Introduction to the

Theory of Laser-Atom Interactions

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To
Sondra

Preface

This book grew out of a graduate course given in the Physics Department of the City College of New York for the first time during the 1976–1977 academic year and a series of lectures given at the Catholic University of Louvain, at Louvain-la-Neuve, Belgium during the Spring and Summer of 1977. I am indebted to Professor F. Brouillard and the DYMO group at that institution for the stimulation and hospitality provided during that period. In both cases, the lectures were at a level that assumed only a knowledge of elementary quantum mechanics of a typical first-year graduate course. I have tried to continue that level of discussion in this book and to make it self-contained for any discussions that go beyond that level. In some sections of the book, the problems dealt with are too complicated to provide the entire description here. In that case, references to the original work are given.

I have also tried to keep the book only slightly longer than a size that could be covered in a one-semester course in order to allow an instructor some options. An attempt was also made to cover a wide range of topics in atomic physics as modified by lasers with particular emphasis on scattering and reactions. These two aims proved to be incompatible, so some topics have been totally omitted (multiphoton spectroscopy, for example), as have some methodologies (the momentum translation method for multiphoton ionization, for example). Topics covered in standard first-year quantum mechanics courses have been avoided where possible, which is the reason for the absence of many first-order Born approximation calculations which could have been included. I have tried to carry the calculations to a point where the laser no longer plays a role and standard atomic physics methods take over. This is not meant to imply that the remaining problem is simple, but only that it falls outside the scope of this book. I have also tried to choose what I believe will be the most enduring treatments, but that is a subjective choice which is bound to be wrong in some cases. Chapter 3, on States of an Atom in a Laser, may well be an example of this situation.

The first chapter deals with standard material, much of which is included in most first-year graduate quantum mechanics courses, but is included here with a point of view that is adaptable to the remainder of the book. The two-level atom, the subject of the second chapter, has been treated extensively from several different viewpoints in other places (notably in the excellent book *Optical Resonances and Two-Level Atoms* by L. Allen and J. H. Eberly, Wiley-Interscience, New York, 1975), so it is treated only to the extent that is necessary for subsequent chapters. Similarly, spontaneous radiation, the subject of the fourth chapter, is treated only briefly in order to obtain results that are necessary for the discussion of other processes. In dealing with both of these subjects I have stopped short of the extensive theoretical discussions and comparison with experiment which are available in other sources, since these are not intended to be the primary emphasis of this book.

The book draws from the work of many people, and a list of references is given at the end of each chapter. However, the perspective presented is my own, and I should acknowledge the effects of discussions with my colleagues at City College, who helped me to straighten things out in my own mind. Professors Kenneth Rubin and Joel Gersten were the principal contributors in that way. Two (at that time) graduate students, Drs. J. Banerji and P. Krstic, also aided greatly with a critical reading of the manuscript and many useful suggestions. It was put in its final form during the Summer of 1981 at the University of California at Berkeley. I am indebted to Professor Richard Marrus for his hospitality there. My own research which occasionally appears in this book was supported by a U.S. Office of Naval Research Contract, which is gratefully acknowledged.

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1

Basic Ideas

1.1. Introduction

Since the object of this exercise is the treatment of the interaction of atoms with lasers, which are very intense electromagnetic fields, our starting point must be the Schrödinger equation describing the time evolution of coupled matter and electromagnetic fields. We shall be interested in relativistic effects only in a peripheral way, so the matter field will be described by a conventional Schrödinger wave function. However, we must allow for creation and destruction of photons, the particles of the electromagnetic field. This is most conveniently done by resorting to the quantum electrodynamic description of this field.[1] In that case the Schrödinger equation reads

$$\left(i\hbar \frac{\partial}{\partial t} - H \right) \Psi = 0 \tag{1.1.1}$$

where

$$H = H_{\text{rad}} + \sum_{j=1}^{j_{\max}} \frac{\left[\mathbf{P}_j - \dfrac{e_j}{c} \mathbf{A}(\mathbf{r}_j) \right]^2}{2m_j} + V(r_1 \cdots r_{j_{\max}}) \,. \tag{1.1.2}$$

The first term is the energy operator of the noninteracting electromagnetic field

$$H_{\text{rad}} = \sum_{k,\lambda} \hbar\, \omega_k \mathbf{n}_{k\lambda} \tag{1.1.3}$$

where the sum k, λ is a sum over a complete set of modes of the electromagnetic field. Usually $\mathbf{k}$ is the momentum of the mode and λ numbers the two possible polarizations of the transverse field, $\mathbf{A}(\mathbf{r})$. $\hbar\omega_k$ is the energy of the mode and $\mathbf{n}_{k\lambda}$ is the operator whose eigenvalues are the occupation numbers (or number of photons) in each mode. This operator is given by

$$\mathbf{n}_{k\lambda} = a_{k\lambda}^{\dagger} a_{k\lambda} \tag{1.1.4}$$

where $a^{\dagger}_{k\lambda}(a_{k\lambda})$ is a creation (destruction) operator of photons in the $\mathbf{k}, \lambda$ mode. They obey the commutation relations

$$[a_{k\lambda}, a^{\dagger}_{k'\lambda'}] = \delta_{kk'}\delta_{\lambda\lambda'} \tag{1.1.5}$$

and all other commutators vanish.

The transverse vector potential of the electromagnetic field is given in terms of these operators as

$$\mathbf{A}(\mathbf{r}) = \sum_{k,\lambda} \left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2} (a_{k\lambda}\hat{\varepsilon}_{k\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} + a^{\dagger}_{k\lambda}\hat{\varepsilon}^{*}_{k\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}}) \tag{1.1.6}$$

where $\hat{\varepsilon}_{k\lambda}$ is the unit polarization vector of the mode and V is the quantization volume of the field. We have specialized to a plane wave complete set for the eigenmodes since this is a convenient expansion but any other complete set could have been used. The transversality condition is realized by the following constraints on the polarization vectors:

$$\mathbf{k} \cdot \hat{\varepsilon}_{k\lambda} = 0 \tag{1.1.7}$$

and the remaining quantities in the Hamiltonian are conventional. The j sum runs over all particles, and the last term in (1.1.2) is the interaction among the particles other than those mediated by the quantum electromagnetic field. For our purposes it will usually be taken to be the sum of the two-body Coulomb interactions among charged pairs. The wave function in (1.1.1) is therefore a function in the configuration space of the particles and the Hilbert space of all of the modes of the quantum-electrodynamic field. This is the usual starting point for the treatment of the interaction of radiation and matter, which is a much broader problem than the one of interest here.

1.2. Transition to a Classical Description of the Laser

We are interested in the interaction of lasers, a very special kind of radiation, with matter. The crucial point which distinguishes lasers from other radiation, for our purposes, is their high intensity and their coherence properties. More specifically, it is the large number of photons in a laser mode. For example, a laser with 1-eV photons with a single mode flux of 1 mW/cm^2 in a typical coherence volume of 1 cm^3 has about 2×10^5 photons in the field:

$$N = \frac{\text{flux}}{\hbar\omega}\frac{V}{c} \simeq \frac{10^{-3}\,\text{W/cm}^2}{1\,\text{eV}}\,\frac{1\,\text{cm}^3}{3\times 10^{10}\,\text{cm/sec}} \simeq 2\times 10^5.$$

This very high quantum number makes it likely that the laser is accurately describable as a classical electromagnetic field. This will now be demonstrated.

We first transform to an interaction representation in which the time evolution due to the field energy, H_{rad}, is absorbed into the wave function

$$\Psi = \exp(-iH_{\text{rad}}t/\hbar)\,\Psi_I \tag{1.2.1}$$

then

$$\left[i\hbar\frac{\partial}{\partial t} - H_I(t)\right]\Psi_I = 0 \tag{1.2.2}$$

where

$$H_I(t) = \sum_{j=1}^{j_{\max}} \frac{\left[\mathbf{P}_j - \dfrac{e_j}{c}\mathbf{A}(\mathbf{r}_j, t)\right]^2}{2m_j} + V(r_1 \cdots r_{j_{\max}}) \tag{1.2.3}$$

and the time dependence introduced into the vector potential by this transformation is given by

$$\begin{aligned}\mathbf{A}(\mathbf{r}, t) &= \exp(iH_{\text{rad}}t/\hbar)\,\mathbf{A}(\mathbf{r})\exp(-iH_{\text{rad}}t/\hbar)\\ &= \sum_{k\lambda}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2}\\ &\quad\times\{a_{k\lambda}\hat{\varepsilon}_{k\lambda}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)] + a_{k\lambda}\hat{\varepsilon}^*_{k\lambda}\exp[-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)]\}.\end{aligned} \tag{1.2.4}$$

Now we transform to the phase representation[2] for the field. Independent coordinates, $\phi_{k\lambda}$, with range $0 \le \phi_{k\lambda} \le 2\pi$, are introduced for each mode and a state with $n_{k\lambda}$ photons in the k, λ mode is described in this coordinate space by

$$n_{k\lambda}\rangle = \frac{1}{(2\pi)^{1/2}}\,e^{in_{k\lambda}\phi_{k\lambda}}. \tag{1.2.5}$$

The number operator is then

$$\mathbf{n}_{k\lambda} = \frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}} \tag{1.2.6}$$

and the creation and destruction operators are

$$a_{k\lambda} = e^{-i\phi_{k\lambda}}\left(\frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}}\right)^{1/2}, \qquad a_{k\lambda}^{\dagger} = \left(\frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}}\right)^{1/2} e^{i\phi_{k\lambda}} \tag{1.2.7}$$

which are readily shown to satisfy the commutation relations (1.1.5). This transformation is not a particularly useful one in the general case since the square root of a derivative is difficult to work with, but when the mode occupation numbers are high, as they are for lasers operating well above their lasing threshold,[3] then it can be exploited to great use. For the laser modes only, we let

$$n_{k\lambda} = N_{k\lambda} + \nu_{k\lambda} \tag{1.2.8}$$

where $N_{k\lambda}$ is some average value for the laser mode occupation number during the process of interest and $\nu_{k\lambda}$ is the variation about that number. We shall be interested in $\nu_{k\lambda}$ as large as perhaps 10^3 for some processes but that is still much smaller than typical values of $N_{k\lambda}$ of interest. For this reason we make the unitary transformation which relabels the laser mode states by the index $\nu_{k\lambda}$ rather than $n_{k\lambda}$

$$|n_{k\lambda}\rangle = e^{iN_{k\lambda}\phi_{k\lambda}}|\nu_{k\lambda}\rangle \tag{1.2.9}$$

so that the operators are changed to

$$\begin{aligned} a_{k\lambda} &= e^{-i\phi_{k\lambda}}\left(N_{k\lambda} + \frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}}\right)^{1/2} \\ a_{k\lambda}^{\dagger} &= \left(N_{k\lambda} + \frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}}\right)^{1/2} e^{i\phi_{k\lambda}}\,. \end{aligned} \tag{1.2.10}$$

Then noting that in this representation, $(1/i)(\partial/\partial\phi_{k\lambda})$ is of the order of $\nu_{k\lambda}$ these may be expanded

$$\begin{aligned} a_{k\lambda} &= e^{-i\phi_{k\lambda}}\sqrt{N_{k\lambda}}\left(1 + \frac{1}{2N_{k\lambda}}\frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}} + \cdots\right) \\ a_{k\lambda}^{\dagger} &= \sqrt{N_{k\lambda}}\left(1 + \frac{1}{2N_{k\lambda}}\frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}} + \cdots\right) e^{i\phi_{k\lambda}}. \end{aligned} \tag{1.2.11}$$

If only the leading terms are kept then

$$\begin{aligned} \mathbf{A} = \sum_{k\lambda}\left(\frac{2\pi\hbar c^2 N_{k\lambda}}{\omega_{k\lambda}V}\right)^{1/2} &\{\hat{\varepsilon}_{k\lambda}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega_k t - \phi_{k\lambda})] \\ &+ \hat{\varepsilon}_{k\lambda}^{*}\exp[-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t - \phi_{k\lambda})]\} + \delta\mathbf{A} \end{aligned} \tag{1.2.12}$$

where

$$\delta \mathbf{A} = \frac{1}{2}\sum_{k\lambda}\left(\frac{2\pi\hbar c^2}{N_{k\lambda}\omega_k V}\right)^{1/2}\left\{\hat{\varepsilon}_{k\lambda}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega_k t - \phi_{k\lambda}) \frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}} + \hat{\varepsilon}^*_{k\lambda}\frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}}\exp[-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t - \phi_{k\lambda})]\right\}. \quad (1.2.13)$$

The first term in (1.2.12) is exactly the form of a classical electromagnetic vector potential with a mode amplitude

$$|A_{k\lambda}| = \frac{c}{\omega_k}|E_{k\lambda}|, \qquad \mathbf{E}_{k\lambda} = \left(\frac{8\pi\hbar\omega_{k\lambda}N_{k\lambda}}{V}\right)^{1/2}\hat{\varepsilon}_{k\lambda}\,. \quad (1.2.14)$$

This relates the amplitude of a classical electromagnetic field to the more fundamental description in terms of the occupation numbers of the field or the density of photons in the field. The term $\delta\mathbf{A}$ is the first quantum correction which is seen to be small for lasers oscillating well above their lasing thresholds, $N_{k\lambda} \gg 1$.

With the prescription

$$\mathbf{A}(\mathbf{r}, t) = \sum_{k\lambda}\frac{c}{2\omega_k}\{\mathbf{E}_{k\lambda}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega_k t - \phi_{k\lambda})] + \mathbf{E}^*_{k\lambda}\exp[-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t - \phi_{k\lambda})]\} \quad (1.2.15)$$

the Schrödinger equation, (1.2.2), describes the particles in the field of an operator which looks like a classical prescribed electromagnetic field with the phase parameters, $\phi_{k\lambda}$ which are still operators.

For a single-mode laser the one phase parameter, which occurs as $(\omega_k t + \phi_{k\lambda})$, can be absorbed into a translation of t and so will not enter into any physical results. However, in a multimode laser, only one of the phase parameters can be eliminated in this way and so we can expect that there are physical results which will depend on the relative values of these parameters. They are usually unknown and so ensemble averages over them are necessary. This is discussed briefly in Section 1.6 of this chapter.

The preceding discussion dealt with the transition to a classical description of the laser field for the case when the numbers of photons in each of the laser modes were large. A complementary derivation of this transition was given by Mollow.[4] It is based on the coherent states of the electromagnetic field. These states have been shown[5] to be the quantum electrodynamic states which most closely approximate the classical state of the field. They can be defined as eigenstates of the photon destruction operator, $a_{k\lambda}$, for the mode under discussion. These states of the radiation field for the (k, λ) mode are

$$\psi_{k\lambda}(t) = \sum_{n=0}^{\infty} e^{-in\omega_k t}\xi_n|n\rangle \tag{1.2.16}$$

and the requirement that this be a normalized eigenstate of $a_{k\lambda}$ results in

$$\xi_n = e^{-\langle n_{k\lambda}\rangle/2}\frac{\alpha_{k\lambda}^n}{(n!)^{1/2}} \tag{1.2.17}$$

where

$$\alpha_{k\lambda} = (\langle n_{k\lambda}\rangle)^{1/2}\, e^{-i\phi_{k\lambda}} \tag{1.2.18}$$

and $\langle n_{k\lambda}\rangle$ is the average occupation number of the $(k\lambda)$ mode. In that case

$$a_{k\lambda}\psi_{k\lambda}(t) = e^{-i\omega_k t}\alpha_{k\lambda}\psi_{k\lambda}(t) \tag{1.2.19}$$

and the expectation value of the vector potential operator for the $(k\lambda)$ mode, (1.1.6), is

$$\langle \psi_{k\lambda}(t), A_{k\lambda}(\mathbf{r}, t)\, \psi_{k\lambda}(t)\rangle = \frac{c\mathbf{E}_{k\lambda}}{\omega_k}\cos(\omega_k t - \mathbf{k}\cdot\mathbf{r} + \phi'_{k\lambda}). \tag{1.2.20}$$

Here $\mathbf{E}_{k\lambda}$ is the electric field amplitude in this mode, (1.2.14). This is precisely the classical value of the vector potential. However, it can be shown that the correspondence goes even further. If the initial state $(t = 0)$ of the radiation field for this mode is a coherent state then the occupation of that state can be expressed as a unitary transformation of the vacuum which starts from the expression of (1.2.17) as

$$\psi_{k\lambda}(0) = e^{-\langle n_{k\lambda}\rangle/2}e^{a_{k\lambda}^\dagger \alpha_{k\lambda}}|0\rangle \tag{1.2.21}$$

where use has been made of

$$a_{k\lambda}^\dagger|n\rangle = (n + 1)^{1/2}\,|n + 1\rangle\,. \tag{1.2.22}$$

Then the unitary operator

$$D(\alpha_{k\lambda}) = \exp(a_{k\lambda}^\dagger\alpha_{k\lambda} - a_{k\lambda}\alpha_{k\lambda}^*) \tag{1.2.23}$$

can be rearranged to give

$$D(\alpha_{k\lambda}) = e^{a_{k\lambda}^\dagger\alpha_{k\lambda}}e^{-a_{k\lambda}\alpha_{k\lambda}^*}e^{-|\alpha_{k\lambda}|^2/2} \tag{1.2.24}$$

and

$$\psi_{k\lambda}(0) = D(\alpha_{k\lambda})|0\rangle\,. \tag{1.2.25}$$

This unitary transformation has the property

$$D^{-1}(\alpha_{k\lambda})\, a_{k\lambda}D(\alpha_{k\lambda}) = a_{k\lambda} + \alpha_{k\lambda} \tag{1.2.26}$$

and its Hermitian conjugate. These can be combined to transform the Hamiltonian (1.2.3) in such a way that the only change is the replacement

$$\mathbf{A}(\mathbf{r}, t) \rightarrow \mathbf{A}(\mathbf{r}, t) + \left(\frac{2\pi\hbar c^2}{V\omega_k}\right)^{1/2}$$

$$\times \{\alpha_{k\lambda}\hat{\varepsilon}_{k\lambda} \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)] + \alpha^*_{k\lambda}\hat{\varepsilon}^*_{k\lambda} \exp[-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)]\}. \quad (1.2.27)$$

Then the use of (1.2.18), (1.2.15), and (1.2.14) shows that this addition to $\mathbf{A}$ is just the classical value of the vector potential for that mode. The results shows that the transformation (1.2.24) changes the initial state, occupied as a coherent photon state, into a vacuum state and compensates by adding a classical electromagnetic potential to the quantum electrodynamic operator. The operator [the first term on the right-hand side of (1.2.27)] will cause fluctuations about the classical field, but if these fluctuations are small compared to the classical field then the operator may be dropped and we arrive at a result similar to that obtained in (1.2.15).

1.3. Dipole Approximation, Center-of-Mass Transformation, Ponderomotive Potential

The particle coordinates of the Hamiltonian (1.2.3) can be transformed into a center-of-mass coordinate, $\boldsymbol{\rho}$, and relative coordinates $\boldsymbol{\chi}_i$. Such that

$$\mathbf{r}_i = \boldsymbol{\rho} + \boldsymbol{\chi}_i \quad (1.3.1)$$

and the vector potential $\mathbf{A}(\mathbf{r}_i, t)$ is then a function of this coordinate through the factors $\exp[\pm i\mathbf{k}\cdot(\boldsymbol{\rho} + \boldsymbol{\chi}_i)]$. The relative coordinate, $\boldsymbol{\chi}_i$, will usually be limited to the size of the atom in any matrix elements that occur and this will make the factor $\exp(\pm i\mathbf{k}\cdot\boldsymbol{\chi}_i)$ only slightly different from unity since

$$\langle \mathbf{k}\cdot\boldsymbol{\chi}_i\rangle \simeq ka_0 \simeq \alpha_F \simeq 137^{-1}.$$

An expansion in powers of $\mathbf{k}\cdot\boldsymbol{\chi}_i$ is then indicated and usually only the leading term is kept. This is the dipole approximation.[6] In cases when such matrix elements vanish the expansion can be carried further but we shall not be concerned with that here.

The center-of-mass coordinate can couple to the laser through the remaining factors $e^{\pm i\mathbf{k}\cdot\boldsymbol{\rho}}$ so that the motion of the atom as a whole can be affected by the laser. We shall return to this in Chapter 5.

In case the forces on the center of mass are small (as is not unusual) then the coordinate, $\boldsymbol{\rho}$, is a constant and it may be removed by a redefinition of the phases of the modes. Moreover, when $\boldsymbol{\rho}$ is not exactly a constant but is instead a slowly varying function of time, its effect can be seen to be the same as a slow drift in the mode phases as a function of time.

We introduce a new set of coordinates in terms of the nuclear coordinate, $\mathbf{R}_N$ and the electron coordinates, $\mathbf{r}_i$

$$\begin{aligned} \boldsymbol{\rho} &= \left(M_N \mathbf{R}_N + \sum_{i=1}^{Z} m\mathbf{r}_i \right) \Big/ M_A \\ \boldsymbol{\chi}_i &= \mathbf{r}_i - \mathbf{R}_N \qquad (1.3.2) \\ M_A &= M_N + Zm \end{aligned}$$

where M_N is the nuclear mass so that M_A is the atomic mass. Then, in these variables, the Hamiltonian (1.2.3) in the dipole approximation becomes (we now drop the subscripts on H_I and Ψ_I)

$$H = \frac{P_\rho^2}{2M_A} + \frac{1}{2\mu} \sum_{i=1}^{Z} \left[\mathbf{P}_i + \frac{e}{c}\mathbf{A}(\boldsymbol{\rho}, t) \right]^2 + \frac{1}{M_N} \sum_{i>j=1}^{Z} \left[\mathbf{P}_i + \frac{e}{c}\mathbf{A}(\boldsymbol{\rho}, t) \right] \cdot \left[\mathbf{P}_j + \frac{e}{c}\mathbf{A}(\boldsymbol{\rho}, t) \right] + V(\chi, \rho) \qquad (1.3.3)$$

where the reduced mass is

$$\mu = \frac{mM_N}{M_N + m} \cdot \qquad (1.3.4)$$

The Hamiltonian can be rewritten as

$$H = \left[\frac{P_\rho^2}{2M_A} + \frac{Ze^2}{2\mu c^2} A^2(\boldsymbol{\rho}, t) \right] + \left[\frac{1}{2\mu} \sum_{i=1}^{Z} P_i^2 + \frac{1}{M_N} \sum_{i>j=1}^{Z} \mathbf{P}_i \cdot \mathbf{P}_j + V(x, \rho) \right] + \left[\frac{e}{\mu c} \mathbf{A}(\boldsymbol{\rho}, t) \cdot \sum_{i=1}^{Z} \mathbf{P}_i \right]. \qquad (1.3.5)$$

The first two terms describe the kinetic energy of the center-of-mass motion and a time-dependent potential acting on the center-of-mass motion. If we write (for a single-mode laser)

$$\mathbf{A}(\boldsymbol{\rho}, t) = \frac{c\mathbf{E}(\rho)}{\omega} \cos(\omega t + \mathbf{k} \cdot \boldsymbol{\rho}) \qquad (1.3.6)$$

where $\mathbf{E}(\rho)$ is the amplitude of the electric field of the laser which is explicitly position dependent, then

$$A^2(\boldsymbol{\rho}, t) = \frac{c^2}{2\omega^2} E^2(\boldsymbol{\rho}) + \frac{c^2}{2\omega} E^2(\boldsymbol{\rho}) \cos 2(\omega t + \mathbf{k} \cdot \boldsymbol{\rho}) \tag{1.3.7}$$

The effect of the second term on the center-of-mass motion is usually negligible since the mass of the atom is too large for it to respond significantly to this rapidly oscillating term. The first term of (1.3.7) will, however, contribute a time-independent potential which acts on the center-of-mass motion, the center-of-mass ponderomotive potential

$$U_P^{\text{CM}}(\boldsymbol{\rho}) = \frac{Ze^2}{4m\omega^2} E^2(\boldsymbol{\rho}). \tag{1.3.8}$$

(We have set $M_A/M_N = 1 + Zm/M_N \simeq 1$.) Notice that the mass that enters here is the electron mass so that (1.3.8) is not negligible in some cases. There will be other contributions to potentials which act on the center-of-mass motion via the second-order contribution of the last term of (1.3.5). This is a subject which we will return to in Chapters 5 and 6.

The second bracket of (1.3.5) is the atomic Hamiltonian in the absence of the laser and the last is the coupling of the laser to the internal coordinates of the atom. The effect of this coupling on the center-of-mass motion may not be negligible (Chapter 5). However, in almost all cases the center-of-mass motion can be described classically (Chapter 5, Sec. 2) so that $\boldsymbol{\rho}$ becomes a function of t and moreover it will be a slowly varying function of time on the time scale of the internal motion of the atom. (A typical period for internal atomic motion is 10^{-16} sec.) A fast atom moving at 10^7 cm/sec will travel 10^{-9} cm in that time and so the external environment in which the atom finds itself will be a very slowly varying function on its own time scale. The external conditions, $\mathbf{E}(\boldsymbol{\rho})$, can be treated as constant or adiabatically varying and the origin may be taken at $\rho = 0$ for the purposes of the next section in which $\mathbf{A}(\boldsymbol{\rho}, t)|_{\rho=0} \equiv \mathbf{A}(t)$.

1.4. Gauge Transformations

The Schrödinger equation (as described in Section 1.3) can be written

$$\left[i\hbar \frac{\partial}{\partial t} - H_1(t) \right] \Psi_1 = 0 \tag{1.4.1}$$

where our new notation is