FOUNDATIONS OF GEOMETRY

EUCLIDEAN AND BOLYAI-LOBACHEVSKIAN GEOMETRY PROJECTIVE GEOMETRY

by

KAROL BORSUK

and

WANDA SZMIELEW

University of Warsaw, Poland

Revised English Translation

1960

NORTH-HOLLAND PUBLISHING COMPANY, AMSTERDAM

PREFACE TO THE POLISH EDITION†

This book was planned, at first, to be a textbook in the foundations of geometry, specifically adapted to the present program of studies in this subject in Polish universities. In the process of writing, however, the conception of the work underwent some change in order that the material included constitute a consistent, closed entity which—as it seems to the authors—may be of value independently of any possible changes in the program of the course. With this aim in mind some portions of the book (e.g. Bolyai-Lobachevskian geometry) have been treated much more extensively than in the original outline while some others—dealing with rather marginal problems—have been entirely omitted. The presentation of the material is quite elementary. Even in those portions of Bolyai-Lobachevskian geometry in which the apparatus of differential geometry is usually applied the authors have used exclusively the most elementary notions and methods of the calculus. On the other hand the authors considered it purposeful to introduce the general topological notions at a very early stage of the discussion. The reader who is not familiar with topology will find the necessary information in the Introduction (Section 9).

The book is organized as follows.

In Part I the authors develop Euclidean and Bolyai-Lobachevskian geometry on the basis of an axiom system due, in principle, to Hilbert. It should be noted at once, however, that the authors develop these geometries, in principle, as far as necessary to be able to prove them categorical, i.e., to show that the Cartesian space known from analytic geometry is up to isomorphism the only model of Euclidean geometry, and Klein space (constructed with the help of notions known from the analytic geometry of projective space) is up to isomorphism the only model of Bolyai-Lobachevskian geometry. In this way two aims are achieved. First, it is shown that each of the theories constitutes a uniquely determined

---

† The Polish original of this work was published under the title *Podstawy Geometrii* by Panstwowe Wydawnictwo Naukowe (Warsaw, 1955).

scheme embracing the geometrical properties of physical space. Secondly, the course in geometry is developed to the point where, by the introduction of coordinates (rectangular coordinates in Euclidean geometry and Beltrami coordinates in Bolyai-Lobachevskian geometry), it becomes possible to employ analytic methods.

Besides the full proof of categoricity, Part I also contains proofs of the consistency of both geometries and—as an example of independence proofs—a proof of the independence of the Axiom of Continuity.

In Part II the authors develop projective geometry on the basis of an axiom system also due, in principle, to Hilbert. In projective geometry use is made of the theorems of Euclidean geometry, this being done by an interpretation of a part of Euclidean geometry in projective geometry. The organization of the material is the same as in Part I. The theory is developed only so far as the introduction of homogeneous projective coordinates in space. This gives the possibility of proving categoricity, which, in this case, reduces to showing that the projective space known from analytic geometry is up to isomorphism the only model of projective geometry.

In spite of such a restricted aim the book is rather extensive. This is so because the authors have set themselves the task of achieving a presentation which does not contain any essential gaps. In this way the reader, after finishing the book, can be confident that the axiom systems adopted are indeed a sufficient basis for the construction of the entire geometry. Such an approach has somewhat encumbered the book, forcing the authors—especially in the initial chapters—to give rigorous proofs of theorems generally well known and frequently trivial. The reader well-versed in the axiomatic treatment of elementary geometry is advised to glance lightly through the first two chapters, and devote attention principally to the conceptional aspects of the work.

It is worth noting that the authors have not taken up in this book the problem of constructing arithmetic on the basis of geometry. Their task was made easier by assuming the arithmetic of real numbers as a theory preceding geometry. Analytic geometry of Cartesian space and projective space is regarded as a branch of arithmetic.

In conclusion, the authors take pleasure in cordially thanking Professor Adam Bielecki for his careful study of the manuscript and for making many valuable and keen remarks which have enabled them to eliminate several errors.

PREFACE TO THE ENGLISH EDITION

The preparation of the English edition of Foundations of Geometry has given the authors the opportunity of introducing essential changes and additions (as well as minor corrections).

In the Polish edition absolute geometry (the common part of Euclidean and Bolyai-Lobachevskian geometries) was based on the Hilbert system of primitive notions and axioms; in the English edition the Hilbert system has been modified in several respects. Lines and planes are now treated as sets of points; therefore the primitive relation of incidence is replaced by the set-theoretical membership relation and does not appear as a primitive notion of geometry. This change has led to a considerable simplification in the logical structure of the discussion of models and categoricity. Furthermore, the primitive relation of congruence of segments is treated in the present edition as a four-termed relation among points; the relation of congruence of angles has been entirely removed from the system of primitive notions, and this has necessitated certain changes in the axioms of congruence. Since the authors wished to avoid any radical changes in the arrangement of the material, they did not avail themselves of the possibility of further limiting the system of primitive notions.

Other essential changes were made to bring out the role of the Axiom of Continuity and the Archimedean Postulate. The authors endeavored to transfer as much material as possible to absolute geometry.

The most essential change introduced in Bolyai-Lobachevskian geometry involves the introduction of the natural basic segment and the natural measure of segments which goes with it. This permits the calculation of the numerical value of the constant $\varkappa$ which appears in a number of formulas of this geometry.

In projective geometry, just as in absolute geometry, lines and planes are now regarded as sets of points, whereas in the original edition, following the traditional approach, the points, the lines, and the planes are treated as three fundamental domains of discourse with equal status in the logical structure of the theory. It should be noticed that the traditional approach, as opposed to the present one, leads to a much simpler and more convenient formulation of the duality law; hence this approach would have been advantageous if the authors had attempted a systematic development of projective geometry. In this work, however, the authors are primarily interested in carrying through simple and precise proofs of categoricity, and from this point of view the new approach is much preferable.

We wish to express our deep gratitude to Professor Alfred Tarski (University of California, Berkeley) for his penetrating comments and criticism of the Polish edition of Foundations of Geometry at the time it appeared. To a large extent these remarks contributed to the changes introduced in the English edition. During the final revision of the English manuscript, Professor Tarski's warm advice has helped us again at every turn. To Professor Stanislaw Jaśkowski (University of Toruń) we are indebted for a remark which led to some simplification of the axiom system for projective geometry.

We are very grateful to Mr. Erwin Marquit (University of Warsaw) for his effort and care in preparing the English translation. We also sincerely appreciate the help extended to us by Professors Henry Helson (University of California, Berkeley), Leon Henkin (University of California, Berkeley) and Steven Orey (University of Minnesota), in connection with the final stylistic revision of the English text.

The final revision of the manuscript of this book, both in material and formal respects, was carried through during the academic year 1957–58. The work was performed partly at the University of Warsaw, Poland, by Karol Borsuk, and partly at the University of California, Berkeley, U.S.A., by Wanda Szmielew, who was then engaged in a research project on the foundations of mathematics sponsored by the U.S. National Science Foundation.

University of Warsaw, Poland — KAROL BORSUK

Institute for Basic Research in Science, University of California, Berkeley, USA — WANDA SZMIELEW

May, 1958

CONTENTS

CHAPTER VI. BOLYAI-LOBACHEVSKIAN GEOMETRY

Part Two

PROJECTIVE GEOMETRY

INTRODUCTION TO PART TWO

CHAPTER VII. AXIOMS OF INCIDENCE AND ORDER

Introduction

1. Geometry before Euclid

The foundations of geometry have attained their present clarity only as a result of a very long process of development which began in very ancient times and concluded in the 20th century. The need for such geometrical notions as segment, line, angle, triangle, circle, length, area, volume, etc. appeared as early as civilization. Even in early antiquity we can find systematic attempts to establish relations between these notions, and this is the beginning of geometry. The famous Rhind papyrus, a copy of which has been preserved from the Hyksos epoch (about 1700 B.C.) testifies to the fact that at that time geometry in Egypt already stood at a rather high level, nevertheless was, in principal, limited to empirically found instructions for calculating the area of a plane figure or the volume of a solid. But only among the Greeks we find a conscious striving to give geometry the form of a science in today's meaning of the word. Beginning with THALES of Milet and PYTHAGORAS of Samos (6th century B.C.) and ending with the Greek mathematicians in Alexandria in the period of the decline of the Roman empire (e.g. PAPPUS of Alexandria, 4th century A.D.), a long list of outstanding Greek mathematicians contributed in an essential way to the development of geometry.

2. Elements of Euclid

One of the most important events in the development of geometry was the systematic treatment of geometry in the form of a uniform deductive system in the work of EUCLID entitled *Elements* (στοιχεῖα) written in Alexandria about 300 B.C. If the value of a scientific work can be measured by the length of time during which it maintains its importance then *Elements* of Euclid is the most valuable scientific book of all time. It has appeared in innumerable editions, and through the entire period of two thousand years has been generally considered a model of rigor and clarity of presentation, and richness of content. Euclid set himself the task of presenting geometry in the form of a system based on a small number of sentences, some of which were called *definitions*, others *axioms*, and still others *postulates*. The remaining statements of his work were to be logical consequences of these three types of initial sentences.

Euclid placed definitions at the beginning of each of the 13 books of his *Elements*. Thus, e.g., the definitions with which he begins the first book are as follows:†

A point is that which has no part. A line is a (breadthless) length. The extremities of a line are points. A straight line is a line which lies evenly with the points on itself. And so on.

As may be seen from these examples, the definitions given by Euclid are not definitions in today's meaning of the word and cannot be used in the construction of a theory. They are rather only explanations of the notions introduced, expressed in imprecise colloquial language and intended to create in the mind of the reader certain intuitive pictures.

The difference between axioms and postulates is not further explained by Euclid. The sentences he called axioms—such as *the whole is greater than the part*—have the character of statements about objects of some very general unspecified kind. The postulates, however, concern specific figures and in character are like the sentences called *axioms* in modern deductive theories. In modern theories the sentences corresponding to Euclid's axioms do not occur, and the terms *axiom* and *postulate* are used interchangeably.

In establishing his system of definitions, axioms, and postulates, Euclid believed that he was creating a sufficient foundation for the construction of geometry, i.e. for the introduction of geometrical notions and for the deduction of their properties. It should be noted that this is not the case. Thus, e.g., Euclid speaks about a point lying between two other points despite the fact that such a relation between points cannot be defined on the basis of his geometry; he makes essential use of the continuity of the space despite the fact that this property cannot be established in his system of geometry. Nevertheless, we are indebted to him for the first attempt known to us to construct an axiomatic theory.

3. Elementary Geometry After Euclid. Euclid's Critics and Commentators

Several years after Euclid the famous Syracusan mathematician ARCHIMEDES supplemented the system of axioms of Euclid by a further system of five axioms which he needed in connection with investigations on the length of curves, the area of surfaces, and the volume of solids. Four of these axioms directly involve the notions of length and area; they are superfluous if these notions are introduced by way of appropriate

† All quotations from Euclid in this book are based on T. C. HEATH'S *The Thirteen Books of Euclid's Elements*, 2nd ed. (Cambridge 1926).

definitions. On the other hand, the fifth (we shall refer to it as the *Postulate of Archimedes*) is an explicit formulation of a property of segments on a line, a property which was already involved in the considerations of Euclid.

The axiomatic treatment of elementary geometry by Euclid was the subject of investigations by many mathematicians over a period of many centuries. Particular interest was attached to the problem of whether the *parallel postulate* (sometimes known as *Euclid's Fifth Postulate*) is necessary for the construction of elementary geometry. This postulate was formulated as follows:

If a straight line falling (in a plane) on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

For a period of over two thousand years many attempts were made to prove that this postulate is a logical consequence of the remaining assumptions, and therefore that it may be omitted with no loss to the theory. There arose a very extensive literature which contains, among other results, various proofs of the parallel postulate using the remaining assumptions of Euclid together with some further assumption. For example, it suffices to add the assumption that there exists at least one rectangle; or that there exists at least one triangle the sum of whose angles is equal to two right angles. Up to the 19th century it was not settled whether there is a proof of the parallel postulate based only on the remaining assumptions of Euclid.

The most penetrating of the commentators on Euclid, the Italian mathematician Gerolamo SACCHERI (1667–1733) and the Swiss mathematician J. H. LAMBERT (1728–1777), consistently developed the domain of geometry which did not employ the parallel postulate. In particular, they drew attention to the theorem concerning the sum of the angles in a triangle. Without the aid of the parallel postulate they proved that this sum cannot be greater than two right angles. A little later the French mathematician A. M. LEGENDRE proved this fact again. SACCHERI also tried to prove, by deriving a contradiction, that the sum of the angles in a triangle cannot be smaller than two right angles, and he believed that in this way he would be able to obtain a proof of the parallel postulate. Lambert tried to show that the negation of the parallel postulate would lead to conclusions departing too greatly from our picture of space. By systematically investigating the logical consequences of the negation of the parallel postulate, both these mathematicians followed a path which subsequently led to the discovery of what we now call *non-Euclidean geometry*.

4. Bolyai-Lobachevskian Geometry

Only in the 19th century was the parallel postulate shown to be independent of the remaining axioms and postulates of Euclid. A decisive step in this direction was made by the Russian mathematician Nikolai Ivanovich LOBACHEVSKI (1793–1856). In 1829 he published in Kasan his work entitled *O načalah geometrii*, which contained an exposition of a new geometry based precisely on the negation of the parallel postulate. In the history of human thought we often meet with the phenomenon that great discoveries are made simultaneously and independently by several people when the state of science and technology reaches the point where it is ready for these developments. In the field of mathematics one may cite as examples the discovery of differential and integral calculus by NEWTON and LEIBNIZ, and of analytic geometry by FERMAT and DESCARTES. Another typical case is the discovery of non-Euclidean geometry. Simultaneously with Lobachevski and completely independently of him, the Hungarian mathematician Janos BOLYAI (1802–1860) arrived at similar conceptions; the work embodying his ideas, *Appendix scientiam spatii absolute veram exhibens*, appeared in 1832.

It should be mentioned that still earlier the eminent German mathematician Karl Friedrich GAUSS (1777–1855) had arrived at the idea of non-Euclidean geometry, but he never published the results of his investigations. He feared the criticism which might be evoked by an idea which departed so far from the ideas then accepted and sanctified by the tradition of many centuries. Through such an attitude, Gauss lost priority for the discovery of non-Euclidean geometry. Gauss's fears, however, were not without basis. The works of Lobachevski and Bolyai did not receive recognition in the lifetime of their creators. On the contrary, they were regarded as eccentric and pathological; a Russian mathematician well known in this period went so far as to call the work of Lobachevski a satire directed against mathematicians.

At the beginning of the 19th century the idea of non-Euclidean geometry appears quite plainly also in the works of the German mathematicians F. K. SCHWEIKART (1780–1859) and F. A. TAURINUS (1794–1874). But only Lobachevski and Bolyai have made the systematic study of what we now call *Bolyai-Lobachevskian* (or *hyperbolic*) *geometry*.

5. Consistency of Geometry

Lobachevski was convinced that there were no inconsistencies in his geometry. To show the consistency he pointed out that between the formulas of his trigonometry and the formulas of the spherical trigonometry a one-to-one correspondence can be established. By means of this correspondence the problem of consistency of his geometry can, in principle, be reduced to the problem of consistency of spherical geometry which can be treated as a part of arithmetic. However a rigorous proof of consistency was an impossible task at that time since, on the one hand, the foundations of geometry were not yet sufficiently established, and on the other, such general notions as axiomatic theory and consistency were not yet precisely formulated and investigated. (These notions belong to the methodology of deductive sciences, the systematic development of which began only in the final years of the 19th century.)

In the period preceding the final axiomatic approach to geometry the basic idea of the precise proof of the consistency of Bolyai-Lobachevskian geometry was given in 1871 by the German mathematician Felix KLEIN (1849–1925), who, on the basis of the earlier ideas of the Italian mathematician Eugenio BELTRAMI (1835–1900), constructed in his work *Über die sogenannte Nicht Euklidische Geometrie*, Mathematische Annalen 4, the arithmetic model of Bolyai-Lobachevskian geometry.

It was only in the year 1899, however, that David HILBERT, in the work *Grundlagen der Geometrie*, gave the system of primitive notions and axioms of Euclidean geometry and a full proof of the consistency of this axiom system (under the assumption of the consistency of arithmetic). In 1903, in the work *Neue Begründung der Bolyai-Lobatchefskyschen Geometrie*, Mathematische Annalen 57, he proved the consistency of Bolyai-Lobachevskian geometry in a similar manner. Thus the two geometries, Euclidean and Bolyai-Lobachevskian, are equally correct from the standpoint of logic. The question of whether Euclidean or Bolyai-Lobachevskian geometry better describes real space can be settled, if at all, only by way of experiment. It seems, however, that experiment could at most confirm Bolyai-Lobachevskian geometry, but not Euclidean geometry. This is because Euclidean geometry is, in a sense, the limiting case of Bolyai-Lobachevskian geometry, and by means of experiments based only on approximate measurements, we cannot distinguish the limiting case from a very close approximation.

In the same period, and independently of Hilbert, investigations were made in Italy on the foundations of geometry. In particular Mario PIERI in his works *Della geometria elementare come sistema ipotetico-deduttivo: Monografia del punto e del moto*, Memorie della R. Accademia delle Scienze di Torino, 1899, and *La geometria elementare instituita sulle nozioni di "punto" e "sfera"*, Memorie di Matematica e di Fisica della Societa Italiana delle Scienze, ser. 3,15, 1908, published two axiomatic systems of Euclidean geometry each of which is based upon only one primitive notion.

6. Riemann Spaces

Simultaneously with the foundations of elementary and Bolyai-Lobachevskian geometry, two other branches of geometry, *differential geometry* and *projective geometry* were cultivated. The development of differential geometry led the German mathematician Bernhard RIEMANN (1826–1866) to introduce (in 1854) a very general class of spaces now called *Riemann spaces*.

Among the general Riemann spaces there stand out, in particular, *spaces with constant curvature* embracing the *parabolic type*, corresponding to the Euclidean space, the *hyperbolic type*, corresponding to the Bolyai-Lobachevskian space, and finally, the *elliptic type*, corresponding to the projective space with suitably chosen metric.

7. Axiomatic Theory

The purpose of axiomatic theory is to approach reality in an abstract form so as to permit the highest possible degree of rigor. In constructing such a theory, the following procedure is adopted:

First of all, a certain system of *primitive notions* is chosen. It is desirable that these notions have the clearest possible intuitive sense. It is taken as a principle that one may employ other notions only when they are defined in terms of the primitive notions, either directly or indirectly by means of previously defined notions.

Next, a certain system of sentences, called *axioms*, is chosen in which there are formulated some properties of the primitive notions. The axioms should state in abstract form some relations holding between the real objects from which the primitive notions were abstracted. It is desirable that the intuitive sense of the axioms not give rise to any doubts. The *theorems* of the theory are the axioms, and those sentences which are logical consequences of the axioms, definitions, and the theorems previously proved.

The geometry in this book will be based on a system of primitive notions and axioms which is a modification of the Hilbert system.

In constructing an axiomatic theory T we usually make use of other axiomatic theories, which are *presupposed* in the following sense: all the primitive notions of those presupposed theories are included in the system of primitive notions of T, and all the axioms of those theories are included in the axiom system of T. Mathematical theories presuppose as a rule mathematical logic and usually also set theory (to a larger or smaller extent). In developing geometry in this book we presuppose mathematical logic, set theory and the arithmetic of real numbers (which can either be treated as an independent theory or can be constructed as a portion of set theory). An axiomatic treatment of these three theories can be found in various special works. We shall not list here the primitive notions and axioms of these theories. In the next section, however, we shall discuss briefly all the basic set-theoretical notions which are relevant to our discussion.

8. Sets and Relations

The basic notion of set theory is that of *membership*. The membership symbol $\in$ occurs in formulas like $x \in X$, which is read *x is an element* (or *a member*) *of the set X*, or *x belongs to the set X*, or, finally, *set X contains x (as an element)*. The set with no elements will be denoted by 0 and will be called the *empty set*. The set consisting of elements $x_1, x_2, \ldots, x_n$ will be denoted by $\{x_1, x_2, \ldots, x_n\}$. If the elements of a set X are sets themselves we refer to set X also as a *family of sets* or a *class*.

If every element of a set X is also an element of a set Y we write $X \subset Y$ and we say that *set X is included in set Y* or *set Y includes set X (as a part)* or *set X is a subset of set Y*. We have:

$$X \subset X,$$

$$\text{if } X \subset Y \text{ and } Y \subset X, \text{ then } X \subset Z,$$

$$\text{if } X \subset Y \text{ and } Y \subset X, \text{ then } X = Y.$$

For any class $\mathfrak{X}$, the *set-theoretical sum* (or *union*) of the sets of class $\mathfrak{X}$ is the set consisting of all elements which belong to at least one of the sets of $\mathfrak{X}$, while the *set-theoretical product* (or *intersection* or *common part*) of the sets of class $\mathfrak{X}$ is the set consisting of all elements which belong to each of the sets of $\mathfrak{X}$. If $\mathfrak{X} = \{X, Y\}$, then the sum of the sets of class $\mathfrak{X}$ is denoted by $X \cup Y$, and their product by $X \cap Y$. In the case $X \cap Y = 0$ we say that sets X and Y are *disjoint*. If to each natural number n there corresponds a set X_n, then the sum of sets $X_1, X_2, \ldots$ is denoted by $\bigcup_{n=1}^{\infty} X_n$ and the product by $\bigcap_{n=1}^{\infty} X_n$. By the *difference* $X - Y$ we understand the set composed of all elements belonging to X, but not belonging to Y. In case z is an element (but not a set) we shall write $X \cup z$ instead of $X \cup \{z\}$, $X - z$ instead of $X - \{z\}$, and $X \cap Y = z$ instead of $X \cap Y = \{z\}$.

Let us now assume that all the sets under consideration are subsets of some fixed set E, and let us denote by X' the set $E - X$ (called the *complement* of set X to set E). We then have: