PLANE WAVES AND SPHERICAL MEANS

APPLIED TO PARTIAL DIFFERENTIAL EQUATIONS

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INTRODUCTION

This tract contains a somewhat heterogeneous collection of results on partial differential equations. The unifying element is the use of certain elementary identities for plane and spherical integrals of an arbitrary function. It is the aim of the author to show how a variety of results on fairly general differential equations follows from those identities.

The use of ordinary euclidean planes and spheres in connection with general differential equations represents a departure from the idea that it is best to work with geometric entities like characteristic conoids, which are associated in an invariant manner with the differential equation. It is probably true that the finer structure of the solutions is only brought out by using an invariant approach, adjusted to the individual equation. On the other hand experience shows that many results have been obtained more easily by employing cruder unspecific tools, such as power series, Fourier integrals, finite difference approximations, or L^2-norms. The characteristic properties of the individual equation then enter only through *inequalities* instead of equations, and the considerable difficulties inherent in the use of singular integrals over characteristic conoids are avoided. In the same way it will be seen here that integrals over ordinary spheres and planes can be used to advantage even for equations that are not related to the ordinary euclidean metric. In such cases the use of these euclidean objects will introduce certain artificial (non-invariant) features. This is compensated for by the simplicity and symmetries of ordinary planes and spheres compared with the corresponding objects (if any) in whatever geometry might be associated *naturally* with the differential equation.

Most of the results given here can be found scattered in the literature, though possibly with different degrees of generality. A conscientious effort has been made to give appropriate credit to other authors and to provide references to related material.

No attempt has been made to give a historical survey and to decide more intricate questions of priority. This would represent a formidable task in the field of partial differential equations, where the actual results of various workers often do not differ as much (and perhaps are not of as much interest) as their emphasis on some specific unifying point of view. The results given here have been selected so as to demonstrate best the usefulness of plane waves and spherical means. As far as possible the treatment has been made elementary and self-contained. It is clear that this imposes severe restrictions on the choice of topics and precludes an exhaustive treatment of any one subject.

The basic identities applied in this monograph are contained in Chapters I and IV. Chapter I deals with the *decomposition of arbitrary functions into functions of the type of plane waves*, i.e. into functions that have parallel planes as level surfaces. Fourier analysis provides one such decomposition, namely into *plane waves of exponential type*. For many applications the exponential character is not essential, and more elementary ways of decomposing a function into plane waves can be used. One such method of decomposition is given here. It consists of expressing a function by spherical means of integrals of the function over hyper-planes, and is due to J. Radon [1]. [1] The resulting formulae are closely related to those giving the solution of the initial value problem of the wave equation. [2] Their connection with more general hyperbolic equations with constant coefficients was indicated by G. Herglotz [3], p. 18. This type of decomposition of a function into plane waves could be called the *Radon transform* in contrast to the *Fourier* transform.

Chapter II brings as the first application of the Radon transformation the solution of the *initial value problem for homogeneous hyperbolic equations with constant coefficients*. ("Homogeneous" here refers to the assumption that all derivatives occurring in the equation are of the same order. [3]) The formulae we arrive

[1] Numbers in brackets refer to the bibliography at the end of the tract.

[2] See Mader [1].

[3] For a complete discussion (including inhomogeneous equations) by means of symbolic calculus the reader is referred to Leray [1], [2].

at go back to Herglotz [4] (though with some restrictions of generality) and were extended by Bureau, Gårding, Leray and Petrovskii. They can be obtained in principle by starting from Cauchy's solution [5] by Fourier integrals and by "evaluating" the kernel arising from interchange of integrations. This method of approach is bound to run into convergence difficulties, especially in the case, where the order of the equation is less than the number of dimensions. In the latter case there just does not exist an "integral representation" of the solution in terms of the data in the ordinary sense, since the solution does not depend ***continuously*** on the data, if the ***maximum norm*** is adopted. All one has a right to expect is that the solution of the initial value problem for a hyperbolic equation depends continuously on the initial data ***and*** on a finite number of their derivatives. [6] If an integral representation is to be used, it has to be interpreted in some generalized sense, say as "finite" or "logarithmic" part of an improper integral, as in Hadamard's theory of second order equations and in the work of Bureau, [7] or following M. Riesz [1] by analytic continuation of proper integrals, or as a distribution in the sense of L. Schwartz [2]. All these generalized integral representations can be made "concrete" in the form of derivatives of ordinary integrals, though the transition may require rather unwieldy computations with singular integrals. In contrast to that the method employed here (for homogeneous equations) avoids all convergence difficulties by working with the simpler Radon decomposition into plane waves instead of the Fourier integrals. The solution is then obtained immediately in the form of an iterated Laplacean applied to a perfectly regular integral operator acting on the initial data. It is only when one attempts to simplify the expression by carrying out some of the differentiations explicitly that the classical difficulties re-appear. [7a]

[4] See Herglotz [1], [2]. Herglotz also gave an exposition of the subject in his course on "Mechanik der Kontinua," Göttingen, 1931 (see [3]).

[5] See Cauchy [1], Courant-Hilbert [1], vol. II, ch. III, Bureau [9].

[6] See Hadamard [1], Gårding [1].

[7] See Bureau [3], [4], [9].

[7a] For a related procedure for general hyperbolic systems with constant coefficients see R. Courant and A. Lax [1].

Chapter III gives the construction of the *fundamental solution* for a linear elliptic equation, and more generally for a linear elliptic system, with analytic coefficients. The problem amounts to finding a solution of the symbolic equation

$$L[u] = \delta,$$

where δ is the Dirac function. The method of solution given here amounts to decomposing the Dirac function into plane waves and thus to reducing the problem to that of finding a solution of $L[u]=f$, where f is a plane wave function. The latter problem in turn can be solved as a consequence of the theorem of Cauchy and Kowalewski. In this way a fundamental solution in the small is obtained in a form for which it is easy to analyze the nature of the singularity to any order of magnitude desired.[8] For equations with constant coefficients one finds explicit expressions for the fundamental solution, which only involve quadratures. This special case is of importance, because it furnishes the *parametrix* solutions that can be used for general linear elliptic equations with non-analytic coefficients.[9]

Chapter IV derives *expressions for an arbitrary function f in terms of spherical integrals of f.* For the applications it is important that the radii of the spheres occurring in those expressions are bounded away from zero. These formulae form the principal tool used in the remaining chapters. They can be looked at as generalizations of the formulae of Chapter I, which express f in terms of its integrals over planes. The resulting formulae for f in terms of its spherical means are not particularly elegant. Fortunately it is only the general form of the expression that matters for the later applications. The identities in Chapter IV are closely related to Huygens' principle for the wave equation and to certain identities for Bessel functions. They can also be looked at as the analytic counterpart of the geometrical observation that spherical shells can be swept out by spheres in two different ways

[8] For the case of equations of second order the fundamental solution had been constructed by Hadamard [1]; book II, Ch. III). See also Thomas and Titt [1], Bureau [2], Miranda [1].

[9] See E. E. Levi [1], John [7], pp. 155—162.

(see Fig. 8), just as the identities of Chapter I are connected with the fact that the exterior of a sphere can be swept out by planes.[10]

Chapter V brings the *identity of Asgeirsson* together with a somewhat more general *identity due to A. Howard.* The latter identity illuminates Asgeirsson's theorem by relating it to the geometry of linear families of quadrics in tangential coordinates. The theorem of Mrs. Howard exceeds the bounds set to this monograph by its title, in so far as it deals with *ellipsoidal* means instead of *spherical* ones. It has the interesting application that it permits to transform a homogeneous differential equation of order $2m$ with constant coefficients into a similar equation of order m in more independent variables.

Chapter VI deals mostly with the problem of determining a function from its integrals over spheres of radius 1. The problem can be solved on the basis of the identities of Chapter IV. The solution by decomposition into plane waves of more general problems for "mean-periodic" functions is indicated.

Chapter VII gives the main application of the identities on spherical means derived in Chapter IV. It presents proofs for the differentiability of solutions of linear or non-linear elliptic equations or of systems of equations, provided the coefficients are sufficiently regular. It also contains a proof for the analyticity of solutions of linear elliptic equations with analytic coefficients. (Another proof is implicit in the construction of analytic fundamental solutions of such equations in Chapter III.)

Chapter VIII contains an extension of the results of Chapter VII to linear *non-elliptic* equations. Here not the regularity of the *solutions* but of certain *integral transforms of the solution* is established. More precisely integrals of a solution over a family of *time-like* curves with common endpoints are shown to depend as regular on the parameter distinguishing the members of the family as the coefficients of the differential equation and the regularity assumptions on the curves of the family permit.

[10] Similarly Asgeirsson's theorem in $2+2$ dimensions corresponds to the geometrical fact that a hyperboloid of one sheet is covered by straight lines in two distinct ways. See John [6].

CHAPTER I

Decomposition of an Arbitrary Function into Plane Waves

Explanation of notation

In what follows the letters $x, y, z, X, Y, Z, \xi, \eta, \zeta$ will always stand for the *vectors* $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n), \ldots, (\zeta_1, \ldots, \zeta_n)$ in n-dimensional space where $n \geqq 2$. All other letters will stand for *scalar* variables. The *scalar product* $\sum_{i=1}^{n} x_i y_i$ of the vectors x and y will be denoted by $x \, . \, y$, the length $(x \cdot x)^{1/2}$ of the vector x by $|x|$. The volume element $dx_1 \ldots dx_n$ will be abbreviated to dx, while dS_x will denote the surface element of a hyper-surface in n-dimensional space. The spherical surface of radius 1 about the origin in x-space will be denoted by Ω_x, its surface element by $d\omega_x$, its total surface measure by ω_n. The volume of the unit-sphere in n-space is then $(1/n)\omega_n$. Integrations are carried out over the whole range of a variable, unless other limits are indicated.

The spherical mean of a function of a single coordinate

Let $g(s)$ be a continuous function of the scalar variable s. Denoting by y a fixed vector, we have in $g(y \cdot x)$ a function of $x = (x_1, \ldots, x_n)$, which is constant along the hyperplanes perpendicular to the direction of y; (such a function will be called a "plane wave" function). We form the integral of $g(y \cdot x)$ over the solid sphere of radius r about the origin by decomposing the sphere into plane sections perpendicular to the y-direction. On the plane $y \cdot x = |y| p$ of distance $|p|$ from the origin the function $g(x \, . \, y)$ has the value $g(|y| p)$. The $(n-1)$-dimensional intersection of that plane with the sphere has the volume (see Fig. 1)

$$\frac{\omega_{n-1}}{n-1}(r^2-p^2)^{(n-1)/2}.$$

It follows that

$$(1.1)\qquad \int\limits_{|x|<r} g(y\cdot x)dx = \frac{\omega_{n-1}}{n-1}\int_{-r}^{+r}(r^2-p^2)^{\frac{n-1}{2}}g(|y|p)dp.$$

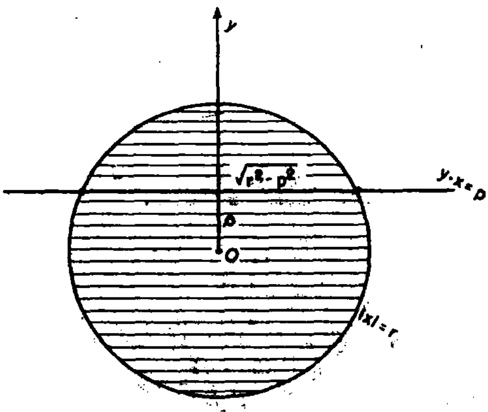

Figure 1

Differentiating with respect to r and putting $r = 1$ we obtain the fundamental identity

$$(1.2)\qquad \int\limits_{\Omega_n} g(y\cdot x)d\omega_x = \omega_{n-1}\int_{-1}^{+1}(1-p^2)^{(n-3)/2}g(|y|p)dp = \omega_n h(|y|)$$

for the integral of a plane wave function over the unitsphere, valid for $n \geqq 2$. [Here h is defined by (1.2).]

For $g(s) = \text{const.} = 1$ we have $h = 1$, and (1.2) yields the recursion formula

$$(1.3)\qquad \frac{\omega_n}{\omega_{n-1}} = \int_{-1}^{1}(1-p^2)^{(n-3)/2}dp = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

From this formula one derives the well known value

$$\omega_n = \frac{2\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2}\right)} \tag{1.4}$$

for the surface area of the unit sphere in n-space. [11]

For $g(s) = e^{is}$ we find

$$h(s) = \frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}(1-p^2)^{\frac{n-3}{2}} e^{isp}\,dp = \frac{2^\nu \Gamma(\nu+1)}{s^\nu} J_\nu(s), \tag{1.5}$$

where J_ν is the Bessel function of index $\nu = (n-2)/2$. [12]

Taking $g(s) = |s|^k$ and $g(s) = |s|^k \log|s|$ in (1.2) yields respectively the identities

$$\int_{\Omega_x} |y\cdot x|^k\, d\omega_x = \frac{2\sqrt{\pi}^{n-1}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} |y|^k \tag{1.6}$$

$$\int_{\Omega_x} |y\cdot x|^k \log|y\cdot x|\, d\omega_x = \frac{2\sqrt{\pi}^{n-1}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} |y|^k(\log|y| + c_{nk}) \tag{1.7}$$

with a certain numerical constant c_{nk}. Formulae (1.6), (1.7) have been derived under the assumption that $g(s)$ is continuous, and hence that $k > 0$; they obviously then also hold for $k = 0$. They form the basis of the decomposition of an arbitrary function into plane waves, discussed in the next section.

Representation of a function by its plane integrals

We consider an arbitrary function $f(x)$ of class C_1, which

[11] See Courant-Hilbert [1], vol. II, p. 223.

[12] This is essentially Poisson's representation of the Bessel functions. See Magnus-Oberhettinger [1], p. 26, § 5.

vanishes outside a bounded set.[13] Then

$$u(z) = \int f(y) \frac{|y-z|^{2-n}}{(2-n)\omega_n} dy \tag{1.8}$$

is a function of z of class C_2, which satisfies Poisson's differential equation

$$\Delta_z u(z) = f(z) \tag{1.9}$$

where Δ_z denotes the Laplacean with respect to the variables $z_1, \ldots, z_n$. (For $n = 2$ the kernel has to be replaced by $(1/2\pi) \log |y - z|$.) For the proof[14] of (1.9) we observe that

$$\begin{aligned}
\Delta_z u &= \frac{-1}{\omega_n} \sum_i \frac{\partial}{\partial z_i} \int f(y)\,(y_i - z_i)\,|y-z|^{-n} dy \\
&= \frac{-1}{\omega_n} \sum_i \frac{\partial}{\partial z_i} \int f(y+z) y_i\,|y|^{-n} dy \\
&= -\frac{1}{\omega_n} \sum_i \int f_{y_i}(y+z) y_i\,|y|^{-n} dy \\
&= -\frac{1}{\omega_n} \lim_{r\to 0} \sum_i \int_{|y|>r} f_{y_i}(y+z) y_i\,|y|^{-n} dy \\
&= -\frac{1}{\omega_n} \lim_{r\to 0} \sum_i \left[\int_{|y|=r} \frac{-y_i^2}{r} |y|^{-n} f(y+z) dS_y - \int_{|y|>r} f(y+z) \frac{\partial}{\partial y_i} (y_i |y|^{-n}) dy \right] \\
&= \frac{1}{\omega_n} \lim_{r\to 0} r^{1-n} \int_{|y|=r} f(y+z) dS_y = f(z)
\end{aligned}$$

The proof of Poisson's equation has been given here in detail, because of its fundamental importance for what follows, since most of the differentiations of singular integrals that will have to be carried out will be reduced to this one formula. It may be mentioned that the same equation can be established under the milder assumption that $f(x)$ satisfies a Hölder condition.[15]

[13] $f(x)$ is of class C_m, if f and all its derivatives of orders $\leq m$ are continuous.
[14] See Courant-Hilbert [1], vol. II, p. 228.
[15] See Kellog [1], p. 156.

We now take for even n identity (1.7), for odd n identity (1.6), replace y by $y - z$, multiply with $f(y)$ and integrate over all y. (We still assume that f is of class C_1 and vanishes outside a bounded set.) We choose for k a non-negative integer such that $n + k$ is an even number, and apply the operator Δ_z to the resulting equation $(n + k)/2$-times. Observing that

$$\Delta_z \mid y - z \mid^k = k(k + n - 2) \mid y - z \mid^{k-2}$$

we find respectively for odd and even $n > 2$

$$\text{(1.9a)} \quad (\Delta_z)^{(n+k-2)/2} | y - z |^k = \frac{2^{n+k-1}\,\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{k+n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{(2-n)\pi} (-1)^{(n-1)/2} | y - z |^{2-n}$$

$$\text{(1.9b)} \quad (\Delta_z)^{(n+k-2)/2} | y - z |^k \log | y - z | = \frac{2^{n+k-2}\,\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{k+n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{2-n} (-1)^{(n-2)/2} | y - z |^{2-n}$$

Hence from (1.6), (1.7), (1.9)

$$\text{(1.10)} \quad (\Delta_z)^{(n+k)/2} \int \left(\int_{\Omega_x} f(y) \,|\, (y - z) \cdot x \,|^k d\omega_x \right) dy = 4(2\pi i)^{n-1} k! f(z)$$

for odd n and $k = 1, 3, 5, \ldots$

$$\text{(1.11)} \quad (\Delta_z)^{(n+k)/2} \int \left(\int_{\Omega_x} f(y) ((y - z) \cdot x)^k \log | (y - z) \cdot x | \, d\omega_x \right) dy = - (2\pi i)^n k! f(z)$$

r even n and $k = 0, 2, 4, \ldots$ (also for $n = 2$).

e can formally combine these formulae for even and odd n

$$f(z) = (\Delta_z)^{(n+k)/2} \mathscr{Re} \left[- \frac{1}{k!(2\pi i)^n} \int \left(\int_{\Omega_x} f(y) [(y - z) \cdot x]^k \times \log \left[\frac{1}{i} (y - z) \cdot x \right] d\omega_x \right) dy \right]$$

where log s denotes the principal branch of that function defined in the complex s-plane slit along the negative real axis.

Formulae (1.10), (1.11) represent a solution of the problem of obtaining a function $f(z)$ as a linear combination of "plane wave" functions of z. These plane waves here either have the form $|(y-z)\cdot x|^k$ or $((y-z)\cdot x)^k \log|(y-z)\cdot x|$. A different solution of the same problem is of course given by the Fourier integral representing f:

$$f(z) = \int g(y) e^{iz\cdot y}\, dy,$$

which decomposes $f(z)$ into the plane wave functions $e^{iy\cdot z}$. The advantage of the formulae (1.10), (1.11) is that the integrals contain f itself instead of its Fourier transform.

Formulae (1.10), (1.11) can also be interpreted as expressing $f(z)$ in terms of the integrals of f over hyper-planes. For $|x| = 1$

$$J(x, p) = \int_{y\cdot x = p} f(y)\, dS_y \tag{1.12}$$

represents the integral of f over the hyperplane with unit normal x and (signed) distance p from the origin. By definition (1.12) $J(x, p) = J(-x, -p)$. Taking for an odd n formula (1.10) with $k = 1$ we have

$$\begin{aligned}\int\!\!\int_{\Omega_x} f(y)\,|(y-z)\cdot x|\, d\omega_x\, dy &= \int_{\Omega_x} d\omega_x \int_{-\infty}^{+\infty} |p|\, dp \int_{(y-z)\cdot x = p} f(y)\, dS_y \\ &= \int_{\Omega_x} d\omega_x \int_{-\infty}^{+\infty} |p|\, J(x, p + z\cdot x)\, dp\end{aligned} \tag{1.13}$$

Observing that for $|x| = 1$

$$\Delta_z \int_{-\infty}^{+\infty} |p|\, J(x, p + z\cdot x)\, dp$$

$$= \Delta_z \left[\int_{z\cdot x}^{\infty} (p - z\cdot x) J(x, p)\, dp - \int_{-\infty}^{z\cdot x} (p - z\cdot x) J(x\cdot p)\, dp \right]$$

we find from (1.10) that for odd n

$$2(2\pi^{1}f(z)i)^{n-} = (\Delta_x)^{(n-1)/2}\int_{\Omega_x} J(x,\ x\cdot z)d\omega_x \tag{1.14}$$

Here the integral represents (except for a constant factor ω_n) the average of the plane integrals of f for the planes passing through the point z. A similar formula can be derived for even n from (1.11) with $k = 0$. We notice here that for $|x| = 1$

$$\Delta_x \int_{-\infty}^{+\infty} \log|p|\, J(x, p + z\cdot x)dp = \int_{-\infty}^{+\infty} (\log|p|)J_{pp}(x, p+z\cdot x)dp$$

$$= \int_{-\infty}^{+\infty} (\log|p - z\cdot x|)\, J_{pp}(x, p)dp = -\int_{-\infty}^{+\infty} \frac{1}{p-z\cdot x} J_p(x, p)dp$$

$$= -\int_{p=-\infty}^{p=+\infty} \frac{dJ(x, p)}{p - x\cdot z}$$

where in the last two integrals the Cauchy principal value is to be taken. Then from (1.11) for even n

$$(2\pi i)^n f(z) = (\Delta_x)^{(n-2)/2}\int_{\Omega_x} d\omega_x \int_{p=-\infty}^{p=+\infty} \frac{dJ(x, p)}{p - z\cdot x} \tag{1.15}$$

Expressions equivalent to (1.14), (1.15) for a function $f(z)$ in terms of its plane integrals J were first given by J. Radon [16]. Expressions of a different type, related to the solution of the wave equation have been given by Ph. Mader. [17]

[16] See Radon [1], Bureau [9], ch. IX.
[17] See Mader [1].