal{F}$, then also $F^c \in \mathcal{F}$, that is, if the set $F$ is in the collection $\mathcal{F}$, then so is its complement $F^c = \{\omega : \omega \notin F\}$.
2. If $F_i \in \mathcal{F}$, $i = 1, 2, \ldots$, then also $\bigcup_i F_i \in \mathcal{F}$, that is, if a (possibly infinite) collection of sets $F_i$ belong to the collection $\mathcal{F}$, then so does the union $\bigcup_i F_i \in \mathcal{F} = \{\omega : \omega \in F_i \text{ for some } i\}$.

It follows from the above conditions that $\Omega \in \mathcal{F}$, i.e., the set "something happens" is an event, and the null set $\emptyset = \Omega^c$ ("nothing happens") is an event. A collection $\mathcal{F}$ of subsets of $\Omega$ with these properties is called an *event space* or a *σ-field*. It is a standard result that the above two conditions imply that any sequence of complements, unions, or intersections of events (members of $\mathcal{F}$) yields another event. This provides a useful algebraic structure to the collection of sets for which we wish to define a probability measure.

$P$ A *probability measure* on an event space $\mathcal{F}$ of subsets of a sample space $\Omega$ is an assignment of a real number $P(F)$ to every $F$ in $\mathcal{F}$ which satisfies the following rules (often called the *axioms of probability*):

1. $P(F) \geq 0$ for all $F \in \mathcal{F}$, that is, probabilities are nonnegative.
2. $P(\Omega) = 1$, that is, probability of the entire sample space ("something happens") is 1.
3. If events $F_i$, $i = 1, 2, \ldots$ are disjoint, that is, if $F_i \bigcap F_j = \{\omega : \omega \in F_i \text{ and } \omega \in F_j\} = \emptyset$ for all $i \neq j$, then

$$P(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} P(F_i);$$

that is, the probability of the union of a sequence of disjoint events is the sum of the probabilities.

It is important to note that probabilities need be defined only for events and not for *all* subsets of the sample space. The abstract setup of a probability space becomes more concrete if we consider the most important special cases. One basic example is the case where $\Omega = \mathcal{R} = (-\infty, \infty)$, the real line. The most useful event space in this case is called the *Borel field* and is denoted $\mathcal{B}(\mathcal{R})$. We will not delve into the theoretical details of Borel fields, but we shall point out that it contains all of the subsets of the real line that are sufficiently "nice" to have consistent probabilities defined on them, e.g., all intervals and sets constructable by sequences of complements, unions, and intersections of intervals. It is a basic result of measure theory, however, that the Borel field does not contain *all* subsets of the real line. The members of the Borel field, that is, the events in the real line, are called *Borel sets.*

Suppose that we have a real valued function $f$ defined on $\mathcal{R}$ with the following properties:

1. $f(r) \geq 0$ , all $r \in \mathcal{R}$,
2. $\int_{-\infty}^{\infty} f(r)\, dr = 1$.

Then the set function $P$ defined by

$$P(F) = \int_F f(r)\, dr$$

is a probability measure and the function $f$ is called a *probability density function* or *pdf* since it is integrated to find probability. There is more going on here than meets the eye and a few words of explanation are in order. Strictly speaking, the above claim is true only if the integral is considered as a Lebesgue integral rather than as the Riemann integral familiar to most engineers. As is discussed in some length in [40], however, this can be considered as a technical detail and Riemann calculus can be used without concern provided that the Riemann integrals make sense, that is, can be evaluated. If the Riemann integral is not well defined, then appropriate limits must be considered. Some of the more common pdf's are listed below. The pdf's are 0 outside the listed domain. $b > a$, $\lambda > 0$, $m$, and $\sigma > 0$ are real-valued parameters which specify the pdf's.

**The uniform pdf** $f(r) = 1/(b-a)$ for $r \in [a, b] = \{r : a \leq r \leq b\}$.

**The exponential pdf** $f(r) = \lambda e^{-\lambda r}$; $r \geq 0$.

**The doubly exponential or Laplacian pdf** $f(r) = \frac{\lambda}{2} e^{-\lambda |r|}$; $r \in \mathcal{R}$.

**The Gaussian pdf** $f(r) = (2\pi\sigma^2)^{-1/2} e^{-(r-m)^2/2\sigma^2}$; $r \in \mathcal{R}$.

A similar construction works when $\Omega$ is some subset of the real line instead of the entire real line. In that case the appropriate event space comprises all the Borel sets in $\Omega$. For example, we could define a probability measure on $([0,\infty),\mathcal{B}([0,\infty)))$, where $\mathcal{B}([0,\infty))$ denotes the Borel sets in $[0,\infty)$, using the exponential pdf. Obviously we could either define this experiment on the entire real line using a pdf that is 0 for negative numbers and exponential on nonnegative numbers or we could define it on the nonnegative portion of the real line using a pdf that is an exponential everywhere. This is strictly a matter of convenience. Another common construction of a probability measure arises when all of the probability sits on a discrete subset of the real line (or any other sample space). Suppose that $\Omega'$ is an arbitrary sample space and $\mathcal{F}$ a corresponding event space. Suppose that an event $\Omega \subset \Omega'$ consists of a finite or countably infinite collection of points. (By countably infinite we mean a set that can be put into one-to-one correspondence with the nonnegative integers, e.g., the nonnegative integers, the integers, the even integers, and the rational numbers.) Suppose further that we have a function $p$ defined for all points in $\Omega$ which has the following properties:

1. $p(\omega) \geq 0$ all $\omega \in \Omega$.

2. $\sum_{\omega\in\Omega} p(\omega) = 1$.

Then the set function $P$ defined by

$$P(F) = \sum_{\omega \in F \bigcap \Omega} p(\omega)$$

is a probability measure and the function $p$ is called a *probability mass function* or *pmf* since one adds the probability masses of points to find the overall probability. That $P$ defined as a sum over a pmf is indeed a probability measure follows from the properties of sums. (It is also a special case of the Lebesgue integral properties.) Some of the more common pmf's are listed below. The pmf's $p(\omega)$ are specified in terms of parameters: $p$ is a real number in $(0,1)$, $n$ is a positive integer, and $\lambda$ is a positive real number.

**The binary pmf** $\Omega = \{0,1\}$. $p(1) = p$, $p(0) = 1-p$.

**The uniform pmf** $\Omega = \{0,1,\ldots,n-1\}$. $p(k) = 1/n$; $k \in \Omega$.

**The geometric pmf** $\Omega = \{1,2,\ldots\}$. $p(k) = p(1-p)^{k-1}$; $k \in \Omega$.

**The Poisson pmf** $\Omega = \{0,1,2,\ldots\}$. $p(k) = \lambda^k e^{-\lambda}/k!$; $k \in \Omega$.