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FOREWORD

Every year since 1984 the conference of Universal and Applied Algebra has been held in Poland. It has been organized cyclically by the Departament of Universal and Applied Algebra of the Institute of Mathematics of the Nicholas Copernicus University in Toruń, the Departament of Algebra in the Institute of Mathematics in the Pedagogical University of Opole and the Departament of Algebraic Methodology of the Istitute of Mathematics of the Warsaw Technical University.

The material contained in these Proceedings represents rather faithfully the scope of the V Symposium. What is not mirrored here is the interactive and friendly atmosphere of the Symposium.

We wish to thank all the lecturers and all the participants for creating this atmosphere. We would like to extend special thanks to Professor Stanisław Kochman - the Rector of the Pedagogical University of Opole and Professor Witold Wacławek vice -Rector of the University óf Opole for their constant help in the organization of the Conference.

Finally, this book could not have been produced without the active co-operation of the members of the Editorial Board particullary Professor Milan Kolibiar for his contribution to having this volume printed.

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Katarzyna Hałkowska

Bogusław Stawski

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TERMAL ALGEBRAS - AN APPROACH TO INFERENCE
ON INCOMPLETE DATA

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Abstract. The notion of a termal (partial) algebra (introduced informally by I.Fleischer) is defined in an intent to create a general algebraic model for systems of inference (or action)over incomplete data. We introduce the notion of a consistent and strongly consistent termal algebra and we prove that inconsistency is a local property of finite character i.e. a termal algebra is inconsistent iff it contains an inconsistent finite relative subalgebra. We also prove that the poset of all consistent congruences of a termal algebra (i.e.of those congruences which yield consistent quotients) is an algebraic lattice.

In [Fl1] and [Fl2] I.Fleischer considered informally defined partial algebras with operations induced by terms rather than by fundamental operation symbols. Such algebras (called termal algebras here) could be viewed on a very general level as algebraic models of systems of inference (or actions) on incomplete data. In this interpretation the fundamental operations correspond to primary deduction rules (or elementary actions),while term operations

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correspond to secondary deduction rules (or complex actions), which can yield a result even when not all the intermediate steps are defined. For instance, the result of an application of **if** α **then** t_I **else** t_2 could be defined as t_I for α true, even when t_2 is undefined at a given valuation of variables. In this approach one may ask whether such an extension of rules is consistent i.e. whether there is some world of reference in which all values (including the "default" ones) can be interpreted. In an abstract setting total (termal) algebras seem to be good candidates for such worlds of reference. The existence of an embedding of a termal algebra into a total one could be understood as a possibility of completing the partial knowledge represented by the termal algebra or, putting it in other words, as the existence of a complete context in which this knowledge is valid. This leads to the definition of a consistent termal algebra, which is the main subject of this paper.

Basic Definitions

Let (F, n) be an algebraic type i.e. F is a set of fundamental operation symbols and $n: F \longrightarrow N$ is the arity function into the set N of all non-negative integers (for convenience we consider finitary operations only). Let $T_F(X)$ be the set of all F-terms over a fixed countably infinite set X of variables and denote by T_F the quotient of $T_F(X)$ obtained via an identification of terms which differ only by a permutation on the set X. This means that we identify two terms $t_I, t_2 \in T_F(X)$ with $t_I = t_I(x_I, \ldots, x_n)$ iff there is a bijection $\pi: X \longrightarrow X$ such that t_2 is obtained from t_I by substituting $\pi(x_i)$ for x_i for $i = 1, \ldots, n$. Thus any $t \in T_F$ represents an "abstract" construction of a term, independent of the names of variables. We shall use objects in T_F as if they were just terms.

Note that the arity function n of a type (F,n) induces a new function $n^*:T_F\longrightarrow N$ such that for any $t\in T_F$, $n^*(t)=|var(t)|$ (the number of different variables in t) is the arity of the term t. We may now consider a new type (T_F,n^*), which in turn induces a new set of terms $T_{T_F}(X)$ (which shall be abbreviated to $T_T(X)$) Clearly any term $\tau\in T_{T_F}(X)$ has a unique decomposition into a term $t\in T_F(X)$, which will be denoted by $\tau^{\#}$. Hence e.g. if F consists of a single unary symbol f, then T_F can be identified with the set $\{f^n(x):n\in N\}$, where $f^0(x)=x$, $f^{n+1}(x)=f(f^n(x))$ for $n\in N$. Further,

$$T_T(X)=\{f^k(f^l(\ldots(f^m(x))\ldots):k,l,\ldots,m\in N,\ x\in X\}$$

and, for instance,

$$f^4(x)^{\#}=f^2(f^2(x))^{\#}=f^3(f(x))^{\#}=f(f(f(f(x))))\neq f^3(f(y))\ (\text{if } x\neq y)$$

We are now ready to define the basic notion: **given a** type (F,n), a partial algebra $\mathfrak{U}=(A,(t^{\mathfrak{U}})_{t\in T_F})$ of type (T_F,n^*) is a *termal algebra* iff all equations of the form

$$\tau\approx\sigma$$

hold weakly in $\mathfrak{U}$ for all $\tau,\sigma\in T_T(X)$ with $\tau^{\#}=\sigma^{\#}$.

Recall that for any $t\in T_F$, $t^{\mathfrak{U}}$ denotes a partial n^*-ary function on A (an interpretation of the operation symbol t in the algebra $\mathfrak{U}$) and then for any $\tau\in T_T(X)$, $\tau^{\mathfrak{U}}$ will be the induced term function on A. An equation $\tau\approx\sigma$ holds weakly in a partial algebra $\mathfrak{U}$ iff $\tau^{\mathfrak{U}}(\underline{a})=\sigma^{\mathfrak{U}}(\underline{a})$ for any valuation $\underline{a}$ of variables, which belongs to the domain of both $\tau^{\mathfrak{U}}$ and $\sigma^{\mathfrak{U}}$.

The following example illustrates the notion of a termal algebra:

Ex.1 Let $F=\{f\}$ with $n(f)=1$. Then a termal algebra of type (T_F,n^*) is a partial algebra $\mathfrak{U}=(A,(f^{n\mathfrak{U}})_{n\in N})$, where for each $n\in N$, $f^{n\mathfrak{U}}$ is a unary partial operation on $\mathfrak{U}$ and all the equations of the form

$$f^k(f^l(\ldots(f^m(x))\ldots))=f^p(f^r(\ldots(f^s(x))\ldots))\qquad (I)$$

with $k+l+\ldots+m=p+r+\ldots+s$ $(k,l,\ldots,m,p,r,\ldots,s\ \in\ N)$ and $x\in X$

hold weakly in $\mathfrak{U}$.

In particular, let $A=\{a,b,c,d\}$ be a four-element set and let $\mathfrak{U}=(A,(f^{n\mathfrak{U}})_{n\in N})$ be a unary partial algebra such that:

$$\text{dom } f^{\mathfrak{U}}=\text{dom } f^{3\mathfrak{U}}=\text{dom } f^{5\mathfrak{U}}=\{a\}$$
$$\text{dom } f^{n\mathfrak{U}}=\emptyset \text{ for } n\neq 1,3,5$$
$$f^{\mathfrak{U}}(a)=b, \quad f^{3\mathfrak{U}}(a)=c, \quad f^{5\mathfrak{U}}(a)=d$$

Consider now any equation of the form (1). Then it can be easily verified that for any valuation $\underline{a}$ of variables in A either both sides are identical or at least one of them is undefined. Thus all equations of the form (1) hold weakly in $\mathfrak{U}$ and therefore $\mathfrak{U}$ is a termal algebra.

A natural class of termal algebras is obtained in the following way: take any partial algebra' $\mathfrak{U}=(A,(f^{\mathfrak{U}})_{f\in F})$ of type (F,n). For any $t\in T_F$ define $t^{\mathfrak{U}}$ as the term function induced in the usual way by the fundamental operations of $\mathfrak{U}$. Then we get a termal algebra $(A,(t^{\mathfrak{U}})_{t\in T_F})$, which shall be called the *canonical termal algebra over* $\mathfrak{U}$. Observe that a total termal algebra (of type (T_F,n^*)) is always a canonical termal algebra over its F-reduct. We shall usually omit the distinction between these two algebras.

Basic algebraic notions carry over from "ordinary" partial algebras to termal algebras in a natural way. We only recall some of them for the sake of completeness. A mapping $h:A\longrightarrow B$ is a *homomorphism* of a termal algebra $\mathfrak{U}=(A,(t^{\mathfrak{U}})_{t\in T_F})$ into a similar termal algebra $\mathfrak{B}=(B,(t^{\mathfrak{B}})_{t\in T_F})$ iff for any $t\in T_F$ and $\underline{a}\in\text{dom } t^{\mathfrak{U}}$, $h\cdot\underline{a}\in\text{dom } t^{\mathfrak{B}}$ and then $h(t^{\mathfrak{U}}(\underline{a}))=t^{\mathfrak{B}}(h\cdot\underline{a})$. If moreover $h\cdot\underline{a}\in\text{dom } t^{\mathfrak{B}}$ implies $\underline{a}\in\text{dom } t^{\mathfrak{U}}$, then h is a *closed homomorphism*. An injective homomorphism will be called an *embedding*.

For any termal algebras $\mathfrak{U}=(A,(t^{\mathfrak{U}})_{t\in T_F})$ and $\mathfrak{B}=(B,(t^{\mathfrak{B}})_{t\in T_F})$ with $A\subseteq B$, $\mathfrak{U}$ is a *weak subalgebra* of $\mathfrak{B}$ iff $t^{\mathfrak{U}}\subseteq t^{\mathfrak{B}}$ for all $t\in T_F$. $\mathfrak{U}$ is a *relative subalgebra* of $\mathfrak{B}$ iff $t^{\mathfrak{U}}=t^{\mathfrak{B}}\cap A^{n^*(t)+1}$ for all $t\in T_F$. Finally, $\mathfrak{U}$ is a *(closed)*

subalgebra of $\mathcal{B}$ iff $t^{\mathcal{U}} = t^B \cap (A^{n^*(t)} \times B)$ for all $t \in T_F$.

If $\mathcal{U}$ is a weak subalgebra of $\mathcal{B}$, $A=B$ and $\mathcal{B}$ is total then we shall call $\mathcal{B}$ an *inner completion* of $\mathcal{U}$.

Algebraic constructions used in this paper such as direct product, direct limit, homomorphic image etc. ar generalized similarly (for precise definitions se e.g.[Bur]).
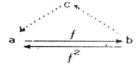
Consistency

Let $\mathcal{U}$ be a termal algebra. $\mathcal{U}$ is said to be *consistent* iff it can be embedded into a total termal algebra. It is *strongly consistent* iff it has an inner completion (to a total termal algebra). $\mathcal{U}$ is *inconsistent* iff it is not consistent. Hence, any canonical termal algebra is both consistent and strongly consistent. Any strongly consistent termal algebra is consistent. The following example shows that the converse may not be true.

Ex 2 Let $A=\{a,b\}$ and let f be a unary operation symbol. Define a termal algebra $\mathcal{U} = (A, (f^{n\mathcal{U}})_{f \in F})$ so that

$$\text{dom } f^{\mathcal{U}} = \{a\} \text{ and dom } f^{2\mathcal{U}} = \{b\}$$
$$\text{dom } f^{n\mathcal{U}} = \emptyset \text{ for } n \neq 1,2$$
$$f^{\mathcal{U}}(a)=b \text{ and } f^{2\mathcal{U}}(b)=a$$

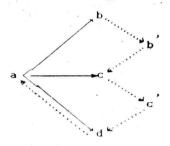As in the previous example it can be easily seen that $\mathcal{U}$ is a termal algebra indeed. It is consistent, since it can be embedded into a total algebra as in the following diagram(the dotted lines represent the completion of $f^{\mathcal{U}}$)

On the other hand, there is no completion on the set $\{a,b\}$, which would extend all the operations $f^{n\mathcal{U}}$ for $n \in N$. Indeed,

if $\mathfrak{C}=(\langle a,b \rangle, f^{\mathfrak{C}})$ were a total algebra completing $\mathfrak{U}$, then for any possible value of $f^{\mathfrak{C}}(b)$ we would have $f^{2\mathfrak{C}}(b)=b \neq f^{2\mathfrak{U}}(b)$. Hence $\mathfrak{U}$ is not strongly consistent.

Ex.3 Let $\mathfrak{U}$ be the algebra of Ex.1. It is consistent, since it can be embedded into a total algebra as indicated below:

However, if we modify $\mathfrak{U}$ into $\mathfrak{U}'$ only by identifying b and c:

then $\mathfrak{U}'$ becomes inconsistent, since for any total algebra $\mathfrak{C}$ which extends $\mathfrak{U}'$ we would have both

$$f^{2\mathfrak{C}}(c)=c$$

(since $f^{2\mathfrak{C}}(c)=f^{2\mathfrak{C}}(f^{\mathfrak{C}}(a))=f^{3\mathfrak{C}}(a)=f^{3\mathfrak{U}'}(a)=c$) and

$$f^{2\mathfrak{C}}(c)=d$$

(since $f^{2\mathfrak{C}}(c)=f^{2\mathfrak{C}}(f^{3\mathfrak{C}}(a)=f^{5\mathfrak{C}}(a)=f^{5\mathfrak{U}'}(a)=d$).

Observe that the mapping $h:(a,b,c,d) \longrightarrow (a, c, d)$ such that $h(b)=c$ and $h(x)=x$ for $x \neq b$ is a closed homomorphism of the consistent termal algebra $\mathfrak{U}$ onto its inconsistent modification $\mathfrak{U}'$. This proves that the class of all consistent termal algebras of a fixed type is not closed with respect to homomorphic images (even if only closed homomorphisms are considered). However, the following holds true:

Theorem 1 The class of all consistent termal algebras

of a fixed type is closed with respect to

(i) isomorphic images

(ii) weak subalgebras

(iii) direct products.

P r o o f.(i) and (ii): Let $\mathfrak{U}$ be a consistent termal algebra and h an embedding of $\mathfrak{U}$ into a total algebra $\mathfrak{B}$. If j is an isomorphism (or an identical embedding, respectively) of $\mathfrak{C}$ into $\mathfrak{U}$, then $h \cdot j$ is an embedding of $\mathfrak{C}$ into $\mathfrak{B}$.

(iii): Let $\mathfrak{U}_i, i \in I$, be a family of consistent termal algebras. Then any family of embeddings h_i of $\mathfrak{U}_i$ into a total algebra $\mathfrak{B}_i$ for $i \in I$ induces an embedding of the direct product $\prod \mathfrak{U}_i$ into the total direct product $\prod \mathfrak{B}_i$. □

Observe that the proofs of (i) and (iii) in the above theorem carry over to the case of strongly consistent termal algebras. Thus

Theorem 2 The class of all strongly consistent termal algebras of a fixed type is closed with respect to

(i) isomorphic images

(ii) direct products. □

A weak subalgebra (nor even a closed one) of a strongly consistent termal algebra need not be strongly consistent (though it will always be consistent):
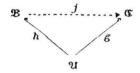
Ex. 4 Let $\mathfrak{U}$ be the algebra of Ex.2. If $\mathfrak{B}=(\{a,b,c\}, f^{\mathfrak{B}})$ differs from $\mathfrak{U}$ only in having one additional (isolated) element c in the universe, then $\mathfrak{U}$ is a subalgebra of $\mathfrak{B}$. Moreover, $\mathfrak{B}$ is strongly consistent, while $\mathfrak{U}$ is not.

The Compactness Theorem

We shall prove here that the consistency of a termal algebra depends on its finite fragments or – equivalently – that a termal algebra is inconsistent iff it contains an inconsistent finite substructure.

Let K be a class of similar partial algebras and let $\mathfrak{U}$

be a partial algebra of the same type as all the algebras in K. A partial algebra $\mathcal{B} \in K$ is a *K-reflection* of $\mathcal{U}$ iff there exists a homomorphism $h: \mathcal{U} \longrightarrow \mathcal{B}$ such that for any algebra $\mathcal{C} \in K$ and any homomorphism $g: \mathcal{U} \longrightarrow \mathcal{C}$ there exists a homomorphism (unique) $j: \mathcal{B} \longrightarrow \mathcal{C}$ such that the diagram is commutative:

An algebra $\mathcal{B} \in K$ is a *K-monoreflection* of $\mathcal{U}$ iff the homomorphism h in the above definition can be chosen to be injective.

A subclass M of K is *(mono)reflective in K* iff every algebra $\mathcal{U}$ in K has an M-(mono)reflection.

The following theorem gives sufficient conditions for M to be reflective in any class of partial algebras which contains it:

<u>Theorem 3</u> ([Sch]) Let $\mathcal{U}$ be an arbitrary partial algebra and let M be a class of partial algebras of the same type as $\mathcal{U}$ such that M is closed with respect to

(i) isomorphic images

(ii) (closed) subalgebras

(iii) direct products.

Then $\mathcal{U}$ has an M-reflection. □

Now let $\mathcal{T}$ be the class of all termal algebras of some fixed type and let $\mathcal{CT}$ be the subclass of all consistent termal algebras, $\mathcal{TT}$ – the class of all total termal algebras of the same type. Then from Theorems 1 and 3 we deduce the following

<u>Corollary</u> The classes $\mathcal{CT}$ and $\mathcal{TT}$ are both reflective in the class $\mathcal{T}$. □

Moreover, a simple argument leads to the following property of consistent termal algebras: