Dynamical Systems

Differential equations, maps and chaotic behaviour

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and
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CHAPMAN & HALL MATHEMATICS

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Preface

In recent years there has been unprecedented popular interest in the chaotic behaviour of discrete dynamical systems. The ease with which a modest microcomputer can produce graphics of extraordinary complexity has fired the interest of mathematically-minded people from pupils in schools to postgraduate students. At undergraduate level, there is a need to give a basic account of the computed complexity within a recognized framework of mathematical theory. In producing this replacement for *Ordinary Differential Equations* (*ODE*) we have responded to this need by extending our treatment of the qualitative behaviour of differential equations.

This book is aimed at second and third year undergraduate students who have completed first courses in Calculus of Several Variables and Linear Algebra. Our approach is to use examples to illustrate the significance of the results presented. The text is supported by a mix of manageable and challenging exercises that give readers the opportunity to both consolidate and develop the ideas they encounter. As in *ODE*, we wish to highlight the significance of important theorems, to show how they are used and to stimulate interest in a deeper understanding of them.

We have retained our earlier introduction and discussion of linear systems (Chapters 1 and 2). Our treatment of non-linear differential equations has been extended to include Poincaré maps and phase spaces of dimension greater than two (Chapters 3 and 4). Applications involving planar phase spaces (covered in Chapter 4 of *ODE*) appear in Chapter 5. Problems involving non-planar phase spaces and families of systems are considered in Chapter 6, where elementary bifurcation theory is introduced and its application to chaotic behaviour is examined. Although ordinary differential equations remain the driving force behind the book, a substantial part of the new material concerns discrete dynamical systems and the title *Ordinary Differential Equations* is no longer appropriate. We have therefore chosen a new title for the extended text that clarifies its connection with the broader field of dynamical systems.

In addition to Professors Brown and Eastham, and Drs Knowles and Smith, who read and commented on the manuscript for *ODE*, we would like to thank those readers who kindly drew our attention to some failings of that book. In relation to the new material appearing in this text we must thank Dr A. Lansbury for her help with the Duffing problem and Mrs G. A. Place for her assistance with the continuity of the assembled manuscript. We are also grateful to Cambridge University Press, the *Quarterly Journal of Applied Mathematics* and the American Institute of Physics for allowing us to use diagrams from some of their publications. Once again we must acknowledge the forbearance of our families and one of us (CMP) would like to thank the Brayshay Foundation for its financial support.

Contents

1

Introduction

In this chapter we illustrate the qualitative approach to differential equations and introduce some key ideas such as phase portraits and qualitative equivalence.

1.1 PRELIMINARY IDEAS

1.1.1 Existence and uniqueness

Definition 1.1.1

Let $X(t,x)$ be a real-valued function of the real variables t and x, with domain $D \subseteq \mathbb{R}^2$. A function $x(t)$, with t in some open interval $I \subseteq \mathbb{R}$, which satisfies

$$\dot{x}(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = X(t, x(t)) \tag{1.1}$$

is said to be a **solution** of the differential equation (1.1).

A necessary condition for $x(t)$ to be a solution is that $(t, x(t)) \in D$ for each $t \in I$; so that D limits the domain and range of $x(t)$. If $x(t)$, with domain I, is a solution to (1.1) then so is its restriction to any interval $J \subset I$. To prevent any confusion, we will always take I to be the largest interval for which $x(t)$ satisfies (1.1). Solutions with this property are called **maximal** solutions. Thus, unless otherwise stated, we will use the word 'solution' to mean 'maximal solution'. Consider the following examples of (1.1) and their solutions; we give

$$\dot{x} = X(t,x), \qquad D, \qquad x(t), \qquad I$$

in each case (C and C' are real numbers):

1.	$\dot{x} = x - t$,	$\mathbb{R}^2$,	$1 + t + Ce^t$,	$\mathbb{R}$;
2.	$\dot{x} = x^2$,	$\mathbb{R}^2$,	$(C - t)^{-1}$,	$(-\infty, C)$
			0,	$\mathbb{R}$
			$(C' - t)^{-1}$,	(C', ∞);

3. $\dot{x} = -x/t$, $\{(t,x)|t \neq 0\}$, C/t, $(-\infty,0)$
C'/t, $(0,\infty)$;

4. $\dot{x} = 2x^{1/2}$, $\{(t,x)|x \geqslant 0\}$, $\begin{cases} 0, & (-\infty, C) \\ (t-C)^2, & [C,\infty) \end{cases}$
0, $\mathbb{R}$;

5. $\dot{x} = 2xt$, $\mathbb{R}^2$, Ce^{t^2}, $\mathbb{R}$;
6. $\dot{x} = -x/\tanh t$, $\{(t,x)|t \neq 0\}$, $C/\sinh t$, $(-\infty,0)$
$C'/\sinh t$, $(0,\infty)$.

The existence of solutions is determined by the properties of X. The following proposition is stated without proof (Petrovski, 1966).

Proposition 1.1.1
If X is continuous in an open domain, $D' \subseteq D$, then given any pair $(t_0, x_0) \in D'$, there exists a solution $x(t)$, $t \in I$, of $\dot{x} = X(t,x)$ such that $t_0 \in I$ and $x(t_0) = x_0$.

For example, consider

$$\dot{x} = 2|x|^{1/2}, \tag{1.2}$$

where $D = \mathbb{R}^2$. Any pair (t_0, x_0) with $x_0 \geqslant 0$ is given by $(t_0, x(t_0))$ when $x(t)$ is the solution

$$x(t) = \begin{cases} 0, & t \in (-\infty, C) \\ (t-C)^2, & t \in [C, \infty) \end{cases} \tag{1.3}$$

and $C = t_0 - \sqrt{x_0}$. A solution can similarly be found for pairs (t_0, x_0) when $x_0 < 0$.

Observe that Proposition 1.1.1 does not exclude the possibility that $x(t_0) = x_0$ for more than one solution $x(t)$. For example, for (1.2) infinitely many solutions $x(t)$ satisfy $x(t_0) = 0$; namely every solution of the form (1.3) for which $C > t_0$ and solution $x(t) \equiv 0$.

The following proposition gives a sufficient condition for each pair in D' to occur in one and only one solution of (1.1).

Proposition 1.1.2
If X and $\partial X/\partial x$ are continuous in an open domain $D' \subseteq D$, then given any $(t_0, x_0) \in D'$ there exists a unique *solution $x(t)$ of $\dot{x} = X(t,x)$ such that $x(t_0) = x_0$.*

Notice that, while $X = 2|x|^{1/2}$ is continuous on $D(= \mathbb{R}^2)$, $\partial X/\partial x (= |x|^{-1/2}$ for $x > 0$ and $-|x|^{-1/2}$ for $x < 0)$ is continuous only on $D' = \{(t,x)|x \neq 0\}$; it is undefined for $x = 0$. We have already observed that the pair $(t_0, 0)$, $t_0 \in \mathbb{R}$ occurs in infinitely many solutions of $\dot{x} = 2|x|^{1/2}$.

On the other hand, $X(t,x) = x - t$ and $\partial X/\partial x = 1$ are continuous throughout the domain $D = \mathbb{R}^2$. Any (t_0, x_0) occurs in one and only one solution of $\dot{x} = x - t$; namely

$$x(t) = 1 + t + Ce^t \tag{1.4}$$

when $C = (x_0 - t_0 - 1)e^{-t_0}$.

Weaker sufficient conditions for existence and uniqueness do exist (Petrovski, 1966). However, Propositions 1.1.1 and 1.1.2 illustrate the kind of properties required for $X(t, x)$.

1.1.2 Geometrical representation

A solution $x(t)$ of $\dot{x} = X(t, x)$ is represented geometrically by the graph of $x(t)$. This graph defines a **solution curve** in the t, x-plane.

If X is continuous in D, then Proposition 1.1.1 implies that the solution curves fill the region D of the t, x-plane. This follows because each point in D must lie on at least one solution curve. The solutions of the differential equation are, therefore, represented by a **family of solution curves** in D (as illustrated in Figs 1.1–1.8).

If both X and $\partial X/\partial x$ are continuous in D then Proposition 1.1.2 implies that there is a unique solution curve passing through every point of D (as shown in Figs 1.1–1.6).

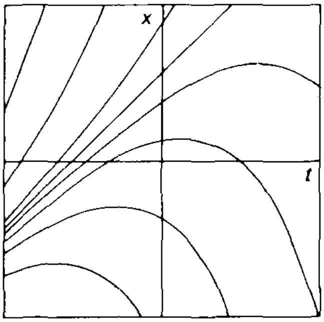

Fig. 1.1. $\dot{x} = x - t$.

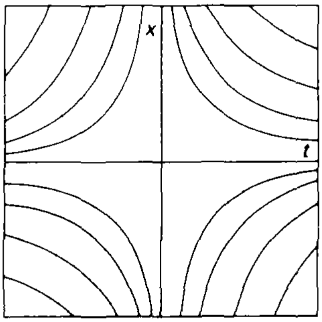

Fig. 1.2. $\dot{x} = -x/t, t \neq 0$.

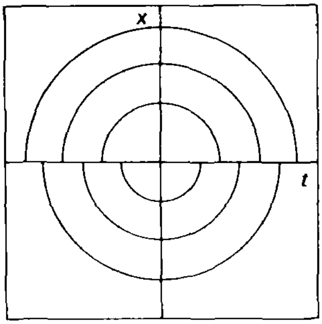

Fig. 1.3. $\dot{x} = -t/x$.

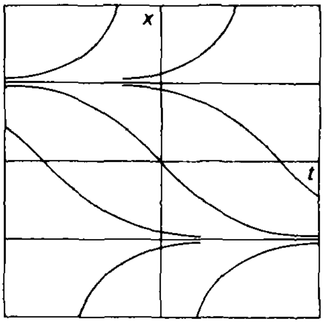

Fig. 1.4. $\dot{x} = \frac{1}{2}(x^2 - 1)$.

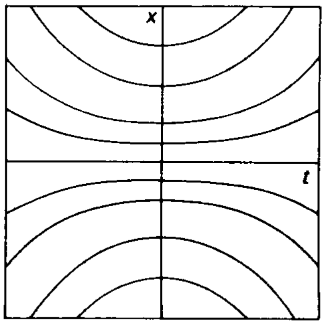

Fig. 1.5. $\dot{x} = 2xt$.

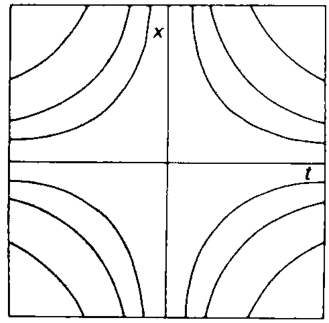

Fig. 1.6. $\dot{x} = -x/\tanh t, t \neq 0$.

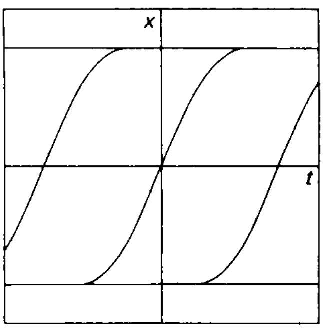

Fig. 1.7. $\dot{x} = \sqrt{(1 - x^2)}, |x| \leqslant 1$.

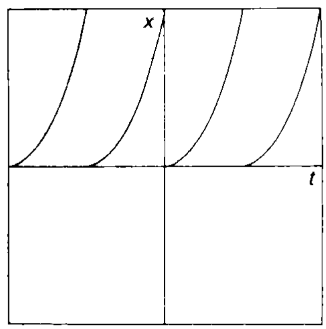

Fig. 1.8. $\dot{x} = 2x^{1/2}, x \geqslant 0$.

Observe that the families of solution curves in Figs 1.2 and 1.6 bear a marked resemblance to one another. Every solution curve in one figure has a counterpart in the other; they are similar in shape, have the same asymptotes, etc., but they are not identical curves. The relationship between these two families of solution curves is an example of what we call **qualitative equivalence** (also described in sections 1.3, 2.4 and 3.3). We say that the **qualitative behaviour** of the solution curves in Fig. 1.2 is the same as those in Fig. 1.6.

Accurate plots of the solution curves are not always necessary to obtain their qualitative behaviour; a sketch is often sufficient. We can sometimes obtain a sketch of the family of solution curves directly from the differential equation.

Example 1.1.1

Sketch the solution curves of the differential equation

$$\dot{x} = t + t/x \tag{1.5}$$

in the region D of the t, x-plane where $x \neq 0$.

Solution

We make the following observations.

1. The differential equation gives the slope of the solution curves at all points of the region D. Thus, in particular, the solution curves cross the curve $t + t/x = k$, a constant, with slope k. This curve is called the **isocline** of slope k. The set of isoclines, obtained by taking different real values for k, is family of hyperbolae
$$x = \frac{t}{k - t}, \tag{1.6}$$
with asymptotes $x = -1$ and $t = k$. A selection of these isoclines is shown in Fig. 1.9.
2. The sign of $\ddot{x}$ determines where in D the solution curves are concave and convex. If $\ddot{x} > 0$ (< 0) then $\dot{x}$ is increasing (decreasing) with t and the solution curve is said to be **convex** (**concave**). The region D can therefore be divided into subsets on which the solution curves are either concave or convex separated by boundaries where $\ddot{x} = 0$. For (1.5) we find
$$\ddot{x} = x^{-3}(x + 1)(x - t)(x + t) \tag{1.7}$$
and D can be split up into regions $P(\ddot{x} > 0)$ and $N(\ddot{x} < 0)$ as shown in Fig. 1.10.
3. The isoclines are symmetrically placed relative to $t = 0$ and so there must also be symmetry of the solution curves. The function $X(t, x) = t + t/x$ satisfies $X(-t, x) = -X(t, x)$ and thus if $x(t)$ is a solution to $\dot{x} = X(t, x)$ then so is $x(-t)$ (cf. Exercise 1.5).

These three observations allow us to produce a sketch of the solution curves for $\dot{x} = t + t/x$ as in Fig. 1.11. Notice that both $X(t, x) = t + t/x$ and

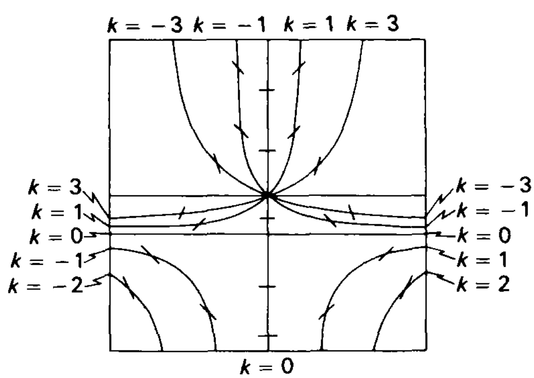

Fig. 1.9. Selected isoclines for the equation $\dot{x} = t + t/x$. The short line segments on the isoclines have slope k and indicate how the solution curves cross them.

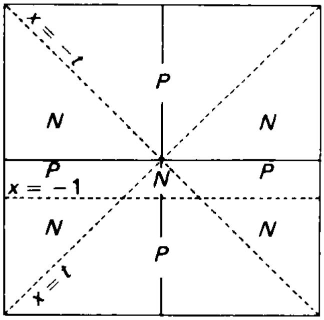

Fig. 1.10. Regions of convexity (P) and concavity (N) for solutions of $\dot{x} = t + t/x$.

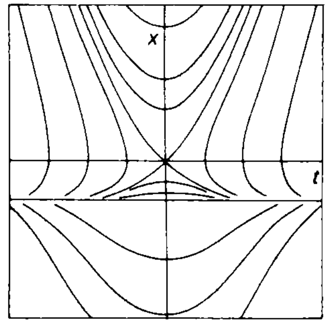

Fig. 1.11. The solution curves of the differential equation $\dot{x} = t + t/x$ in the t, x-plane.

$\partial X/\partial x = -t/x^2$ are continuous on $D = \{(t,x) | x \neq 0\}$, so there is a unique solution curve passing through each point of D. □

It is possible to find the solutions of

$$\dot{x} = t + t/x \tag{1.8}$$

by separation of the variable (defined in Exercise 1.2). We obtain the equation

$$x - \ln|x+1| = \tfrac{1}{2}t^2 + C, \tag{1.9}$$

C a constant, for the family of solution curves as well as the solution $x(t) \equiv -1$. However, to sketch the solution curves from (1.9) is less straightforward than to use (1.8) itself.

The above discussion has introduced two important ideas:

1. that two different differential equations can have solutions that exhibit the same qualitative behaviour; and
2. that the qualitative behaviour of solutions is determined by $X(t, x)$.

We will now put these two ideas together and illustrate the qualitative approach to differential equations for the special case of equations of the form $\dot{x} = X(x)$. We shall see that such equations can be classified into qualitatively equivalent types.

1.2 AUTONOMOUS EQUATIONS

1.2.1 Solution curves and the phase portrait

A differential equation of the form

$$\dot{x} = X(x), \qquad x \in S \subseteq \mathbb{R}, \qquad (D = \mathbb{R} \times S) \tag{1.10}$$

is said to be **autonomous**, because $\dot{x}$ is determined by x alone and so the solutions are, as it were, self-governing.

The solutions of autonomous equations have the following important property. If $\xi(t)$ is a solution of (1.10) with domain I and range $\xi(I)$ then $\eta(t) = \xi(t + C)$, for any real C, is also a solution with the same range, but with domain $\{t \mid t + C \in I\}$. This follows because

$$\dot{\eta}(t) = \dot{\xi}(t + C) = X(\xi(t + C)) = X(\eta(t)). \qquad (1.11)$$

The solution curve $x = \xi(t)$ is obtained by translating the solution curve $x = \eta(t)$ by the amount C in the positive t-direction.

Furthermore if there exists a unique solution curve passing through each point of strip $D' = \mathbb{R} \times \xi\,(I)$ then all solution curves on D' are translations of $x = \xi(t)$. The domain D is therefore divided into strips where the solution curves are all obtained by shifting a single curve in the t-direction (as shown in Figs 1.12–1.15). For example.

$$\dot{x} = x \qquad (1.12)$$

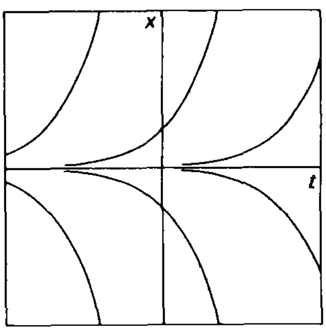

Fig. 1.12. $\dot{x} = x$: strips D' consist of the half-planes $x < 0$ and $x > 0$.

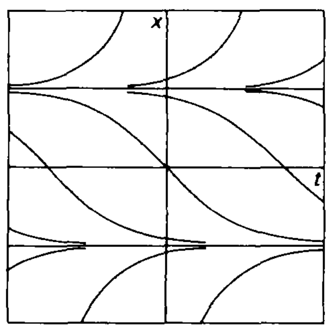

Fig. 1.13. $\dot{x} = \frac{1}{2}(x^2 - 1)$: strips $D' = \mathbb{R} \times \xi(I)$ with $\xi(I) = (-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.

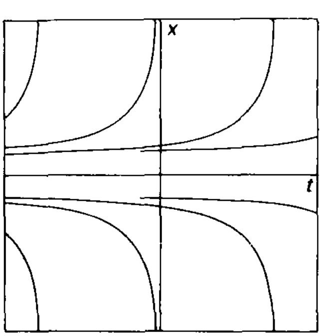

Fig. 1.14. Solution curves for $\dot{x} = x^3$.

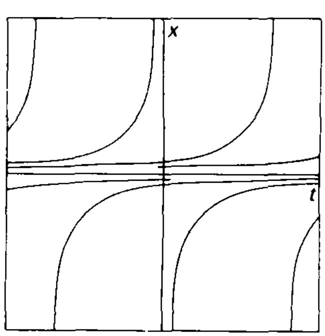

Fig. 1.15. Solution curves for $\dot{x} = x^2$.

has solutions:

$$\xi(t) = e^t, \qquad I = \mathbb{R}, \qquad \xi(I) = (0, \infty) \tag{1.13}$$

$$\xi(t) \equiv 0, \qquad I = \mathbb{R}, \qquad \xi(I) = \{0\} \tag{1.14}$$

$$\xi(t) = -e^t, \qquad I = \mathbb{R}, \qquad \xi(I) = (-\infty, 0). \tag{1.15}$$

All the solution curves in the strip D' defined by $x \in (0, \infty)$, $t \in \mathbb{R}$ are translations of e^t. Similarly, those in $D' = \{(t, x) | x \in (-\infty, 0), t \in \mathbb{R}\}$ are translations of $-e^t$.

For families of solution curves related by translations in t, the qualitative behaviour of the family of solutions is determined by that of any individual member. The qualitative behaviour of such a sample curve is determined by $X(x)$. When $X(x) \neq 0$, then the solution is either increasing or decreasing; when $X(c) = 0$ there is a solution $x(t) \equiv c$.

This information can be represented on the x-line rather than the t, x-plane. If $X(x) \neq 0$ for $x \in (a, b)$ then the interval is labelled with an arrow showing the sense in which x is changing. When $X(c) = 0$, the solution $x(t) \equiv c$ is represented by the point $x = c$. These solutions are called **fixed points** of the equation because x remains at c for all t. This geometrical representation of the qualitative behaviour of $\dot{x} = X(x)$ is called its **phase portrait**. Some examples of phase portraits are shown, in relation to X, in Figs 1.16–1.19. The corresponding families of solution curves are given in Figs 1.12–1.15.

If x is not stationary it must either be increasing or decreasing. Thus for a given finite number of fixed points there can only be a finite number of 'different' phase portraits. By 'different', we mean with distinct assignments of where x is increasing or decreasing. For example, consider a single fixed point $x = c$ (Fig. 1.20). For $x < c$, $X(x)$ must be either positive or negative and similarly for $x > c$. Hence, one of the four phase portraits shown must occur. This means that the qualitative behaviour of any autonomous differential equation with one fixed point must correspond to one of the phase

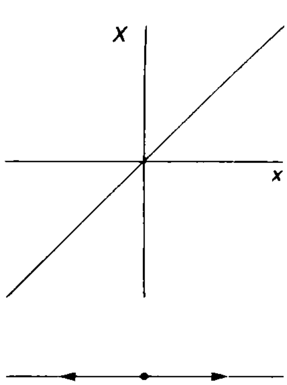

Fig. 1.16. $\dot{x} = x$, $x = 0$ is a fixed point.

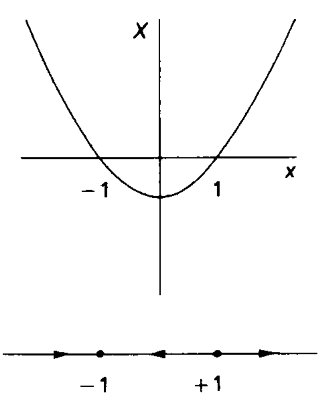

Fig. 1.17. $\dot{x} = \frac{1}{2}(x^2 - 1)$, $x = \pm 1$ are fixed points.

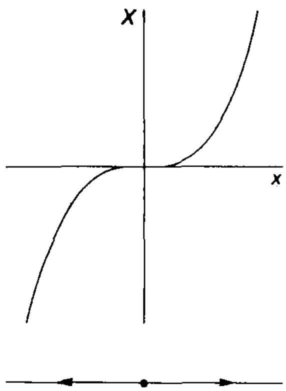

Fig. 1.18. $\dot{x} = x^3$, $x = 0$ is a fixed point.

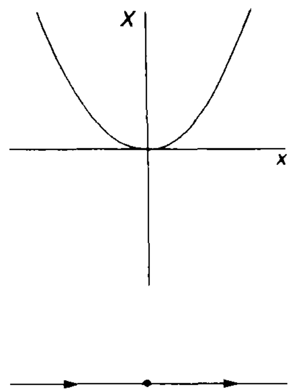

Fig. 1.19. $\dot{x} = x^2$, $x = 0$ is a fixed point.

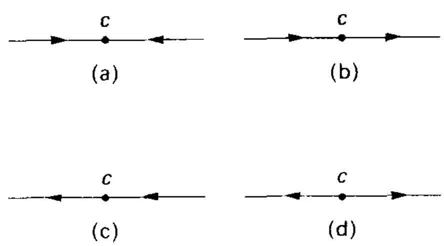

Fig. 1.20. The four possible phase portraits associated with a single fixed point. The fixed point is described as an **attractor** in (a), a **shunt** in (b) and (c) and a **repellor** in (d).

portraits in Fig. 1.20 for some value of c. For example, $\dot{x} = x$, $\dot{x} = x^3$, $\dot{x} = x - a$, $\dot{x} = (x - a)^3$, $\dot{x} = \sinh x$, $\dot{x} = \sinh(x - a)$ all correspond to Fig. 1.20(d) for $c = 0$ or a. Of course, two different equations, each having one fixed point, that correspond to the same phase portrait in Fig. 1.20 have the same qualitative behaviour. We say that two such differential equations are **qualitatively equivalent**.

Now observe that the argument leading to Fig. 1.20 holds equally well if the fixed point at $x = c$ is one of many in a phase portrait. In other words, the qualitative behaviour of x in the neighbourhood of any fixed point must be one of those illustrated in Fig. 1.20(a)–(d). We say that this behaviour determines the **nature of the fixed point** and use the terminology defined in the caption to Fig. 1.20 to describe this.

This is an important step because it implies that the phase portrait of any autonomous equation is determined completely by the nature of its fixed points. We can make the following definition.